



# CHAPTER 4

## Matrices

### BIG Ideas

- Organize data in matrices.
- Perform operations with matrices and determinants.
- Transform figures on a coordinate plane.
- Find the inverse of a matrix.
- Use matrices to solve systems of equations.

### Key Vocabulary

determinant (p. 194)

identity matrix (p. 208)

inverse (p. 209)

matrix (p. 162)

scalar multiplication (p. 171)

### Real-World Link

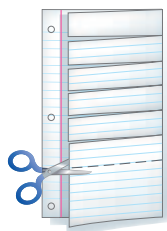
**Data Organization** Matrices are often used to organize data. If the number of male and female students who participate in various sports are organized in separate matrices, the total number of participants can be found by adding the matrices.



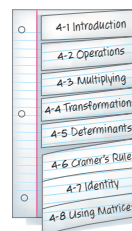
### FOLDABLES Study Organizer

**Matrices** Make this Foldable to help you organize your notes. Begin with one sheet of notebook paper.

- 1** **Fold** lengthwise to the holes. Cut eight tabs in the top sheet.



- 2** **Label** each tab with a lesson number and title.



# GET READY for Chapter 4

**Diagnose Readiness** You have two options for checking Prerequisite Skills.

## Option 2



Take the Online Readiness Quiz at [algebra2.com](http://algebra2.com).

## Option 1

Take the Quick Check below. Refer to the Quick Review for help.

### QUICK Check

Name the additive inverse and the multiplicative inverse for each number. (Lesson 1-2)

1. 3      2. -11      3. 8      4. -0.5

5. 1.25      6.  $\frac{5}{9}$       7.  $-\frac{8}{3}$       8.  $-1\frac{1}{5}$

9. **FOOTBALL** After the quarterback from Central High takes a snap from the center, he drops back 4 yards. How many yards forward does Central High have to go to make it back to the line of scrimmage? (Lesson 1-2)

Solve each system of equations by using either substitution or elimination.

(Lesson 3-2)

10.  $x = y + 5$       11.  $3x - 2y = 1$   
 $3x + y = 19$        $4x + 2y = 20$

12.  $5x + 3y = 25$       13.  $y = x - 7$   
 $4x + 7y = -3$        $2x - 8y = 2$

14. **MONEY** Last year the chess team paid \$7 per hat and \$15 per shirt for a total purchase of \$330. This year they spent \$360 to buy the same number of shirts and hats because the hats now cost \$8 and the shirts cost \$16. Write and solve a system of two equations that represents the number of hats and shirts bought each year. (Lesson 3-2)

### QUICK Review

#### EXAMPLE 1

Name the additive inverse and the multiplicative inverse for  $-\frac{1}{2}$ .

The additive inverse of  $-\frac{1}{2}$  is a number  $x$  such that  $-\frac{1}{2} + x = 0$ .

$$x = \frac{1}{2} \quad \text{Add } \frac{1}{2} \text{ to each side.}$$

The multiplicative inverse of  $-\frac{1}{2}$  is a number  $x$ , such that  $-\frac{1}{2}x = 1$ .

$$x = -2 \quad \text{Multiply each side by } -2.$$

#### EXAMPLE 2

Solve the following system of equations by using either substitution or elimination.

$$\begin{aligned} 2y &= -x + 3 \\ 6x + 7y &= 8 \end{aligned}$$

Since  $x$  has a coefficient of  $-1$  in the first equation, use the substitution method. First solve that equation for  $x$ .

$$2y = -x + 3 \rightarrow x = -2y + 3$$

$$6(-2y + 3) + 7y = 8 \quad \text{Substitute } -2y + 3 \text{ for } x.$$

$$-12y + 18 + 7y = 8 \quad \text{Distributive Property}$$

$$-5y = -10 \quad \text{Combine like terms.}$$

$$y = 2 \quad \text{Divide each side by } -5.$$

To find  $x$ , use  $y = 2$  in the first equation.

$$2(2) = -x + 3 \quad \text{Substitute 2 for } y.$$

$$4 = -x + 3 \quad \text{Multiply.}$$

$$x = -1 \quad \text{Subtract 4 from and add } x \text{ to each side.}$$

The solution is  $(-1, 2)$ .

**Main Ideas**

- Organize data in matrices.
- Solve equations involving matrices.

**New Vocabulary**

matrix  
 element  
 dimension  
 row matrix  
 column matrix  
 square matrix  
 zero matrix  
 equal matrices

**GET READY for the Lesson**

There are many types of sport-utility vehicles (SUVs) in many prices and styles. So, Oleta makes a list of qualities to consider for some top-rated models. She organizes the information in a matrix to easily compare the features of each vehicle.

	Base Price (\$)	Horse-power	Exterior Length (in.)	Cargo Space (ft <sup>3</sup> )	Fuel Economy (mpg)
Hybrid SUV	19,940	153	174.9	66.3	22
Standard SUV	31,710	275	208.4	108.8	15
Mid-Size SUV	27,350	255	188.0	90.3	17
Compact SUV	21,295	165	175.2	64.1	21

Source: cars.com

**Reading Math**

**Matrices** The plural of *matrix* is *matrices*.

**Organize Data** A **matrix** is a rectangular array of variables or constants in horizontal rows and vertical columns, usually enclosed in brackets.

**Real-World EXAMPLE****Organize Data into a Matrix**

The prices for two cable companies are listed below. Use a matrix to organize the information. When is each company's service less expensive?

Metro Cable	
Basic Service (26 channels)	\$11.95
Standard Service (53 channels)	\$30.75
Premium Channels (in addition to Standard Service)	
• One Premium	\$10.00
• Two Premiums	\$19.00
• Three Premiums	\$25.00

Cable City	
Basic Service (26 channels)	\$9.95
Standard Service (53 channels)	\$31.95
Premium Channels (in addition to Standard Service)	
• One Premium	\$8.95
• Two Premiums	\$16.95
• Three Premiums	\$22.95

Organize the costs into labeled columns and rows.

	Basic	Standard	Standard Plus One Premium	Standard Plus Two Premiums	Standard Plus Three Premiums
Metro Cable	11.95	30.75	40.75	49.75	55.75
Cable City	9.95	31.95	40.90	48.90	54.90

Metro Cable has the best price for standard service and standard plus one premium channel. Cable City has the best price for the other categories.



### CHECK Your Progress

1. Use a matrix to organize and compare the following information about some roller coasters.

Roller Coaster	Batman the Escape	Great White	Mr. Freeze
Speed (mph)	55	50	70
Height (feet)	90	108	218
Length (feet)	2300	2562	1300

### Reading Math

**Element** The elements of a matrix can be represented using double subscript notation. The element  $a_{ij}$  is the element in row  $i$  column  $j$ .

In a matrix, numbers or data are organized so that each position in the matrix has a purpose. Each value in the matrix is called an **element**. A matrix is usually named using an uppercase letter.

$$A = \begin{bmatrix} 2 & 6 & 1 \\ 7 & 1 & 5 \\ 9 & 3 & 0 \\ 12 & 15 & 26 \end{bmatrix}$$

4 rows

3 columns

The element 15 is in row 4, column 2.

A matrix can be described by its **dimensions**. A matrix with  $m$  rows and  $n$  columns is an  $m \times n$  matrix (read “ $m$  by  $n$ ”). Matrix  $A$  above is a  $4 \times 3$  matrix since it has 4 rows and 3 columns.

### EXAMPLE Dimensions of a Matrix

- 2 State the dimensions of matrix  $B$  if  $B = \begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix}$ .

$$B = \begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix}$$

3 rows

2 columns

Since matrix  $B$  has 3 rows and 2 columns, the dimensions of matrix  $B$  are  $3 \times 2$ .

### CHECK Your Progress

2. State the dimensions of matrix  $L$  if  $L = \begin{bmatrix} -2 & 1 & 3 & -4 \\ 0 & 3 & 0 & 7 \end{bmatrix}$ .

Certain matrices have special names. A matrix that has only one row is called a **row matrix**, while a matrix that has only one column is called a **column matrix**. A matrix that has the same number of rows and columns is called a **square matrix**. Another special type of matrix is the **zero matrix**, in which every element is 0. The zero matrix can have any dimension.







**Equations Involving Matrices** Two matrices are considered **equal matrices** if they have the same dimensions and if each element of one matrix is equal to the corresponding element of the other matrix.

Example:  $\begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix}$  The matrices have the same dimensions and the corresponding elements are equal.  
The matrices are equal.

Non-example:  $\begin{bmatrix} 6 & 3 \\ 0 & 9 \\ 1 & 3 \end{bmatrix} \neq \begin{bmatrix} 6 & 0 & 1 \\ 3 & 9 & 3 \end{bmatrix}$  The matrices have different dimensions.  
They are not equal.

Non-example:  $\begin{bmatrix} 1 & 2 \\ 8 & 5 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 2 & 5 \end{bmatrix}$  Not all corresponding elements are equal.  
The matrices are not equal.



The definition of equal matrices can be used to find values when elements of equal matrices are algebraic expressions.

### EXAMPLE Solve an Equation Involving Matrices

**3** Solve  $\begin{bmatrix} y \\ 3x \end{bmatrix} = \begin{bmatrix} 6 - 2x \\ 31 + 4y \end{bmatrix}$  for  $x$  and  $y$ .

Since the matrices are equal, the corresponding elements are equal. When you write the sentences to show this equality, two linear equations are formed.

$$y = 6 - 2x$$

$$3x = 31 + 4y$$

This system can be solved using substitution.

$$3x = 31 + 4y \quad \text{Second equation}$$

$$3x = 31 + 4(6 - 2x) \quad \text{Substitute } 6 - 2x \text{ for } y.$$

$$3x = 31 + 24 - 8x \quad \text{Distributive Property}$$

$$11x = 55 \quad \text{Add } 8x \text{ to each side.}$$

$$x = 5 \quad \text{Divide each side by 11.}$$

To find the value for  $y$ , substitute 5 for  $x$  in either equation.

$$y = 6 - 2x \quad \text{First equation}$$

$$y = 6 - 2(5) \quad \text{Substitute 5 for } x.$$

$$y = -4 \quad \text{Simplify.}$$

The solution is  $(5, -4)$ .

### CHECK Your Progress






**3.** Solve  $\begin{bmatrix} 5x + 2 & y - 4 \\ 0 & 4z + 6 \end{bmatrix} = \begin{bmatrix} 12 & -8 \\ 0 & 2 \end{bmatrix}$ .

**Online Personal Tutor at** [algebra2.com](http://algebra2.com)



**Example 1**  
(pp. 162-163)

**WEATHER** For Exercises 1 and 2, use the table that shows a five-day forecast indicating high (H) and low (L) temperatures.

Fri	Sat	Sun	Mon	Tue
				
H 88	H 88	H 90	H 86	H 85
L 54	L 54	L 56	L 53	L 52

1. Organize the temperatures in a matrix.
2. Which day will be the warmest?

**Example 2**  
(p. 163)

**State the dimensions of each matrix.**

3.  $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \end{bmatrix}$

4.  $\begin{bmatrix} 10 & -6 & 18 & 0 \\ -7 & 5 & 2 & 4 \\ 3 & 11 & 9 & 7 \end{bmatrix}$

**Example 3**  
(p. 164)

**Solve each equation.**

5.  $\begin{bmatrix} x + 4 \\ 2y \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$

6.  $\begin{bmatrix} 9 & 13 \end{bmatrix} = \begin{bmatrix} x + 2y & 4x + 1 \end{bmatrix}$

## Exercises

HOMEWORK	HELP
For Exercises	See Examples
7-8	1
9-14	2
15-20	3

**Organize the information in a matrix.**

7.

Ocean	Area (mi <sup>2</sup> )	Average Depth (ft)
Pacific	60,060,700	13,215
Atlantic	29,637,900	12,880
Indian	26,469,500	13,002
Southern	7,848,300	16,400
Arctic	5,427,000	3,953

Source: factmonster.com

8.

Top Hockey Goalies				
Goalie	Games	Wins	Losses	Ties
Roy	1029	551	315	131
Sawchuk	971	447	330	172
Plante	837	435	247	146
Esposito	886	423	306	152
Hall	906	407	326	163

Source: factmonster.com

**State the dimensions of each matrix.**

9.  $\begin{bmatrix} 6 & -1 & 5 \\ -2 & 3 & -4 \end{bmatrix}$

10.  $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

11.  $\begin{bmatrix} 0 & 0 & 8 \\ 6 & 2 & 4 \\ 1 & 3 & 6 \\ 5 & 9 & 2 \end{bmatrix}$

12.  $\begin{bmatrix} -3 & 17 & -22 \\ 9 & 31 & 16 \\ 20 & -15 & 4 \end{bmatrix}$

13.  $\begin{bmatrix} 17 & -2 & 8 & -9 & 6 \\ 5 & 11 & 20 & -1 & 4 \end{bmatrix}$

14.  $\begin{bmatrix} 16 & 8 \\ 10 & 5 \\ 0 & 0 \end{bmatrix}$

**Solve each equation.**

15.  $\begin{bmatrix} 4x & 3y \end{bmatrix} = \begin{bmatrix} 12 & -1 \end{bmatrix}$

16.  $\begin{bmatrix} 2x & 3 & 3z \end{bmatrix} = \begin{bmatrix} 5 & 3y & 9 \end{bmatrix}$

17.  $\begin{bmatrix} 4x \\ 5 \end{bmatrix} = \begin{bmatrix} 15 + x \\ 2y - 1 \end{bmatrix}$

18.  $\begin{bmatrix} x + 3y \\ 3x + y \end{bmatrix} = \begin{bmatrix} -13 \\ 1 \end{bmatrix}$

19.  $\begin{bmatrix} 2x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$

20.  $\begin{bmatrix} 4x - 3 & 3y \\ 7 & 13 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ 7 & 2z + 1 \end{bmatrix}$



### Real-World Link

Adjusting for inflation, *Cleopatra* (1963) is the most expensive movie ever made. Its \$44 million budget is equivalent to \$306,867,120 today.

Source: *The Guinness Book of Records*

### DINING OUT

For Exercises 21 and 22, use the following information. A newspaper rated several restaurants by cost, level of service, atmosphere, and location using a scale of ★ being low and ★★★★★ being high.

Restaurant	Cost	Service	Atmosphere	Location
Catalina Grill	★★	★	★	★
Oyster Club	★★★★	★★	★	★★
Casa di Pasta	★★★★★	★★★★	★★★★	★★★★
Mason's Steakhouse	★★	★★★★★	★★★★★	★★★

- Write a  $4 \times 4$  matrix to organize this information.
- Which restaurant would you select based on this information, and why?

### MOVIES

For Exercises 23 and 24, use the advertisement shown at the right.

- Write a matrix for the prices of movie tickets for adults, children, and seniors.
- What are the dimensions of the matrix?



### HOTELS

For Exercises 25 and 26, use the costs for an overnight stay at a hotel that are given below.

Single Room: \$60 weekday;  
\$79 weekend

Double Room: \$70 weekday;  
\$89 weekend

Suite: \$75 weekday; \$95 weekend

- Write a  $3 \times 2$  matrix that represents the cost of each room.
- Write a  $2 \times 3$  matrix that represents the cost of each room.

**EXTRA PRACTICE**  
See pages 897, 929.

**Math online**  
Self-Check Quiz at [algebra2.com](http://algebra2.com)

### H.O.T. Problems

- RESEARCH** Use the Internet or other resource to find the meaning of the word *matrix*. How does the meaning of this word in other fields compare to its mathematical meaning?
- OPEN ENDED** Give examples of a row matrix, a column matrix, a square matrix, and a zero matrix. State the dimensions of each matrix.

### CHALLENGE

For Exercises 29 and 30, use the matrix at the right.

- Study the pattern of numbers. Complete the matrix for column 6 and row 7.
- In which row and column will 100 occur?

1	3	6	10	15	...
2	5	9	14	20	...
4	8	13	19	26	...
7	12	18	25	33	...
11	17	24	32	41	...
16	23	31	40	50	...
⋮	⋮	⋮	⋮	⋮	⋮

- Writing in Math** Use the information about SUVs on page 162 to explain how a matrix can help Sabrina decide which SUV to buy.

- 32. ACT/SAT** The results of a recent poll are organized in the matrix.

	For	Against
Proposition 1	1553	771
Proposition 2	689	1633
Proposition 3	2088	229

Based on these results, which conclusion is NOT valid?

- A There were 771 votes cast against Proposition 1.
- B More people voted against Proposition 1 than voted for Proposition 2.
- C Proposition 2 has little chance of passing.
- D More people voted for Proposition 1 than for Proposition 3.

- 33. REVIEW** The chart shows an expression evaluated for four different values of  $x$ .

$x$	$x^2 + x + 1$
1	3
2	7
3	13
5	31

A student concludes that for all values of  $x$ ,  $x^2 + x + 1$  produces a prime number. Which value of  $x$  serves as a counterexample to prove this conclusion false?

- F -4
- G -3
- H -2
- J 4

## Spiral Review

Solve each system of equations. (Lesson 3-5)

**34.**  $3x - 3y = 6$   
 $-6y = -30$   
 $5z - 2x = 6$

**35.**  $3a + 2b = 27$   
 $5a - 7b + c = 5$   
 $-2a + 10b + 5c = -29$

**36.**  $3r - 15s + 4t = -57$   
 $9r + 45s - t = 26$   
 $-6r + 10s + 3t = -19$

- 37. BUSINESS** A factory is making skirts and dresses from the same fabric. Each skirt requires 1 hour of cutting and 1 hour of sewing. Each dress requires 2 hours of cutting and 3 hours of sewing. The cutting department can cut up to 120 hours each week and the sewing department can sew up to 150 hours each week. If profits are \$12 for each skirt and \$18 for each dress, how many of each should the factory make for maximum profit? (Lesson 3-4)

- 38.** Write an equation in slope-intercept form of the line that passes through the points indicated in the table. (Lesson 2-4)

$x$	$y$
-3	-1
2	$\frac{7}{3}$
3	3

- 39.** Write an equation in standard form of the line that passes through the points indicated in the table. (Lesson 2-1)

Find each value if  $f(x) = x^2 - 3x + 2$ . (Lesson 2-1)

**40.**  $f(3)$

**41.**  $f(0)$

**42.**  $f(2)$

**43.**  $f(-3)$

## GET READY for the Next Lesson

Find the value of each expression. (Lesson 1-2)

**44.**  $8 + (-5)$

**45.**  $6(-3)$

**46.**  $\frac{1}{2}(34)$

**47.**  $-5(3 - 18)$



# Spreadsheet Lab

## Organizing Data

You can use a computer **spreadsheet** to organize and display data. Similar to a matrix, data in a spreadsheet are entered into rows and columns. Then you can use the data to create graphs or perform calculations.

### ACTIVITY

Enter the data on free throws (FT) and 2- and 3-point field goals (FG) in Big Twelve Conference Men's Basketball into a spreadsheet.

Big Twelve Conference 2004–2005 Men's Basketball							
Team	FT	2-PT FG	3-PT FG	Team	FT	2-PT FG	3-PT FG
Baylor	366	423	217	Nebraska	409	487	174
Colorado	382	548	223	Oklahoma	450	694	214
Iowa St.	431	671	113	Oklahoma St.	521	671	240
Kansas	451	603	198	Texas	509	573	243
Kansas St.	412	545	167	Texas A&M	517	590	195
Missouri	473	506	213	Texas Tech	526	787	145

Source: SportsTicker

Use Column A for the team names, Column B for the numbers of free throws, Column C for the numbers of 2-point field goals, and Column D for the numbers of 3-point field goals.

Big Twelve Conference				
	A	B	C	D
1	Baylor	366	423	217
2	Colorado	382	548	223
3	Iowa St.	431	671	113
4	Kansas	451	603	198
5	Kansas St.	412	545	167
6	Missouri	473	506	213
7	Nebraska	409	487	174
8	Oklahoma	450	694	214
9	Oklahoma St.	521	671	240
10	Texas	509	573	243
11	Texas A&M	517	590	195
12	Texas Tech	526	787	145

Each **row** contains data for a different team. Row 2 represents Colorado.

Each **cell** of the spreadsheet contains one piece of data. Cell 10D contains the value 243, representing the number of 3-point field goals made by Texas.

### MODEL AND ANALYZE

1. Enter the data about sport-utility vehicles on page 162 into a spreadsheet.
2. Compare and contrast how data are organized in a spreadsheet and how they are organized in a matrix.

**Main Ideas**

- Add and subtract matrices.
- Multiply by a matrix scalar.

**New Vocabulary**

scalar

scalar multiplication

**GET READY for the Lesson**

Eneas, a hospital dietician, designs weekly menus for his patients and tracks nutrients for each daily diet. The table shows the Calories, protein, and fat in a patient's meals over a three-day period.

Day	Breakfast			Lunch			Dinner		
	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)
1	566	18	7	785	22	19	1257	40	26
2	482	12	17	622	23	20	987	32	45
3	530	10	11	710	26	12	1380	29	38

These data can be organized in three matrices representing breakfast, lunch, and dinner. The daily totals can then be found by adding the three matrices.

**Add and Subtract Matrices** Matrices can be added if and only if they have the same dimensions.

**KEY CONCEPT****Addition and Subtraction of Matrices**

**Words** If  $A$  and  $B$  are two  $m \times n$  matrices, then  $A + B$  is an  $m \times n$  matrix in which each element is the sum of the corresponding elements of  $A$  and  $B$ . Also,  $A - B$  is an  $m \times n$  matrix in which each element is the difference of the corresponding elements of  $A$  and  $B$ .

**Symbols**

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a-j & b-k & c-l \\ d-m & e-n & f-o \\ g-p & h-q & i-r \end{bmatrix}$$

**EXAMPLE Add Matrices**

**1** a. Find  $A + B$  if  $A = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix}$ .

$$A + B = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix} \quad \text{Definition of matrix addition}$$

$$= \begin{bmatrix} 4 + (-3) & -6 + 7 \\ 2 + 5 & 3 + (-9) \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ 7 & -6 \end{bmatrix} \quad \text{Simplify.}$$

(continued on the next page)



b. Find  $A + B$  if  $A = \begin{bmatrix} 3 & -7 & 4 \\ 12 & 5 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 9 \\ 4 & -6 \end{bmatrix}$ .

Since the dimensions of  $A$  are  $2 \times 3$  and the dimensions of  $B$  are  $2 \times 2$ , you cannot add these matrices.

### CHECK Your Progress

1. Find  $A + B$  if  $A = \begin{bmatrix} -5 & 7 \\ -1 & 12 \end{bmatrix}$  and  $B = \begin{bmatrix} 11 & 3 \\ -4 & -5 \end{bmatrix}$ .

### EXAMPLE Subtract Matrices


2 Find  $A - B$  if  $A = \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix}$ .

$$\begin{aligned} A - B &= \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix} && \text{Substitution} \\ &= \begin{bmatrix} 9 - 3 & 2 - 6 \\ -4 - 8 & 7 - (-2) \end{bmatrix} && \text{Subtract corresponding elements.} \\ &= \begin{bmatrix} 6 & -4 \\ -12 & 9 \end{bmatrix} && \text{Simplify.} \end{aligned}$$

### CHECK Your Progress

2. Find  $A - B$  if  $A = \begin{bmatrix} 12 & -4 \\ -5 & 8 \end{bmatrix}$  and  $B = \begin{bmatrix} 7 & 3 \\ -3 & -2 \end{bmatrix}$ .

### Real-World EXAMPLE

 **3 ANIMALS** The table below shows the number of endangered and threatened species in the United States and in the world. How many more endangered and threatened species are there on the world list than on the U.S. list?

Type of Animal	Endangered and Threatened Species			
	United States		World	
	Endangered	Threatened	Endangered	Threatened
Mammals	68	10	319	27
Birds	77	13	252	19
Reptiles	14	22	78	37
Amphibians	11	10	19	11
Fish	71	43	82	44

**Source:** Fish and Wildlife Service, U.S. Department of Interior

The data in the table can be organized in two matrices. Find the difference of the matrix that represents species in the world and the matrix that represents species in the U.S.



### Real-World Link

The rarest animal in the world today is a giant tortoise that lives in the Galapagos Islands. "Lonesome George" is the only remaining representative of his species (*Geochelone elephantopus abingdoni*). With virtually no hope of discovering another specimen, this species is now effectively extinct.

**Source:** ecoworld.com



World	U.S.	Endangered	Threatened	
$\begin{bmatrix} 319 & 27 \\ 252 & 19 \\ 78 & 37 \\ 19 & 11 \\ 82 & 44 \end{bmatrix}$	$\begin{bmatrix} 68 & 10 \\ 77 & 13 \\ 14 & 22 \\ 11 & 10 \\ 71 & 43 \end{bmatrix}$	$=$	$\begin{bmatrix} 319 - 68 & 27 - 10 \\ 252 - 77 & 19 - 13 \\ 78 - 14 & 37 - 22 \\ 19 - 11 & 11 - 10 \\ 82 - 71 & 44 - 43 \end{bmatrix}$	Subtract corresponding elements.
		$=$	$\begin{bmatrix} 251 & 17 \\ 175 & 6 \\ 64 & 15 \\ 8 & 1 \\ 11 & 1 \end{bmatrix}$	

The first column represents the difference in the number of endangered species on the world and U.S. lists. There are 251 mammals, 175 birds, 64 reptiles, 8 amphibians, and 11 fish species in this category.

The second column represents the difference in the number of threatened species on the world and U.S. lists. There are 17 mammals, 6 birds, 15 reptiles, 1 amphibian, and 1 fish species in this category.

### CHECK Your Progress

3. Refer to the data on page 169 and use matrices to show the difference of Calories, protein, and fat between lunch and breakfast.

**Online Personal Tutor at** [algebra2.com](http://algebra2.com)

**Scalar Multiplication** You can multiply any matrix by a constant called a **scalar**. This operation is called **scalar multiplication**.

### KEY CONCEPT

### Scalar Multiplication

**Words** The product of a scalar  $k$  and an  $m \times n$  matrix is an  $m \times n$  matrix in which each element equals  $k$  times the corresponding elements of the original matrix.

**Symbols**  $k \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

### EXAMPLE Multiply a Matrix by a Scalar

4 If  $A = \begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix}$ , find  $3A$ .

$$3A = 3 \begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 3(2) & 3(8) & 3(-3) \\ 3(5) & 3(-9) & 3(2) \end{bmatrix} \text{ or } \begin{bmatrix} 6 & 24 & -9 \\ 15 & -27 & 6 \end{bmatrix} \quad \text{Simplify.}$$

### CHECK Your Progress

4. If  $A = \begin{bmatrix} 7 & -4 & 10 \\ -2 & 6 & -9 \end{bmatrix}$ , find  $-4A$ .





Many properties of real numbers also hold true for matrices.

## CONCEPT SUMMARY

### Properties of Matrix Operations

For any matrices  $A$ ,  $B$ , and  $C$  with the same dimensions and any scalar  $c$ , the following properties are true.

**Commutative Property of Addition**

$$A + B = B + A$$

**Associative Property of Addition**

$$(A + B) + C = A + (B + C)$$

**Distributive Property**

$$c(A + B) = cA + cB$$

## EXAMPLE

### Combination of Matrix Operations

5 If  $A = \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$ , find  $5A - 2B$ .

Perform the scalar multiplication first. Then subtract the matrices.

$$5A - 2B = 5 \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix} - 2 \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$$

Substitution

$$= \begin{bmatrix} 5(7) & 5(3) \\ 5(-4) & 5(-1) \end{bmatrix} - \begin{bmatrix} 2(9) & 2(6) \\ 2(3) & 2(10) \end{bmatrix}$$

Multiply each element in the first matrix by 5 and multiply each element in the second matrix by 2.

$$= \begin{bmatrix} 35 & 15 \\ -20 & -5 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 6 & 20 \end{bmatrix}$$

Simplify.

$$= \begin{bmatrix} 35 - 18 & 15 - 12 \\ -20 - 6 & -5 - 20 \end{bmatrix} \text{ or } \begin{bmatrix} 17 & 3 \\ -26 & -25 \end{bmatrix}$$

Subtract corresponding elements.

## CHECK Your Progress

5. If  $A = \begin{bmatrix} 4 & -2 \\ 5 & -9 \end{bmatrix}$  and  $B = \begin{bmatrix} 8 & 2 \\ -1 & -3 \end{bmatrix}$ , find  $6A - 3B$ .

## Study Tip

### Additive Identity

The matrix  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  is called a *zero matrix*. It is the *additive identity matrix* for any  $2 \times 2$  matrix. How is this similar to the additive identity for real numbers?

## GRAPHING CALCULATOR LAB

### Matrix Operations

On the TI-83/84 Plus, **2nd** **[MATRIX]** accesses the matrix menu. Choose **EDIT** to define a matrix. Press **1** or **ENTER** and enter the dimensions of the matrix  $A$  using the **►** key. Then enter each element by pressing **ENTER** after each entry. To display and use the matrix, exit the editing mode and choose the matrix under **NAMES** from the **[MATRIX]** menu.

### THINK AND DISCUSS

1. Enter  $A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$ . What do the two numbers separated by a comma in the bottom left corner of the screen represent?
2. Enter  $B = \begin{bmatrix} 1 & 9 & -3 \\ 8 & 6 & -5 \end{bmatrix}$ . Find  $A + B$ . What is the result and why?

## Study Tip

### Matrix Operations

The order of operations for matrices is similar to that of real numbers. Perform scalar multiplication before matrix addition and subtraction.

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

**Example 1**  
(pp. 169–170)

1.  $\begin{bmatrix} 5 & 8 & -4 \end{bmatrix} + \begin{bmatrix} 12 & 5 \end{bmatrix}$

2.  $\begin{bmatrix} 12 & 6 \\ -8 & -3 \end{bmatrix} + \begin{bmatrix} 14 & -9 \\ 11 & -6 \end{bmatrix}$

**Example 2**  
(p. 170)

3.  $\begin{bmatrix} 3 & 7 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}$

4.  $\begin{bmatrix} 4 & 12 \\ -3 & -7 \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ -4 & -4 \end{bmatrix}$

**Example 3**  
(pp. 170–171)

**SPORTS** For Exercises 5–7, use the table below that shows high school participation in various sports.

Sport	Males		Females	
	Schools	Participants	Schools	Participants
Basketball	17,389	544,811	17,061	457,986
Track and Field	15,221	504,801	15,089	418,322
Baseball/Softball	14,984	457,146	14,181	362,468
Soccer	10,219	349,785	9,490	309,032
Swimming and Diving	5,758	96,562	6,176	144,565



Source: National Federation of State High School Associations

- Write two matrices that represent these data for males and females.
- Find the total number of students that participate in each individual sport expressed as a matrix.
- Could you add the two matrices to find the total number of schools that offer a particular sport? Why or why not?

**Example 4**  
(p. 171)

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

8.  $3 \begin{bmatrix} 6 & -1 & 5 & 2 \\ 7 & 3 & -2 & 8 \end{bmatrix}$

9.  $-5 \begin{bmatrix} 2 & -4 \\ -6 & 3 \\ -9 & -1 \end{bmatrix}$

**Example 5**  
(p. 172)

Use matrices  $A$ ,  $B$ ,  $C$ , and  $D$  to find the following.

$A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 7 \\ 0 & -4 \end{bmatrix}$

$C = \begin{bmatrix} 9 & -4 \\ -6 & 5 \end{bmatrix}$

$D = \begin{bmatrix} 2 & -5 \end{bmatrix}$

10.  $A + B + C$

11.  $3B - 2C$

12.  $4A + 2B - C$

13.  $B + 2C + D$

## Exercises

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

14.  $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ -5 \\ 8 \end{bmatrix}$

15.  $\begin{bmatrix} -11 & 4 \\ -3 & 6 \end{bmatrix} + \begin{bmatrix} -2 & -5 \\ 5 & -3 \end{bmatrix}$

HOMEWORK HELP	
For Exercises	See Examples
14–17	1
18–21	2
22–24	3
25, 26	4
27, 28	5

Perform the indicated matrix operations. If the matrix does not exist, write impossible.

$$16. [-5 \ 2 \ -1] + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$18. \begin{bmatrix} -5 & 7 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -2 \\ 9 & 0 & 1 \end{bmatrix}$$

$$20. \begin{bmatrix} 3 \\ -8 \\ -2 \end{bmatrix} - \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$$

$$17. \begin{bmatrix} 2 & 5 & 3 \\ -7 & -1 & 11 \\ 4 & -4 & 0 \end{bmatrix} + \begin{bmatrix} -9 & 2 & -5 \\ 1 & 6 & -3 \\ -9 & -12 & 8 \end{bmatrix}$$

$$19. \begin{bmatrix} 12 & 0 & 8 \\ 9 & 15 & -11 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 4 \\ 9 & 2 & -6 \end{bmatrix}$$

$$21. \begin{bmatrix} -9 & 2 & -7 \\ 8 & 10 & 3 \\ -7 & 4 & 15 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 6 \\ -7 & -3 & 5 \\ 2 & 11 & -4 \end{bmatrix}$$

**BUSINESS** For Exercises 22–24, use the following information.

An electronics store records each type of entertainment device sold at three of their branch stores so that they can monitor their purchases of supplies. Two weeks of sales are shown in the spreadsheets at the right.

	A	B	C	D	E
1	Week 1	Televisions	DVD players	Video game units	CD players
2	Store 1	325	215	147	276
3	Store 2	294	221	79	152
4	Store 3	175	191	100	146

22. Write a matrix for each week's sales.
23. Find the sum of the two weeks' sales expressed as a matrix.
24. Express the difference in sales from Week 1 to Week 2 as a matrix.

	A	B	C	D	E
1	Week 2	Televisions	DVD players	Video game units	CD players
2	Store 1	306	162	145	257
3	Store 2	258	210	84	165
4	Store 3	188	176	99	112

Perform the indicated matrix operation. If the matrix does not exist, write impossible.

$$25. -2 \begin{bmatrix} 2 & -4 & 1 \\ -3 & 5 & 8 \\ 7 & 6 & -2 \end{bmatrix}$$

$$26. 3 \begin{bmatrix} 5 & -3 \\ -10 & 8 \\ -1 & 7 \end{bmatrix}$$

$$27. 5[0 \ -1 \ 7 \ 2] + 3[5 \ -8 \ 10 \ -4]$$

$$28. 5 \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + 6 \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ 8 \\ -4 \end{bmatrix}$$

Use matrices  $A$ ,  $B$ ,  $C$ , and  $D$  to find the following.

$$A = \begin{bmatrix} 5 & 7 \\ -1 & 6 \\ 3 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} 8 & 3 \\ 5 & 1 \\ 4 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 4 \\ -2 & 5 \\ 7 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 2 \\ 9 & 0 \\ -3 & 0 \end{bmatrix}$$

$$29. A + B$$

$$30. D - B$$

$$31. 4C$$

$$32. 6B - 2A$$

$$33. 3C - 4A + B$$

$$34. C + \frac{1}{3}D$$

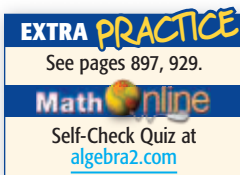


### Real-World Link

Jenny Thompson won her record setting twelfth Olympic medal by winning the silver in the  $4 \times 100$  Medley Relay at the 2004 Athens Olympics.

Source:

athens2004.com



### H.O.T. Problems

Perform the indicated matrix operation. If the matrix does not exist, write *impossible*.

$$35. \begin{bmatrix} 1.35 & 5.80 \\ 1.24 & 14.32 \\ 6.10 & 35.26 \end{bmatrix} + \begin{bmatrix} 0.45 & 3.28 \\ 1.94 & 16.72 \\ 4.31 & 21.30 \end{bmatrix}$$

$$36. 8 \begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix} - 2 \begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix}$$

$$37. \frac{1}{2} \begin{bmatrix} 4 & 6 \\ 3 & 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 9 & 27 \\ 0 & 3 \end{bmatrix}$$

$$38. 5 \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 2 & \frac{1}{3} & -1 \end{bmatrix} + 4 \begin{bmatrix} -2 & \frac{3}{4} & 1 \\ \frac{1}{6} & 0 & \frac{5}{8} \end{bmatrix}$$

**SWIMMING** For Exercises 39–41, use the table that shows some of the world, Olympic, and U.S. women's freestyle swimming records.

Distance (meters)	World	Olympic	U.S.
50	24.13 s	24.13 s	24.63 s
100	53.52 s	53.52 s	53.99 s
200	1:56.54 min	1:57.65 min	1:57.41 min
800	8:16.22 min	8:19.67 min	8:16.22 min

Source: hickoksports.com

39. Find the difference between U.S. and World records expressed as a column matrix.
40. Write a matrix that compares the total time of all four events for World, Olympic, and U.S. record holders.
41. In which events were the fastest times set at the Olympics?

**RECREATION** For Exercises 42 and 43, use the following price list for one-day admissions to the community pool.

42. Write the matrix that represents the additional cost for nonresidents.
43. Write a matrix that represents the difference in cost if a child or adult goes to the pool after 6:00 P.M.

Daily Admission Fees			
Residents			
Time of day	Child	Adult	
Before 6:00 P.M.	\$3.00	\$4.50	
After 6:00 P.M.	\$2.00	\$3.50	
Nonresidents			
Time of day	Child	Adult	
Before 6:00 P.M.	\$4.50	\$6.75	
After 6:00 P.M.	\$3.00	\$5.25	

44. **CHALLENGE** Determine values for each variable if  $d = 1$ ,  $e = 4d$ ,  $z + d = e$ ,  $f = \frac{x}{5}$ ,  $ay = 1.5$ ,  $x = \frac{d}{2}$ , and  $y = x + \frac{x}{2}$ .

$$a \begin{bmatrix} x & y & z \\ d & e & f \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ ad & ae & af \end{bmatrix}$$

45. **OPEN ENDED** Give an example of two matrices whose sum is a zero matrix.

46. **CHALLENGE** For matrix  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , the *transpose* of  $A$  is  $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ .

Write a matrix  $B$  that is equal to its transpose  $B^T$ .

47. **Writing in Math** Use the data on nutrition on page 169 to explain how matrices can be used to calculate daily dietary needs. Include three matrices that represent breakfast, lunch, and dinner over the three-day period, and a matrix that represents the total Calories, protein, and fat consumed each day.



## STANDARDIZED TEST PRACTICE

48. **ACT/SAT** Solve for  $x$  and  $y$  in the matrix

$$\text{equation } \begin{bmatrix} x \\ 7 \end{bmatrix} + \begin{bmatrix} 3y \\ -x \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}.$$

- A  $x = -5, y = 7$   
 B  $x = 7, y = 3$   
 C  $x = 7, y = 5$   
 D  $x = 5, y = 7$

49. **REVIEW** What is the equation of the line that has a slope of 3 and passes through the point  $(2, -9)$ ?

- F  $y = 3x + 11$   
 G  $y = 3x - 11$   
 H  $y = 3x + 15$   
 J  $y = 3x - 15$

## Spiral Review

State the dimensions of each matrix. (Lesson 4-1)

50.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

51.  $[2 \ 0 \ 3 \ 0]$

52.  $\begin{bmatrix} 5 & 1 & -6 & 2 \\ -38 & 5 & 7 & 3 \end{bmatrix}$

53.  $\begin{bmatrix} 7 & -3 & 5 \\ 0 & 2 & -9 \\ 6 & 5 & 1 \end{bmatrix}$

54.  $\begin{bmatrix} 8 & 6 \\ 5 & 2 \\ -4 & -1 \end{bmatrix}$

55.  $\begin{bmatrix} 7 & 5 & 0 \\ -8 & 3 & 8 \\ 9 & -1 & 15 \\ 4 & 2 & 11 \end{bmatrix}$

Solve each system of equations. (Lesson 3-5)

56.  $2a + b = 2$

$5a = 15$

$a + b + c = -1$

57.  $r + s + t = 15$

$r + t = 12$

$s + t = 10$

58.  $6x - 2y - 3z = -10$

$-6x + y + 9z = 3$

$8x - 3y = -16$

Solve each system by using substitution or elimination. (Lesson 3-2)

59.  $2s + 7t = 39$

$5s - t = 5$

60.  $3p + 6q = -3$

$2p - 3q = -9$

61.  $a + 5b = 1$

$7a - 2b = 44$

**SCRAPBOOKS** For Exercises 62 and 63, use the following information. (Lesson 2-7)

Ian has \$6.00, and he wants to buy paper for his scrapbook. A sheet of printed paper costs 30¢, and a sheet of solid color paper costs 15¢.

62. Write and graph an inequality that describes this situation.

63. Does Ian have enough money to buy 14 pieces of each type of paper? Explain.

## GET READY for the Next Lesson

Name the property illustrated by each equation. (Lesson 1-2)

64.  $\frac{7}{9} \cdot \frac{9}{7} = 1$

66.  $3(x + 12) = 3x + 3(12)$

65.  $7 + (w + 5) = (7 + w) + 5$

67.  $6(9a) = 9a(6)$

# 4-3

## Multiplying Matrices

### Main Ideas

- Multiply matrices.
- Use the properties of matrix multiplication

### GET READY for the Lesson

The table shows the scoring summary of the Carolina Panthers for the 2005 season. The team's record can be summarized in the record matrix  $R$ . The values for each type of score can be organized in the point values matrix  $P$ .

Carolina Panthers Regular Season Scoring	
Type	Number
Touchdown	45
Extra Point	43
Field Goal	26
2-Point Conversion	1
Safety	0

Source: National Football League

$$R = \begin{bmatrix} 45 \\ 43 \\ 26 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} \text{touchdown} \\ \text{extra point} \\ \text{field goal} \\ \text{2-point conversion} \\ \text{safety} \end{matrix}$$

### Point Values

$$P = \begin{bmatrix} 6 & 1 & 3 & 2 & 2 \end{bmatrix} \begin{matrix} \text{touchdown} \\ \text{extra point} \\ \text{field goal} \\ \text{2-point conversion} \\ \text{safety} \end{matrix}$$

You can use matrix multiplication to find the total points scored.

**Multiply Matrices** You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix. When you multiply two matrices  $A_{m \times n}$  and  $B_{n \times r}$ , the resulting matrix  $AB$  is an  $m \times r$  matrix.

### EXAMPLE Dimensions of Matrix Products

**1** Determine whether each matrix product is defined. If so, state the dimensions of the product.

a.  $A_{2 \times 5}$  and  $B_{5 \times 4}$

$$A \cdot B = AB$$

$$\begin{matrix} 2 \times 5 & 5 \times 4 & 2 \times 4 \\ \uparrow & \uparrow & \end{matrix}$$

The inner dimensions are equal, so the product is defined. Its dimensions are  $2 \times 4$ .

b.  $A_{1 \times 3}$  and  $B_{4 \times 3}$

$$A \cdot B$$

$$\begin{matrix} 1 \times 3 & 4 \times 3 \\ \uparrow & \uparrow \end{matrix}$$

The inner dimensions are not equal, so the matrix product is not defined.

### CHECK Your Progress

1A.  $A_{4 \times 6}$  and  $B_{6 \times 2}$

1B.  $A_{3 \times 2}$  and  $B_{3 \times 2}$





The product of two matrices is found by multiplying corresponding columns and rows.

## KEY CONCEPT

## Multiplying Matrices

**Words** The element  $a_{ij}$  of  $AB$  is the sum of the products of the corresponding elements in row  $i$  of  $A$  and column  $j$  of  $B$ .

**Symbols** 
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$$

## EXAMPLE

## Multiply Square Matrices

**2** Find  $RS$  if  $R = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$  and  $S = \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$ .

$$RS = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 4 & 7 \end{bmatrix}$$

**Step 1** Multiply the numbers in the first row of  $R$  by the numbers in the first column of  $S$ , add the products, and put the result in the first row, first column of  $RS$ .

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & \\ & \end{bmatrix}$$

**Step 2** Follow the same procedure as in Step 1 using the first row and second column numbers. Write the result in the first row, second column.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ & \end{bmatrix}$$

**Step 3** Follow the same procedure with the second row and first column numbers. Write the result in the second row, first column.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & \end{bmatrix}$$

**Step 4** The procedure is the same for the numbers in the second row, second column.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix}$$

**Step 5** Simplify the product matrix.

$$\begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix} = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$$



## CHECK Your Progress

**2.** Find  $UV$  if  $U = \begin{bmatrix} 5 & 9 \\ -3 & -2 \end{bmatrix}$  and  $V = \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix}$ .

## Study Tip

### Multiplying Matrices

To avoid any miscalculations, find the product of the matrices in order as shown in Example 2. It may also help to cover rows or columns not being multiplied as you find elements of the product matrix.

Concepts  
in Motion

Animation  
[algebra2.com](http://algebra2.com)



### Real-World Link

Swim meets consist of racing and diving competitions.

There are more than 241,000 high schools that participate each year.

Source: NFHS



## Real-World EXAMPLE

3

**SWIM MEET** At a particular swim meet, 7 points were awarded for each first-place finish, 4 points for each second, and 2 points for each third. Which school won the meet?

School	First Place	Second Place	Third Place
Central	4	7	3
Franklin	8	9	1
Hayes	10	5	3
Lincoln	3	3	6

**Explore** The final scores can be found by multiplying the swim results for each school by the points awarded for each first-, second-, and third-place finish.

**Plan** Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

### Results

$$R = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix}$$

### Points

$$P = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

**Solve** Multiply the matrices.

$$RP = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

Write an equation.

$$= \begin{bmatrix} 4(7) + 7(4) + 3(2) \\ 8(7) + 9(4) + 1(2) \\ 10(7) + 5(4) + 3(2) \\ 3(7) + 3(4) + 6(2) \end{bmatrix}$$

Multiply columns by rows.

$$= \begin{bmatrix} 62 \\ 94 \\ 96 \\ 45 \end{bmatrix}$$

Simplify.

The product matrix shows the scores for Central, Franklin, Hayes, and Lincoln in order. Hayes won the swim meet with a total of 96 points.

**Check**  $R$  is a  $4 \times 3$  matrix and  $P$  is a  $3 \times 1$  matrix; so their product should be a  $4 \times 1$  matrix. *Why?*



## CHECK Your Progress

3. Refer to the data in Exercises 22–24 on page 174. If the cost of televisions was \$250, DVD players was \$225, video game units was \$149, and CD players was \$75, use matrices to find the total sales for week 1.





**Multiplicative Properties** Recall that the same properties for real numbers also held true for matrix addition. However, some of these properties do *not* always hold true for matrix multiplication.

### EXAMPLE Commutative Property

**4** Find each product if  $P = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix}$  and  $Q = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix}$ .

a.  $PQ$

$$\begin{aligned} PQ &= \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} && \text{Substitution} \\ &= \begin{bmatrix} 72 - 42 & -24 + 7 & 16 + 35 \\ -18 + 24 & 6 - 4 & -4 - 20 \\ 0 + 18 & 0 - 3 & 0 - 15 \end{bmatrix} \text{ or } \begin{bmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{bmatrix} \end{aligned}$$

b.  $QP$

$$\begin{aligned} QP &= \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} && \text{Substitution} \\ &= \begin{bmatrix} 72 + 6 + 0 & -63 - 12 + 6 \\ 48 + 2 + 0 & -42 - 4 - 15 \end{bmatrix} \text{ or } \begin{bmatrix} 78 & -69 \\ 50 & -61 \end{bmatrix} \end{aligned}$$

### CHECK Your Progress

**4.** Use  $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$  and  $B = \begin{bmatrix} -3 & 6 \\ -4 & 5 \end{bmatrix}$  to determine whether  $AB = BA$  is true for the given matrices.



In Example 4, notice that  $PQ \neq QP$ . This demonstrates that the Commutative Property of Multiplication does not hold for matrix multiplication. The order in which you multiply matrices is very important.

### EXAMPLE Distributive Property

**5** Find each product if  $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix}$ , and  $C = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$ .

a.  $A(B + C)$

$$\begin{aligned} A(B + C) &= \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \left( \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \right) && \text{Substitution} \\ &= \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 6 \\ 1 & 10 \end{bmatrix} && \text{Add corresponding elements.} \\ &= \begin{bmatrix} 3(-1) + 2(1) & 3(6) + 2(10) \\ -1(-1) + 4(1) & -1(6) + 4(10) \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix} && \text{Multiply columns by rows.} \end{aligned}$$

**b.  $AB + AC$**

$$AB + AC = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 3(-2) + 2(6) & 3(5) + 2(7) \\ -1(-2) + 4(6) & -1(5) + 4(7) \end{bmatrix} + \begin{bmatrix} 3(1) + 2(-5) & 3(1) + 2(3) \\ -1(1) + 4(-5) & -1(1) + 4(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 29 \\ 26 & 23 \end{bmatrix} + \begin{bmatrix} -7 & 9 \\ -21 & 11 \end{bmatrix} \quad \text{Simplify.}$$

$$= \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix} \quad \text{Add corresponding elements.}$$

**CHECK Your Progress**

5. Use the matrices  $R = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ ,  $S = \begin{bmatrix} 4 & 6 \\ -2 & 5 \end{bmatrix}$ , and  $T = \begin{bmatrix} -3 & 7 \\ -4 & 8 \end{bmatrix}$  to determine if  $(S + T)R = SR + TR$ .



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Notice that in Example 5,  $A(B + C) = AB + AC$ . This and other examples suggest that the Distributive Property is true for matrix multiplication. Some properties of matrix multiplication are shown below.

**KEY CONCEPT**

**Properties of Matrix Multiplication**

For any matrices  $A$ ,  $B$ , and  $C$  for which the matrix products are defined, and any scalar  $c$ , the following properties are true.

**Associative Property of Matrix Multiplication**  $(AB)C = A(BC)$

**Associative Property of Scalar Multiplication**  $c(AB) = (cA)B = A(cB)$

**Left Distributive Property**  $C(A + B) = CA + CB$

**Right Distributive Property**  $(A + B)C = AC + BC$

To show that a property is true for all cases, you must show it is true for the general case. To show that a property is *not* always true, you only need to find one counterexample.

**CHECK Your Understanding**

**Example 1**  
(p. 177)

Determine whether each matrix product is defined. If so, state the dimensions of the product.

1.  $A_{3 \times 5} \cdot B_{5 \times 2}$

2.  $X_{2 \times 3} \cdot Y_{2 \times 3}$

3.  $R_{3 \times 2} S_{2 \times 22}$

Find each product, if possible.

**Example 2**  
(p. 178)

4.  $\begin{bmatrix} 2 & 1 \\ 7 & -5 \end{bmatrix} \cdot \begin{bmatrix} -6 & 3 \\ -2 & -4 \end{bmatrix}$

5.  $\begin{bmatrix} 10 & -2 \\ -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$

**Example 3**  
(p. 179)

6.  $\begin{bmatrix} 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ -2 & 0 \end{bmatrix}$

7.  $\begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}$

8.  $\begin{bmatrix} 5 & -2 & -1 \\ 8 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$

9.  $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \end{bmatrix}$



**SPORTS** For Exercises 10 and 11, use the table below that shows the number of kids registered for baseball and softball.

The Westfall Youth Baseball and Softball League charges the following registration fees: ages 7–8, \$45; ages 9–10, \$55; and ages 11–14, \$65.

Team Members		
Age	Baseball	Softball
7–8	350	280
9–10	320	165
11–14	180	120

10. Write a matrix for the registration fees and a matrix for the number of players.
11. Find the total amount of money the league received from baseball and softball registrations.

**Examples 4, 5**  
(pp. 180–181)

Use  $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ ,  $B = \begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix}$ , and  $C = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$  to determine whether the following equations are true for the given matrices.

12.  $AB = BA$

13.  $A(BC) = (AB)C$

## Exercises

HOMEWORK	HELP
For Exercises	See Examples
14–19	1
20–27	2, 3
28–30	3
31, 32	4
33, 34	5

Determine whether each matrix product is defined. If so, state the dimensions of the product.

14.  $A_{4 \times 3} \cdot B_{3 \times 2}$

15.  $X_{2 \times 2} \cdot Y_{2 \times 2}$

16.  $P_{1 \times 3} \cdot Q_{4 \times 1}$

17.  $R_{1 \times 4} \cdot S_{4 \times 5}$

18.  $M_{4 \times 3} \cdot N_{4 \times 3}$

19.  $A_{3 \times 1} \cdot B_{1 \times 5}$

Find each product, if possible.

20.  $\begin{bmatrix} 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

21.  $\begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -7 \end{bmatrix}$

22.  $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$

23.  $\begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ 7 & -2 \end{bmatrix}$

24.  $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

25.  $\begin{bmatrix} 4 & -2 & -7 \\ 6 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$

26.  $\begin{bmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{bmatrix}$

27.  $\begin{bmatrix} -4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 \end{bmatrix}$

**BUSINESS** For Exercises 28–30, use the table and the following information.

Solada Fox sells fruit from her three farms. Apples are \$22 a case, peaches are \$25 a case, and apricots are \$18 a case.

Number of Cases in Stock of Each Type of Fruit			
Farm	Apples	Peaches	Apricots
1	290	165	210
2	175	240	190
3	110	75	0

28. Write an inventory matrix for the number of cases for each type of fruit for each farm and a cost matrix for the price per case for each type of fruit.
29. Find the total income of the three fruit farms expressed as a matrix.
30. What is the total income from all three fruit farms combined?

Use  $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}$ ,  $C = \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$ , and scalar  $c = 3$  to determine whether the following equations are true for the given matrices.

31.  $c(AB) = A(cB)$

32.  $(AB)C = (CB)A$

33.  $AC + BC = (A + B)C$

34.  $C(A + B) = AC + BC$

**FUND-RAISING** For Exercises 35 and 36, use the following information.

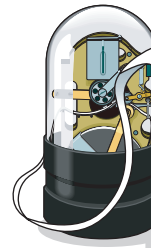
Lawrence High School sold wrapping paper and boxed cards for their fund-raising event. The school gets \$1.00 for each roll of wrapping paper sold and \$0.50 for each box of cards sold.

Total Amounts for Each Class		
Class	Wrapping Paper	Cards
Freshmen	72	49
Sophomores	68	63
Juniors	90	56
Seniors	86	62

35. Use a matrix to determine which class earned the most money.
36. What is the total amount of money the school made from the fund-raiser?

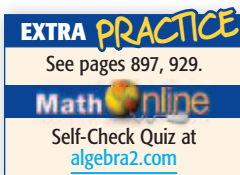
**FINANCE** For Exercises 37–39, use the table below that shows the purchase price and selling price of stock for three companies.

For a class project, Taini “bought” shares of stock in three companies. She bought 150 shares of a utility company, 100 shares of a computer company, and 200 shares of a food company. At the end of the project she “sold” all of her stock.



Company	Purchase Price (per share)	Selling Price (per share)
Utility	\$54.00	\$55.20
Computer	\$48.00	\$58.60
Food	\$60.00	\$61.10

37. Organize the data in two matrices and use matrix multiplication to find the total amount she spent for the stock.
38. Write two matrices and use matrix multiplication to find the total amount she received for selling the stock.
39. Use matrix operations to find how much money Taini “made” or “lost” in her project.

**H.O.T. Problems**

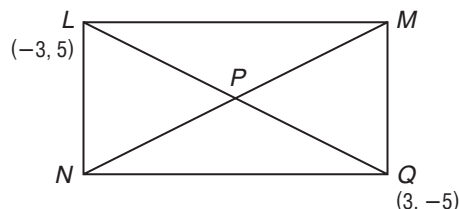
40. **OPEN ENDED** Give an example of two matrices whose product is a  $3 \times 2$  matrix.
41. **REASONING** Determine whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.
- For any matrix  $A_{m \times n}$  for  $m \neq n$ ,  $A^2$  is defined.*
42. **CHALLENGE** Give an example of two matrices  $A$  and  $B$  for which multiplication is commutative so that  $AB = BA$ . Explain how you found  $A$  and  $B$ .
43. **CHALLENGE** Find the values of  $a$ ,  $b$ ,  $c$ , and  $d$  to make the statement  $\begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix}$  true. If matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  was multiplied by any other two-column matrix, what do you think the result would be?
44. **Writing in Math** Use the data on the Carolina Panthers found on page 177 to explain how matrices can be used in sports statistics. Describe a matrix that represents the total number of points scored in the 2005 season, and an example of another sport where different point values are used in scoring.

- 45. ACT/SAT** What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

- A  $1 \times 4$   
B  $3 \times 3$   
C  $4 \times 1$   
D  $4 \times 3$

- 46. REVIEW** Rectangle  $LMNQ$  has diagonals that intersect at point  $P$ .



Which of the following represents point  $P$ ?

- F  $(1, 1)$   
G  $(2, 2)$   
H  $(0, 0)$   
J  $(-1, -1)$

## Spiral Review

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. (Lesson 4-2)

47.  $3 \begin{bmatrix} 4 & -2 \\ -1 & 7 \end{bmatrix}$

48.  $\begin{bmatrix} 3 & 5 & 9 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$

49.  $2 \begin{bmatrix} 6 & 3 \\ -8 & -2 \end{bmatrix} - 4 \begin{bmatrix} 8 & 1 \\ 3 & -4 \end{bmatrix}$

Solve each equation. (Lesson 4-1)

50.  $\begin{bmatrix} 3x + 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 23 \\ -4y - 1 \end{bmatrix}$

51.  $\begin{bmatrix} x + 3y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -22 \\ 19 \end{bmatrix}$

52.  $\begin{bmatrix} x + 3z \\ -2x + y - z \\ 5y - 7z \end{bmatrix} = \begin{bmatrix} -19 \\ -2 \\ 24 \end{bmatrix}$

- 53. VACATIONS** Mrs. Franklin is planning a family vacation. She bought 8 rolls of film and 2 camera batteries for \$23. The next day, her daughter went back and bought 6 more rolls of film and 2 batteries for her camera. This bill was \$18. What are the prices of a roll of film and a camera battery? (Lesson 3-2)

Find the  $x$ -intercept and the  $y$ -intercept of the graph of each equation. Then graph the equation. (Lesson 2-2)

54.  $y = 3 - 2x$

55.  $x - \frac{1}{2}y = 8$

56.  $5x - 2y = 10$

## GET READY for the Next Lesson

**PREREQUISITE SKILL** Graph each set of ordered pairs on a coordinate plane. (Lesson 2-1)

57.  $\{(2, 4), (-1, 3), (0, -2)\}$

58.  $\{(-3, 5), (-2, -4), (3, -2)\}$

59.  $\{(-1, 2), (2, 4), (3, -3), (4, -1)\}$

60.  $\{(-3, 3), (1, 3), (4, 2), (-1, -5)\}$

# Transformations with Matrices

## Main Ideas

- Use matrices to determine the coordinates of a translated or dilated figure.
- Use matrix multiplication to find the coordinates of a reflected or rotated figure.

## New Vocabulary

vertex matrix  
transformation  
preimage  
image  
translation  
dilation  
reflection  
rotation

## Reading Math

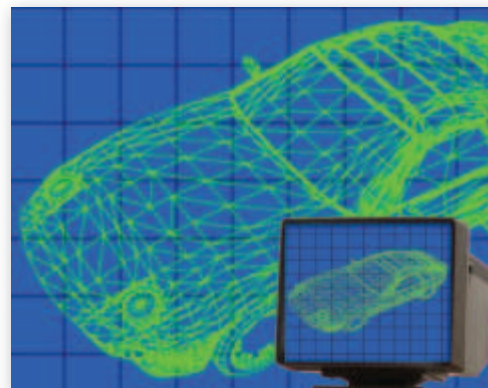
### Coordinate Matrix

A matrix containing coordinates of a geometric figure is also called a *coordinate matrix*.

## GET READY for the Lesson

Computer animation creates the illusion of motion by using a succession of computer-generated still images. Computer animation is used to create movie special effects and to simulate images that would be impossible to show otherwise.

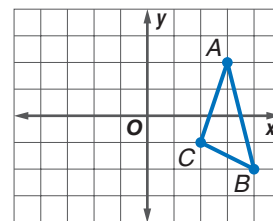
Complex geometric figures can be broken into simple triangles and then moved to other parts of the screen using matrices.



**Translations and Dilations** Points on a coordinate plane can be represented by matrices. The ordered pair  $(x, y)$  can be represented by the column matrix  $\begin{bmatrix} x \\ y \end{bmatrix}$ . Likewise, polygons can be represented by placing all of the column matrices of the coordinates of the vertices into one matrix, called a **vertex matrix**.

Triangle  $ABC$  with vertices  $A(3, 2)$ ,  $B(4, -2)$ , and  $C(2, -1)$  can be represented by the following vertex matrix.

$$\triangle ABC = \begin{bmatrix} A & B & C \\ 3 & 4 & 2 \\ 2 & -2 & -1 \end{bmatrix} \begin{matrix} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{matrix}$$



Notice that the triangle has 3 vertices and the vertex matrix has 3 columns. In general, the vertex matrix for a polygon with  $n$  vertices will have dimensions of  $2 \times n$ .

Matrices can be used to perform transformations. **Transformations** are functions that map points of a **preimage** onto its **image**.

One type of transformation is a translation. A **translation** occurs when a figure is moved from one location to another without changing its size, shape, or orientation. You can use matrix addition and a *translation matrix* to find the coordinates of a translated figure. The dimensions of a translation matrix should be the same as the dimensions of the vertex matrix.



## EXAMPLE Translate a Figure

- 1** Find the coordinates of the vertices of the image of quadrilateral  $QUAD$  with  $Q(2, 3)$ ,  $U(5, 2)$ ,  $A(4, -2)$ , and  $D(1, -1)$  if it is moved 4 units to the left and 2 units up. Then graph  $QUAD$  and its image  $Q'U'A'D'$ .

Write the vertex matrix for quadrilateral  $QUAD$ .  $\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix}$

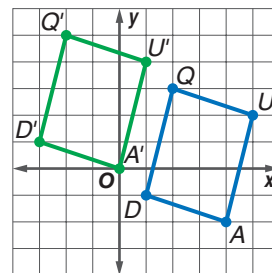
To translate the quadrilateral 4 units to the left, add  $-4$  to each  $x$ -coordinate. To translate the figure 2 units up, add 2 to each  $y$ -coordinate. This can be done by adding the translation

matrix  $\begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$  to the vertex matrix of  $QUAD$ .

$$\begin{array}{c} \text{Vertex Matrix} \\ \text{of } QUAD \end{array} \quad \begin{array}{c} \text{Translation} \\ \text{Matrix} \end{array} \quad \begin{array}{c} \text{Vertex Matrix} \\ \text{of } Q'U'A'D' \end{array}$$

$$\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix} + \begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & -3 \\ 5 & 4 & 0 & 1 \end{bmatrix}$$

The vertices of  $Q'U'A'D'$  are  $Q'(-2, 5)$ ,  $U'(1, 4)$ ,  $A'(0, 0)$ , and  $D'(-3, 1)$ .  $QUAD$  and  $Q'U'A'D'$  have the same size and shape.



## CHECK Your Progress

- 1.** Find the coordinates of the vertices of the image of triangle  $RST$  with  $R(-1, 5)$ ,  $S(2, 1)$ , and  $T(-3, 2)$  if it is moved 3 units to the right and 4 units up. Then graph  $RST$  and its image  $R'S'T'$ .

## STANDARDIZED TEST EXAMPLE

### Find a Translation Matrix

- 2** Rectangle  $A'B'C'D'$  is the result of a translation of rectangle  $ABCD$ . A table of the vertices of each rectangle is shown. Find the coordinates of  $D'$ .

Rectangle $ABCD$	Rectangle $A'B'C'D'$
$A(-4, 5)$	$A'(-1, 1)$
$B(1, 5)$	$B'(4, 1)$
$C(1, -2)$	$C'(4, -6)$
$D(-4, -2)$	$D'$

A  $(-7, 2)$

B  $(-7, -6)$

C  $(-1, -6)$

D  $(-1, 2)$

### Test-Taking Tip

Sometimes you need to solve for unknown value(s) before you can solve for the value(s) requested in the question.

### Read the Test Item

You are given the coordinates of the preimage and image of points  $A$ ,  $B$ , and  $C$ . Use this information to find the translation matrix. Then you can use the translation matrix to find the coordinates of  $D$ .

### Solve the Test Item

**Step 1** Write a matrix equation. Let  $(c, d)$  represent the coordinates of  $D$ .

$$\begin{bmatrix} -4 & 1 & 1 & -4 \\ 5 & 5 & -2 & -2 \end{bmatrix} + \begin{bmatrix} x & x & x & x \\ y & y & y & y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

$$\begin{bmatrix} -4 + x & 1 + x & 1 + x & -4 + x \\ 5 + y & 5 + y & -2 + y & -2 + y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$



**Step 2** The matrices are equal, so corresponding elements are equal.

$$-4 + x = -1 \quad \text{Solve for } x.$$

$$x = 3$$

$$5 + y = 1 \quad \text{Solve for } y.$$

$$y = -4$$

**Step 3** Use the values for  $x$  and  $y$  to find the values for  $D'(c, d)$ .

$$-4 + 3 = c$$

$$-1 = c$$

$$-2 + (-4) = d$$

$$-6 = d$$

So the coordinates for  $D$  are  $(-1, -6)$ , and the answer is  $C$ .

### CHECK Your Progress

2. Triangle  $X'Y'Z'$  is the result of a translation of triangle  $XYZ$ . Find the coordinates of  $Z'$  using the information shown in the table.

Triangle $XYZ$	Triangle $X'Y'Z'$
$X(3, -1)$	$X'(1, 0)$
$Y(-4, 2)$	$Y'(-6, 3)$
$Z(5, 1)$	$Z'$

$$F(3, 2)$$

$$G(7, 2)$$

$$H(7, 0)$$

$$J(3, 0)$$

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## Study Tip

### Dilations

In a dilation, all linear measures of the image change in the same ratio. The image is similar to the preimage.

When a figure is enlarged or reduced, the transformation is called a **dilation**. A dilation is performed relative to its center. Unless otherwise specified, the center is the origin. You can use scalar multiplication to perform dilations.

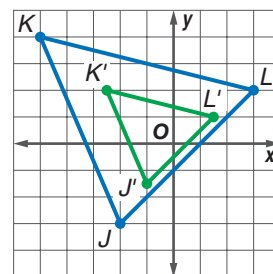
### EXAMPLE Dilation

- 3 Dilate  $\triangle JKL$  with  $J(-2, -3)$ ,  $K(-5, 4)$ , and  $L(3, 2)$  so that its perimeter is half the original perimeter. Find the coordinates of the vertices of  $\triangle J'K'L'$ .

If the perimeter of a figure is half the original perimeter, then the lengths of the sides of the figure will be one-half the measure of the original lengths. Multiply the vertex matrix by the scale factor of  $\frac{1}{2}$ .

$$\frac{1}{2} \begin{bmatrix} -2 & -5 & 3 \\ -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & 2 & 1 \end{bmatrix}$$

The coordinates of the vertices of  $\triangle J'K'L'$  are  $J'(-1, -\frac{3}{2})$ ,  $K'(-\frac{5}{2}, 2)$ , and  $L'(\frac{3}{2}, 1)$ .



### CHECK Your Progress

3. Dilate rectangle  $MNPQ$  with  $M(4, 4)$ ,  $N(4, 12)$ ,  $P(8, 4)$ , and  $Q(8, 12)$  so that its perimeter is one fourth the original perimeter. Find the coordinates of the vertices of rectangle  $M'N'P'Q'$ .

**Reflections and Rotations** A **reflection** maps every point of a figure to an image across a line of symmetry using a *reflection matrix*.

### CONCEPT SUMMARY

#### Reflection Matrices

For a reflection over the:	$x$ -axis	$y$ -axis	line $y = x$
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



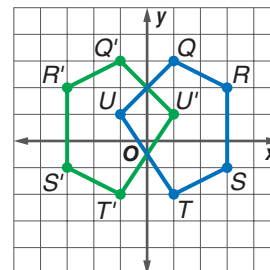
## EXAMPLE Reflection

- 4 Find the coordinates of the vertices of the image of pentagon  $QRSTU$  with  $Q(1, 3)$ ,  $R(3, 2)$ ,  $S(3, -1)$ ,  $T(1, -2)$ , and  $U(-1, 1)$  after a reflection across the  $y$ -axis.

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for the  $y$ -axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 3 & 1 & -1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & -1 & 1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix}$$

Notice that the preimage and image are congruent. Both figures have the same size and shape.



## CHECK Your Progress

4. Find the coordinates of the vertices of the image of pentagon  $QRSTU$  after a reflection across the  $x$ -axis.

A **rotation** occurs when a figure is moved around a center point, usually the origin. To determine the vertices of a figure's image by rotation, multiply its vertex matrix by a *rotation matrix*.

## CONCEPT SUMMARY

### Rotation Matrices

For a counterclockwise rotation about the origin of:	$90^\circ$	$180^\circ$	$270^\circ$
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

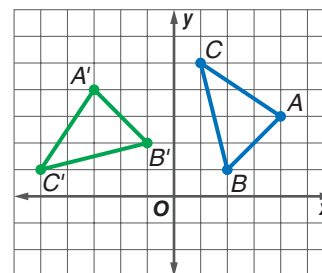
## EXAMPLE Rotation

- 5 Find the coordinates of the vertices of the image  $\triangle ABC$  with  $A(4, 3)$ ,  $B(2, 1)$ , and  $C(1, 5)$  after it is rotated  $90^\circ$  counterclockwise about the origin.

Write the ordered pairs in a vertex matrix. Then multiply the vertex matrix by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -1 & -5 \\ 4 & 2 & 1 \end{bmatrix}$$

The coordinates of the vertices of  $\triangle A'B'C'$  are  $A'(-3, 4)$ ,  $B'(-1, 2)$ , and  $C'(-5, 1)$ . The image is congruent to the preimage.



## CHECK Your Progress

5. Find the coordinates of the vertices of the image of  $\triangle XYZ$  with  $X(-5, -6)$ ,  $Y(-1, -3)$ , and  $Z(-2, -4)$  after it is rotated  $180^\circ$  counterclockwise about the origin.

**Example 1**  
(pp. 185–186)

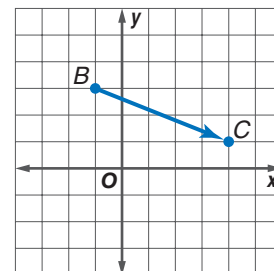
Triangle  $ABC$  with vertices  $A(1, 4)$ ,  $B(2, -5)$ , and  $C(-6, -6)$  is translated 3 units right and 1 unit down.

1. Write the translation matrix.
2. Find the coordinates of  $\triangle A'B'C'$ .
3. Graph the preimage and the image.

**Example 2**  
(pp. 186–187)

- 4. STANDARDIZED TEST PRACTICE** A point is translated from  $B$  to  $C$  as shown at the right. If a point at  $(-4, 3)$  is translated in the same way, what will be its new coordinates?

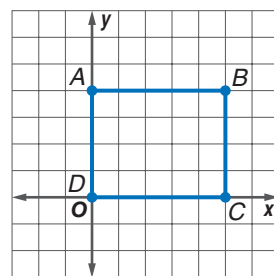
A  $(3, 4)$     B  $(1, 1)$     C  $(-8, 8)$     D  $(1, 6)$



**Example 3**  
(p. 187)

For Exercises 5–11, use the rectangle at the right.

5. Write the coordinates in a vertex matrix.
6. Find the coordinates of the image after a dilation by a scale factor of 3.
7. Find the coordinates of the image after a dilation by a scale factor of  $\frac{1}{2}$ .



**Example 4**  
(p. 188)

8. Find the coordinates of the image after a reflection over the  $x$ -axis.
9. Find the coordinates of the image after a reflection over the  $y$ -axis.
10. Find the coordinates of the image after a rotation of  $180^\circ$ .
11. Find the coordinates of the image after a rotation of  $270^\circ$ .

**Example 5**  
(p. 188)

**Exercises**

HOMEWORK	HELP
For Exercises	See Examples
12, 13	1
14, 15	2
16, 17	3
18, 19	4
20, 21	5

Write the translation matrix for each figure. Then find the coordinates of the image after the translation. Graph the preimage and the image on a coordinate plane.

12.  $\triangle DEF$  with  $D(1, 4)$ ,  $E(2, -5)$ , and  $F(-6, -6)$ , translated 4 units left and 2 units up
13.  $\triangle MNO$  with  $M(-7, 6)$ ,  $N(1, 7)$ , and  $O(-3, 1)$ , translated 2 units right and 6 units down
14. Rectangle  $RSUT$  with vertices  $R(-3, 2)$ ,  $S(1, 2)$ ,  $U(1, -1)$ ,  $T(-3, -1)$  is translated so that  $T'$  is at  $(-4, 1)$ . Find the coordinates of  $R'$  and  $U'$ .
15. Triangle  $DEF$  with vertices  $D(-2, 2)$ ,  $E(3, 5)$ , and  $F(5, -2)$  is translated so that  $D'$  is at  $(1, -5)$ . Find the coordinates of  $E'$  and  $F'$ .

Write the vertex matrix for each figure. Then find the coordinates of the image after the dilation. Graph the preimage and the image on a coordinate plane.

16.  $\triangle ABC$  with  $A(0, 2)$ ,  $B(1.5, -1.5)$ , and  $C(-2.5, 0)$  is dilated so that its perimeter is three times the original perimeter.
17.  $\triangle XYZ$  with  $X(-6, 2)$ ,  $Y(4, 8)$ , and  $Z(2, -6)$  is dilated so that its perimeter is one half times the original perimeter.

Write the vertex matrix and the reflection matrix for each figure. Then find the coordinates of the image after the reflection. Graph the preimage and the image on a coordinate plane.

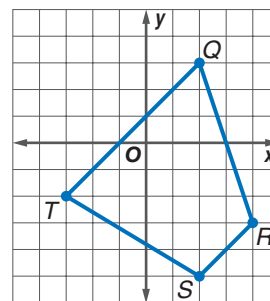
18. The vertices of  $\triangle XYZ$  are  $X(1, -1)$ ,  $Y(2, -4)$ , and  $Z(7, -1)$ . The triangle is reflected over the line  $y = x$ .
19. The vertices of rectangle  $ABDC$  are  $A(-3, 5)$ ,  $B(5, 5)$ ,  $D(5, -1)$ , and  $C(-3, -1)$ . The rectangle is reflected over the  $x$ -axis.

Write the vertex matrix and the rotation matrix for each figure. Then find the coordinates of the image after the rotation. Graph the preimage and the image on a coordinate plane.

20. Parallelogram  $DEFG$  with  $D(2, 4)$ ,  $E(5, 4)$ ,  $F(4, 1)$ , and  $G(1, 1)$  is rotated  $270^\circ$  counterclockwise about the origin.
21.  $\triangle MNO$  with  $M(-2, -6)$ ,  $N(1, 4)$ , and  $O(3, -4)$  is rotated  $180^\circ$  counterclockwise about the origin.

For Exercises 22–24, refer to the quadrilateral  $QRST$  shown at the right.

22. Write the vertex matrix. Multiply the vertex matrix by  $-1$ .
23. Graph the preimage and image.
24. What type of transformation does the graph represent?



25. A triangle is rotated  $90^\circ$  counterclockwise about the origin. The coordinates of the vertices are  $J'(-3, -5)$ ,  $K'(-2, 7)$ , and  $L'(1, 4)$ . What were the coordinates of the triangle in its original position?
26. A triangle is rotated  $90^\circ$  clockwise about the origin. The coordinates of the vertices are  $F'(2, -3)$ ,  $G'(-1, -2)$ , and  $H'(3, -2)$ . What were the coordinates of the triangle in its original position?
27. A quadrilateral is reflected across the  $y$ -axis. The coordinates of the vertices are  $P'(-2, 2)$ ,  $Q'(4, 1)$ ,  $R'(-1, -5)$ , and  $S'(-3, -4)$ . What were the coordinates of the quadrilateral in its original position?

For Exercises 28–31, use rectangle  $ABCD$  with vertices  $A(-4, 4)$ ,  $B(4, 4)$ ,  $C(4, -4)$ , and  $D(-4, -4)$ .

28. Find the coordinates of the image in matrix form after a reflection over the  $x$ -axis followed by a reflection over the  $y$ -axis.
29. Find the coordinates of the image in matrix form after a  $180^\circ$  rotation about the origin.
30. Find the coordinates of the image in matrix form after a reflection over the line  $y = x$ .
31. What do you observe about these three matrices? Explain.

**TECHNOLOGY** For Exercises 32 and 33, use the following information.

As you move the mouse for your computer, a corresponding arrow is translated on the screen. Suppose the position of the cursor on the screen is given in inches with the origin at the bottom left-hand corner of the screen.

32. Write a translation matrix that can be used to move the cursor 3 inches to the right and 4 inches up.
33. If the cursor is currently at  $(3.5, 2.25)$ , what are the coordinates of the position after the translation?



#### Real-World Link

Douglas Engelbart invented the “X-Y position indicator for a display system” in 1964. He nicknamed this invention “the mouse” because a tail came out the end.

Source: about.com



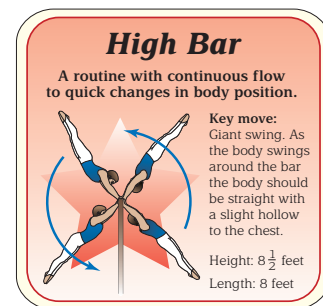


**LANDSCAPING** For Exercises 34 and 35, use the following information.

A garden design is plotted on a coordinate grid. The original plan shows a fountain with vertices at  $(-2, -2)$ ,  $(-6, -2)$ ,  $(-8, -5)$ , and  $(-4, -5)$ . Changes to the plan now require that the fountain's perimeter be three-fourths that of the original.

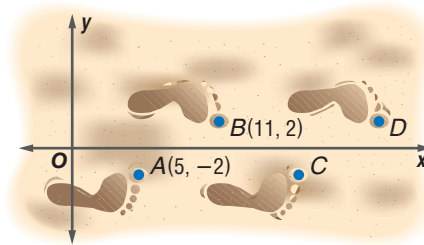
34. Determine the coordinates for the vertices of the fountain.
35. The center of the fountain was at  $(-5, -3.5)$ . What will be the coordinates of the center after the changes in the plan have been made?

36. **GYMNASTICS** The drawing at the right shows four positions of a man performing the giant swing in the high bar event. Suppose this drawing is placed on a coordinate grid with the hand grips at  $H(0, 0)$  and the toe of the figure in the upper right corner at  $T(7, 8)$ . Find the coordinates of the toes of the other three figures, if each successive figure has been rotated  $90^\circ$  counterclockwise about the origin.

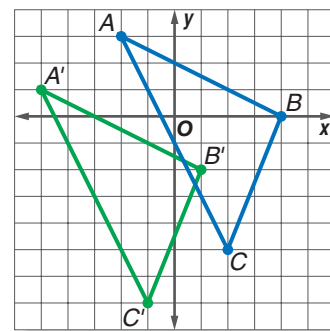
**FOOTPRINTS** For Exercises 37–40, use the following information.

The combination of a reflection and a translation is called a *glide reflection*.

An example is a set of footprints.



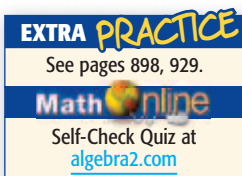
37. Describe the reflection and transformation combination shown at the right.
38. Write two matrix operations that can be used to find the coordinates of point C.
39. Does it matter which operation you do first? Explain.
40. What are the coordinates of the next two footprints?
41. Write the translation matrix for  $\triangle ABC$  and its image  $\triangle A'B'C'$  shown at the right.



42. Compare and contrast the size and shape of the preimage and image for each type of transformation. For which types of transformations are the images congruent to the preimage?

**H.O.T. Problems**

43. **OPEN ENDED** Write a translation matrix that moves  $\triangle DEF$  up and left.
44. **CHALLENGE** Do you think a matrix exists that would represent a reflection over the line  $x = 3$ ? If so, make a conjecture and verify it.
45. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.  
The image of a dilation is congruent to its preimage.
46. **Writing in Math** Use the information about computer animation on page 185 to explain how matrices can be used with transformations in computer animation. Include an example of how a figure with 5 points (coordinates) changes as a result of repeated dilations.





## STANDARDIZED TEST PRACTICE

47. **ACT/SAT** Triangle  $ABC$  has vertices with coordinates  $A(-4, 2)$ ,  $B(-4, -3)$ , and  $C(3, -2)$ . After a dilation, triangle  $A'B'C'$  has coordinates  $A'(-12, 6)$ ,  $B'(-12, -9)$ , and  $C'(9, -6)$ . How many times as great is the perimeter of  $\triangle A'B'C'$  as that of  $\triangle ABC$ ?

A 3  
B 6  
C 12  
D  $\frac{1}{3}$

48. **REVIEW** Melanie wanted to find 5 consecutive whole numbers that add up to 95. She wrote the equation  $(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 98$ . What does the variable  $n$  represent in the equation?

F The least of the 5 whole numbers  
G The middle of the 5 whole numbers  
H The greatest of the 5 whole numbers  
J The difference between the least and the greatest of the 5 whole numbers.

## Spiral Review

Determine whether each matrix product is defined. If so, state the dimensions of the product. (Lesson 4-3)

49.  $A_{2 \times 3} \cdot B_{3 \times 2}$

50.  $A_{4 \times 1} \cdot B_{2 \times 1}$

51.  $A_{2 \times 5} \cdot B_{5 \times 5}$

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. (Lesson 4-2)

52.  $2 \begin{bmatrix} 4 & 9 & -8 \\ 6 & -11 & -2 \\ 12 & -10 & 3 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

53.  $4 \begin{bmatrix} 3 & 4 & -7 \\ 6 & -9 & -2 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -8 & 6 & -4 \\ -7 & 10 & 1 \\ -2 & 1 & 5 \end{bmatrix}$

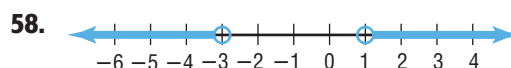
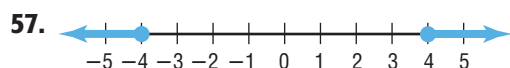
Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function. (Lesson 2-1)

54.  $(3, 5), (4, 6), (5, -4)$

55.  $x = -5y + 2$

56.  $x = y^2$

Write an absolute value inequality for each graph. (Lesson 1-6)



59. **BUSINESS** Reliable Rentals rents cars for \$12.95 per day plus 15¢ per mile. Luis Romero works for a company that limits expenses for car rentals to \$90 per day. How many miles can Mr. Romero drive each day? (Lesson 1-5)

## GET READY for the Next Lesson

**PREREQUISITE SKILL** Use cross products to solve each proportion.

60.  $\frac{x}{8} = \frac{3}{4}$

61.  $\frac{4}{20} = \frac{1}{m}$

62.  $\frac{2}{3} = \frac{a}{42}$

63.  $\frac{2}{y} = \frac{8}{9}$

64.  $\frac{4}{n} = \frac{6}{2n - 3}$

65.  $\frac{x}{5} = \frac{x + 1}{8}$

## Mid-Chapter Quiz

Lessons 4-1 through 4-4

Solve each equation. (Lesson 4-1)

1.  $\begin{bmatrix} 3x + 1 \\ 7y \end{bmatrix} = \begin{bmatrix} 19 \\ 21 \end{bmatrix}$

2.  $\begin{bmatrix} 2x + y \\ 4x - 3y \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \end{bmatrix}$

**BUSINESS** For Exercises 3 and 4, use the table and the following information.

The manager of The Best Bagel Shop keeps records of the types of bagels sold each day at their two stores. Two days of sales are shown below.

Day	Store	Type of Bagel			
		Sesame	Poppy	Wheat	Plain
Monday	East	120	80	64	75
	West	65	105	77	53
Tuesday	East	112	79	56	74
	West	69	95	82	50

3. Write a matrix for each day's sales. (Lesson 4-1)

4. Find the sum of the two days' sales using matrix addition. (Lesson 4-2)

Perform the indicated matrix operations.

(Lesson 4-2)

5.  $\begin{bmatrix} 3 & 0 \\ 7 & 12 \end{bmatrix} - \begin{bmatrix} 6 & -5 \\ 4 & -1 \end{bmatrix}$

6.  $5 \begin{bmatrix} -2 & 4 & 5 \\ 0 & -4 & 7 \end{bmatrix}$

7. **MULTIPLE CHOICE** Solve for  $x$  and  $y$  in the matrix equation

$$\begin{bmatrix} 4x \\ -y \end{bmatrix} + \begin{bmatrix} -3y \\ 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 2 \end{bmatrix} \quad (\text{Lesson 4-2})$$

A  $x = 7, y = 2$       C  $x = -7, y = 2$

B  $x = -7, y = -2$       D  $x = 7, y = -2$

Find each product, if possible. (Lesson 4-3)

8.  $\begin{bmatrix} 4 & 0 & -8 \\ 7 & -2 & 10 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 6 & 0 \end{bmatrix}$

9.  $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & -2 \\ -3 & 5 & 4 \end{bmatrix}$

**RESTAURANTS** For Exercises 10–13, use the table and the following information. (Lesson 4-3)

At Joe's Diner, the employees get paid weekly. The diner is closed on Mondays and Tuesdays. The servers make \$20 per day (plus tips), cooks make \$64 per day, and managers make \$96 per day.

Number of Staff			
Day	Servers	Cooks	Managers
Wed.	8	3	2
Thur.	11	4	2
Fri.	17	6	5
Sat.	18	6	5
Sun.	14	5	3

10. Write a matrix for the number of staff needed for each day at the diner.

11. Write a cost matrix for the cost per type of employee.

12. Find the total cost of the wages for each day expressed as a matrix.

13. What is the total cost of wages for the week?

14. **MULTIPLE CHOICE** What is the product of

$$\begin{bmatrix} 5 & -2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \quad (\text{Lesson 4-3})$$

F  $\begin{bmatrix} 11 \\ -1 \end{bmatrix}$

G  $\begin{bmatrix} 11 & -1 \end{bmatrix}$

H  $\begin{bmatrix} 5 & -10 \\ 0 & -6 \\ 6 & -15 \end{bmatrix}$

J undefined

For Exercises 15 and 16, reflect square  $ABCD$  with vertices  $A(1, 2)$ ,  $B(4, -1)$ ,  $C(1, -4)$ , and  $D(-2, -1)$  over the  $y$ -axis. (Lesson 4-4)

15. Write the coordinates in a vertex matrix.

16. Find the coordinates of  $A'B'C'D'$ . Then graph  $ABCD$  and  $A'B'C'D'$ .

**Main Ideas**

- Evaluate the determinant of a  $2 \times 2$  matrix.
- Evaluate the determinant of a  $3 \times 3$  matrix.

**New Vocabulary**

determinant  
 second-order determinant  
 third-order determinant  
 expansion by minors  
 minor

**GET READY for the Lesson**

The “Bermuda Triangle” is an area located off the southeastern Atlantic coast of the United States that is noted for a high incidence of unexplained losses of ships, small boats, and aircraft. Using the coordinates of the vertices of this triangle, you can find the value of a determinant to approximate the area of the triangle.



**Determinants of  $2 \times 2$  Matrices** Every square matrix has a number associated with it called its **determinant**. The determinant of  $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$  can be represented by  $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$  or  $\det \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ . The determinant of a  $2 \times 2$  matrix is called a **second-order determinant**.

**KEY CONCEPT****Second-Order Determinant**

**Words** The value of a second-order determinant is found by calculating the difference of the products of the two diagonals.

**Symbols**  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

**Example**  $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = 3(5) - (-1)(2) = 17$

**EXAMPLE****Second-Order Determinant**

**1** Find the value of the determinant  $\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix}$ .  
 $\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix} = (-2)(8) - 5(6)$  Definition of determinant  
 $= -16 - 30$  or  $-46$  Multiply.

**CHECK Your Progress**

Find the value of each determinant.

1A.  $\begin{vmatrix} 7 & 4 \\ -3 & 2 \end{vmatrix}$

1B.  $\begin{vmatrix} -4 & 6 \\ -3 & -2 \end{vmatrix}$

## Study Tip

### Determinants

Note that only square matrices have determinants.

**Determinants of  $3 \times 3$  Matrices** Determinants of  $3 \times 3$  matrices are called **third-order determinants**. One method of evaluating third-order determinants is **expansion by minors**. The **minor** of an element is the determinant formed when the row and column containing that element are deleted.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The minor of  $a$  is  $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ .

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The minor of  $b$  is  $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$ .

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The minor of  $c$  is  $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$ .

To use expansion by minors with third-order determinants, each member of one row is multiplied by its minor and its *position sign*, and the results are added together. The position signs alternate between positive and negative, beginning with a positive sign in the first row, first column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

### KEY CONCEPT

### Third-Order Determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The definition of third-order determinants shows an expansion using the elements in the first row of the determinant. However, any row can be used.

### EXAMPLE

### Expansion by Minors

**2** Evaluate  $\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix}$  using expansion by minors.

Decide which row of elements to use for the expansion. For this example, we will use the first row.

$$\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2 \begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7 \begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix}$$

Expansion by minors

$$= 2(0 - (-36)) - 7(0 - (-24)) - 3(-9 - 30)$$

Evaluate determinants.

$$= 2(36) - 7(24) - 3(-39)$$

$$= 72 - 168 + 117 \text{ or } 21$$

Multiply.

### CHECK Your Progress

**2.** Evaluate  $\begin{vmatrix} -2 & 3 & -1 \\ 5 & -3 & 8 \\ 4 & -6 & -5 \end{vmatrix}$  using expansion by minors.



Personal Tutor at [algebra2.com](http://algebra2.com)

Another method for evaluating a third-order determinant is by using diagonals.

**Step 1** Begin by writing the first two columns on the right side of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}$$

**Step 2** Next, draw diagonals from each element of the top row of the determinant downward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}$$

*aei bfg cdh*

Then, draw diagonals from the elements in the third row of the determinant upward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}$$

*gec hfa idb*

**Step 3** To find the value of the determinant, add the products of the first set of diagonals and then subtract the products of the second set of diagonals. The sum is  $aei + bfg + cdh - gec - hfa - idb$ .

### EXAMPLE Use Diagonals

**3** Evaluate  $\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}$  using diagonals.

**Step 1** Rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix} \quad \begin{vmatrix} -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{vmatrix}$$

**Step 2** Find the products of the elements of the diagonals.

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix} \quad \begin{vmatrix} -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{vmatrix}$$

*0 -5 24*  
*4 0 60*

**Step 3** Add the bottom products and subtract the top products.  
 $4 + 0 + 60 - 0 - (-5) - 24 = 45$

The value of the determinant is 45.

### CHECK Your Progress

**3.** Evaluate  $\begin{vmatrix} 1 & -5 & 3 \\ 0 & 2 & -7 \\ 5 & -1 & -2 \end{vmatrix}$  using diagonals.



One very useful application of determinants is finding the areas of polygons. The formula below shows how determinants can be used to find the area of a triangle using the coordinates of the vertices.

## Study Tip

### Area Formula

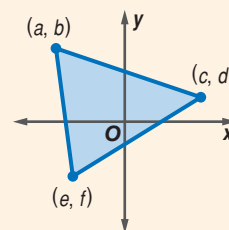
Notice that it is necessary to use the absolute value of  $A$  to guarantee a nonnegative value for the area.

## KEY CONCEPT

### Area of a Triangle

The area of a triangle having vertices at  $(a, b)$ ,  $(c, d)$ , and  $(e, f)$  is  $|A|$ , where

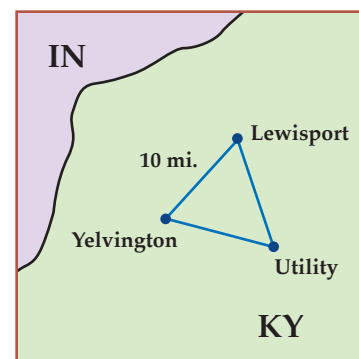
$$A = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}.$$



## Real-World EXAMPLE



**RADIO** A local radio station in Kentucky wants to place a tower that is strong enough to cover the cities of Yelvington, Utility, and Lewisport. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Kentucky with Yelvington at the origin, the coordinates of the three cities are  $(0, 0)$ ,  $(3, 0)$ , and  $(1, 2)$ . Use a determinant to estimate the area the signal must cover.



$$A = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

Area Formula

$$= \frac{1}{2} \begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$(a, b) = (3, 0)$ ,  $(c, d) = (0, 2)$ ,  $(e, f) = (0, 0)$

$$= \frac{1}{2} \left[ 3 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \right]$$

Expansion by minors

$$= \frac{1}{2} [3(2 - 0) - 0(1 - 0) + 1(0 - 0)]$$

Evaluate  $2 \times 2$  determinants.

$$= \frac{1}{2} (6 - 0 - 0)$$

Multiply.

$$= \frac{1}{2} (6) \text{ or } 3$$

Simplify.

Remember that 1 unit equals 10 miles, so 1 square unit =  $10 \times 10$  or 100 square miles. Thus, the area is  $3 \times 100$  or 300 square miles.



## CHECK Your Progress

- Find the area of the triangle whose vertices are located at  $(2, 3)$ ,  $(-4, -3)$ , and  $(1, -2)$ .



**Example 1**  
(p. 194)

Find the value of each determinant.

1.  $\begin{vmatrix} 7 & 8 \\ 3 & -2 \end{vmatrix}$

2.  $\begin{vmatrix} -3 & -6 \\ 4 & 8 \end{vmatrix}$

**Example 2**  
(p. 195)

Evaluate each determinant using expansion by minors.

3.  $\begin{vmatrix} 0 & -4 & 0 \\ 3 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix}$

4.  $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix}$

**Example 3**  
(p. 196)

Evaluate each determinant using diagonals.

5.  $\begin{vmatrix} 1 & 6 & 4 \\ -2 & 3 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

6.  $\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix}$

**Example 4**  
(p. 197)

7. **GEOMETRY** What is the area of  $\triangle ABC$  with  $A(5, 4)$ ,  $B(3, -4)$ , and  $C(-3, -2)$ ?
8. Find the area of the triangle whose vertices are located at  $(2, -1)$ ,  $(1, 2)$ , and  $(-1, 0)$ .

## Exercises

Find the value of each determinant.

9.  $\begin{vmatrix} 10 & 6 \\ 5 & 5 \end{vmatrix}$

10.  $\begin{vmatrix} 8 & 5 \\ 6 & 1 \end{vmatrix}$

11.  $\begin{vmatrix} -7 & 3 \\ -9 & 7 \end{vmatrix}$

12.  $\begin{vmatrix} -2 & 4 \\ 3 & -6 \end{vmatrix}$

13.  $\begin{vmatrix} -6 & -2 \\ 8 & 5 \end{vmatrix}$

14.  $\begin{vmatrix} -9 & 0 \\ -12 & -7 \end{vmatrix}$

15.  $\begin{vmatrix} 7 & 5.2 \\ -4 & 1.6 \end{vmatrix}$

16.  $\begin{vmatrix} -3.2 & -5.8 \\ 4.1 & 3.9 \end{vmatrix}$

17.  $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 6 & 4 \\ 2 & 5 & 1 \end{vmatrix}$

18.  $\begin{vmatrix} 7 & 3 & -4 \\ -2 & 9 & 6 \\ 0 & 0 & 0 \end{vmatrix}$

19.  $\begin{vmatrix} -2 & 7 & -2 \\ 4 & 5 & 2 \\ 1 & 0 & -1 \end{vmatrix}$

20.  $\begin{vmatrix} -3 & 0 & 6 \\ 6 & 5 & -2 \\ 1 & 4 & 2 \end{vmatrix}$

21.  $\begin{vmatrix} 1 & 5 & -4 \\ -7 & 3 & 2 \\ 6 & 3 & -1 \end{vmatrix}$

22.  $\begin{vmatrix} 3 & 7 & 6 \\ -1 & 6 & 2 \\ 8 & -3 & -5 \end{vmatrix}$

23.  $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 9 & 5 \\ 8 & 7 & 4 \end{vmatrix}$

24.  $\begin{vmatrix} 1 & 5 & 2 \\ -6 & -7 & 8 \\ 5 & 9 & -3 \end{vmatrix}$

25.  $\begin{vmatrix} 8 & -9 & 0 \\ 1 & 5 & 4 \\ 6 & -2 & 3 \end{vmatrix}$

26. **GEOGRAPHY** Mr. Cardona is a regional sales manager for a company in Florida. Tampa, Orlando, and Ocala outline his region. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Florida with Tampa at the origin, the coordinates of the three cities are  $(0, 0)$ ,  $(7, 5)$ , and  $(2.5, 10)$ . Estimate the area of his sales territory.





### Real-World Career

#### Archaeologist

Archaeologists attempt to reconstruct past ways of life by examining preserved bones, the ruins of buildings, and artifacts such as tools, pottery, and jewelry.



For more information, go to [algebra2.com](http://algebra2.com).

### H.O.T. Problems

#### EXTRA PRACTICE

See pages 898, 929.

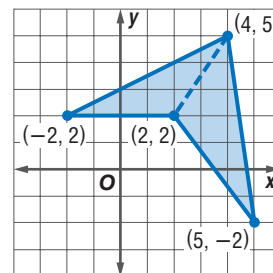


Self-Check Quiz at [algebra2.com](http://algebra2.com)

- 27. ARCHAEOLOGY** During an archaeological dig, a coordinate grid is laid over the site to identify the location of artifacts as they are excavated. Suppose three corners of a building have been unearthed at  $(-1, 6)$ ,  $(4, 5)$ , and  $(-1, -2)$ . If each square on the grid measures one square foot, estimate the area of the floor of the building, assuming that it is triangular.

- 28. GEOMETRY** Find the area of a triangle whose vertices are located at  $(4, 1)$ ,  $(2, -1)$ , and  $(0, 2)$ .

- 29. GEOMETRY** Find the area of the polygon shown at the right.



- 30.** Solve for  $x$  if  $\det \begin{bmatrix} 2 & x \\ 5 & -3 \end{bmatrix} = 24$ .

- 31.** Solve  $\det \begin{bmatrix} 4 & x & -2 \\ -x & -3 & 1 \\ -6 & 2 & 3 \end{bmatrix} = -3$  for  $x$ .

- 32. GEOMETRY** Find the value of  $x$  such that the area of a triangle whose vertices have coordinates  $(6, 5)$ ,  $(8, 2)$ , and  $(x, 11)$  is 15 square units.

- 33. GEOMETRY** The area of a triangle  $ABC$  is 2 square units. The vertices of the triangle are  $A(-1, 5)$ ,  $B(3, 1)$ , and  $C(-1, y)$ . What are the possible values of  $y$ ?

**MATRIX FUNCTION** You can use a TI-83/84 Plus to find determinants of square matrices using the **MATRIX** functions. Enter the matrix under the **EDIT** menu. Then from the home screen choose **det**(, which is option 1 on the **MATH** menu, followed by the matrix name to calculate the determinant.

Use a graphing calculator to find the value of each determinant.

- 34.**  $\begin{bmatrix} 3 & -6.5 \\ 8 & 3.75 \end{bmatrix}$       **35.**  $\begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{bmatrix}$       **36.**  $\begin{bmatrix} 10 & 12 & 4 \\ -3 & 18 & -9 \\ 16 & -2 & -1 \end{bmatrix}$

- 37. OPEN ENDED** Write a matrix whose determinant is zero.

- 38. FIND THE ERROR** Khalid and Erica are finding the determinant of  $\begin{bmatrix} 8 & 3 \\ -5 & 2 \end{bmatrix}$ . Who is correct? Explain your reasoning.

Khalid

$$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - (-15) \\ = 31$$

Erica

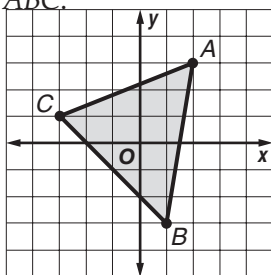
$$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - 15 \\ = 1$$

- 39. REASONING** Find a counterexample to disprove the following statement.  
*Two different matrices can never have the same determinant.*

- 40. CHALLENGE** Find a third-order determinant in which no element is 0, but for which the determinant is 0.

- 41. Writing in Math** Use the information about the “Bermuda Triangle” on page 194 to explain how matrices can be used to find the area covered in this triangle. Then use your method to find the area.

42. **ACT/SAT** Find the area of triangle  $ABC$ .



- A 10 units<sup>2</sup>  
 B 12 units<sup>2</sup>  
 C 14 units<sup>2</sup>  
 D 16 units<sup>2</sup>

43. **REVIEW** Use the table to determine the expression that best represents the number of faces of any prism having a base with  $n$  sides.

Base	Sides of Base	Faces of Prisms
Triangle	3	5
Quadrilateral	4	6
Pentagon	5	7
Hexagon	6	8
Heptagon	7	9
Octagon	8	10

- F  $2(n - 1)$       H  $n + 2$   
 G  $2(n + 1)$       J  $2n$

## Spiral Review

For Exercises 44 and 45, use the following information. (Lesson 4-4)

The vertices of  $\triangle ABC$  are  $A(-2, 1)$ ,  $B(1, 2)$  and  $C(2, -3)$ . The triangle is dilated so that its perimeter is  $2\frac{1}{2}$  times the original perimeter.

44. Write the coordinates of  $\triangle ABC$  in a vertex matrix.  
 45. Find the coordinates of  $\triangle A'B'C'$ . Then graph  $\triangle ABC$  and  $\triangle A'B'C'$ .

Find each product, if possible. (Lesson 4-3)

46.  $\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 9 \\ -1 & 2 \end{bmatrix}$       47.  $\begin{bmatrix} 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ -4 & 2 \end{bmatrix}$       48.  $\begin{bmatrix} 7 & -5 & 4 \\ 6 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ -2 & -8 \\ 1 & 2 \end{bmatrix}$

49. **MARATHONS** The length of a marathon was determined in the 1908 Olympic Games in London, England. The race began at Windsor Castle and ended in front of the royal box at London's Olympic Stadium, which was a distance of 26 miles 385 yards. Determine how many feet the marathon covers using the formula  $f(m, y) = 5280m + 3y$ , where  $m$  is the number of miles and  $y$  is the number of yards. (Lesson 3-4)

Write an equation in slope-intercept form for the line that satisfies each set of conditions. (Lesson 2-4)

50. slope 1, passes through  $(5, 3)$       51. slope  $-\frac{4}{3}$ , passes through  $(6, -8)$   
 52. passes through  $(3, 7)$  and  $(-2, -3)$       53. passes through  $(0, 5)$  and  $(10, 10)$

## GET READY for the Next Lesson

**PREREQUISITE SKILL** Solve each system of equations. (Lesson 3-2)

54.  $x + y = -3$       55.  $x + y = 10$       56.  $2x + y = 5$   
 $3x + 4y = -12$        $2x + y = 11$        $4x + y = 9$

## Main Ideas

- Solve systems of two linear equations by using Cramer's Rule.
- Solve systems of three linear equations by using Cramer's Rule.

## New Vocabulary

Cramer's Rule

## Study Tip

## Look Back

To review **solving systems of equations**, see Lesson 3-2.

## GET READY for the Lesson

Two sides of a triangle are contained in lines whose equations are  $1.4x + 3.8y = 3.4$  and  $2.5x - 1.7y = -10.9$ . To find the coordinates of the vertex of the triangle between these two sides, you must solve the system of equations. One method for solving systems of equations is Cramer's Rule.

**Systems of Two Linear Equations** **Cramer's Rule** uses determinants to solve systems of equations. Consider the following system.

$$ax + by = e \quad a, b, c, d, e, \text{ and } f \text{ represent constants, not variables.}$$

$$cx + dy = f$$

Solve for  $x$  by using elimination.

$$adx + bdy = de$$

Multiply the first equation by  $d$ .

$$(-) \quad bcx + bdy = bf$$

Multiply the second equation by  $b$ .

$$\frac{adx - bcx = de - bf}{(ad - bc)x = de - bf}$$

Subtract.

$$(ad - bc)x = de - bf$$

Factor.

$$x = \frac{de - bf}{ad - bc}$$

Divide. Notice that  $ad - bc$  must not be zero.

Solving for  $y$  in the same way produces the following expression.

$$y = \frac{af - ce}{ad - bc}$$

So the solution of the system of equations is  $\left(\frac{de - bf}{ad - bc}, \frac{af - ce}{ad - bc}\right)$ .

The fractions have a common denominator. It can be written using a determinant. The numerators can also be written as determinants.

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad de - bf = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \quad af - ce = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

## KEY CONCEPT

## Cramer's Rule for Two Variables

The solution of the system of linear equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

is  $(x, y)$ , where  $x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ ,  $y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ , and  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ .

## EXAMPLE System of Two Equations

1 Use Cramer's Rule to solve the system of equations.

$$5x + 7y = 13$$

$$2x - 5y = 13$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Cramer's Rule

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 13 & 7 \\ 13 & -5 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}}$$

$$a = 5, b = 7, c = 2, d = -5, \\ e = 13, \text{ and } f = 13$$

$$= \frac{\begin{vmatrix} 5 & 13 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}}$$

$$= \frac{13(-5) - 13(7)}{5(-5) - 2(7)}$$

Evaluate each determinant.

$$= \frac{5(13) - 2(13)}{5(-5) - 2(7)}$$

$$= \frac{-156}{-39} \text{ or } 4$$

Simplify.

$$= \frac{39}{-39} \text{ or } -1$$

The solution is  $(4, -1)$ .

## CHECK Your Progress

Use Cramer's Rule to solve the systems of equations.

1A.  $4x - 2y = -2$   
 $-x + 3y = 13$

1B.  $2x - 3y = 12$   
 $-6x + y = -20$



## Real-World EXAMPLE



**ELECTIONS** In the 2004 presidential election, George W. Bush received about 10,000,000 votes in California and Texas, while John Kerry received about 9,500,000 votes in those states. The graph shows the percent of the popular vote that each candidate received in those states.

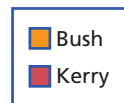
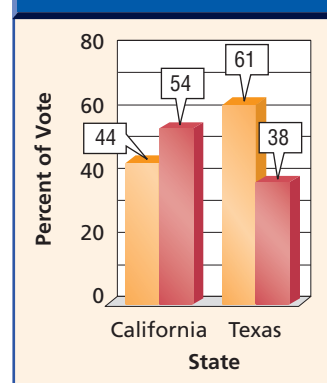
a. Write a system of equations that represents the total number of votes cast for each candidate in these two states.

**Words**

George W. Bush received 44% and 61% of the votes in California and Texas, respectively, for a total of 10,000,000 votes.

John Kerry received 54% and 38% of the votes in California and Texas, respectively, for a total of 9,500,000 votes.

Presidential Elections



## Real-World Link

In 2000, George W. Bush became the first son of a former president to win the presidency since John Quincy Adams did it in 1825.



You know the total votes for each candidate in Texas and California and the percent of the votes cast for each. You need to know the number of votes for each candidate in each state.

**Variables** Let  $x$  represent the total number of votes in California.  
Let  $y$  represent the total number of votes in Texas.

**Equations**  $0.44x + 0.61y = 10,000,000$  Votes for Bush  
 $0.54x + 0.38y = 9,500,000$  Votes for Kerry

**b. Find the total number of popular votes cast in California and Texas.**

Use Cramer's Rule to solve the system of equations.

Let  $a = 0.44$ ,  $b = 0.61$ ,  $c = 0.54$ ,  $d = 0.38$ ,  $e = 10,000,000$ , and  $f = 9,500,000$ .

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{Cramer's Rule} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 10,000,000 & 0.61 \\ 9,500,000 & 0.38 \end{vmatrix}}{\begin{vmatrix} 0.44 & 0.61 \\ 0.54 & 0.38 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} 0.44 & 10,000,000 \\ 0.54 & 9,500,000 \end{vmatrix}}{\begin{vmatrix} 0.44 & 0.61 \\ 0.54 & 0.38 \end{vmatrix}}$$

$$= \frac{10,000,000(0.38) - 9,500,000(0.61)}{0.44(0.38) - 0.54(0.61)} \quad y = \frac{0.44(9,500,000) - 0.54(10,000,000)}{0.44(0.38) - 0.54(0.61)}$$

$$= \frac{-1995000}{-0.1622} \quad y = \frac{-1220000}{-0.1622}$$

$$\approx 12,299,630 \quad \approx 7,521,578$$

The solution of the system is about (12,299,630, 7,521,578).

So, there were about 12,300,000 popular votes cast in California and about 7,500,000 popular votes cast in Texas.

**CHECK** If you add the votes that Bush and Kerry received, the result is  $10,000,000 + 9,500,000$  or 19,500,000. If you add the popular votes in California and Texas, the result is  $12,300,000 + 7,500,000$  or 19,800,000. The difference of 300,000 votes is reasonable considering there were over 19 million total votes.

**CHECK Your Progress**

At the game on Friday, the Athletic Boosters sold chips  $C$  for \$0.50 and candy bars  $B$  for \$0.50 and made \$27. At Saturday's game, they raised the prices of chips to \$0.75 and candy bars to \$1.00. They made \$48 for the same amount of chips and candy bars sold.

- 2A.** Write a system of equations that represents the total number of chips and candy bars sold at the games on Friday and Saturday.
- 2B.** Find the total number of chips and candy bars that were sold on each day.





**Systems of Three Linear Equations** You can also use Cramer's Rule to solve a system of three equations in three variables.

### KEY CONCEPT

### Cramer's Rule for Three Variables

The solution of the system whose equations are

$$ax + by + cz = j$$

$$dx + ey + fz = k$$

$$gx + hy + iz = \ell$$

$$\text{is } (x, y, z), \text{ where } x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \text{ and } \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0.$$

### EXAMPLE

### System of Three Equations

**3** Use Cramer's Rule to solve the system of equations.

$$3x + y + z = -1$$

$$-6x + 5y + 3z = -9$$

$$9x - 2y - z = 5$$

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -1 & 1 & 1 \\ -9 & 5 & 3 \\ 5 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 3 & -1 & 1 \\ -6 & -9 & 3 \\ 9 & 5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 3 & 1 & -1 \\ -6 & 5 & -9 \\ 9 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}}$$

Use a calculator to evaluate each determinant.

$$x = \frac{-2}{-9} \text{ or } \frac{2}{9}$$

$$y = \frac{12}{-9} \text{ or } -\frac{4}{3}$$

$$z = \frac{3}{-9} \text{ or } -\frac{1}{3}$$

The solution is  $(\frac{2}{9}, -\frac{4}{3}, -\frac{1}{3})$ .



### CHECK Your Progress

**3.**  $2x + y - z = -2$

$$-x + 2y + z = -0.5$$

$$x + y + 2z = 3.5$$

### Cross-Curricular Project

**Math** You can use Cramer's Rule to compare home loans. Visit [algebra2.com](http://algebra2.com) to continue work on your project.

**Example 1**  
(p. 202)

Use Cramer's Rule to solve each system of equations.

$$\begin{aligned} 1. \quad x - 4y &= 1 \\ 2x + 3y &= 13 \end{aligned}$$

$$\begin{aligned} 2. \quad 0.2a &= 0.3b \\ 0.4a - 0.2b &= 0.2 \end{aligned}$$

**Example 2**  
(pp. 202–203)

**INVESTING** For Exercises 3 and 4, use the following information.

Jarrold Wright has a total of \$5000 in his savings account and in a certificate of deposit. His savings account earns 3.5% interest annually. The certificate of deposit pays 5% interest annually if the money is invested for one year. He calculates that his interest earnings for the year will be \$227.50.

3. Write a system of equations for the amount of money in each investment.
4. How much money is in his savings account and in the certificate of deposit?

**Example 3**  
(p. 204)

Use Cramer's Rule to solve each system of equations.

$$\begin{aligned} 5. \quad 2x - y + 3z &= 5 \\ 3x + 2y - 5z &= 4 \\ x - 4y + 11z &= 3 \end{aligned}$$

$$\begin{aligned} 6. \quad a + 9b - 2c &= 2 \\ -a - 3b + 4c &= 1 \\ 2a + 3b - 6c &= -5 \end{aligned}$$

**Exercises**

HOMEWORK <b>HELP</b>	
For Exercises	See Examples
7–12	1
13–17	2
18–21	3

Use Cramer's Rule to solve each system of equations.

$$\begin{aligned} 7. \quad 5x + 2y &= 8 \\ 2x - 3y &= 7 \end{aligned}$$

$$\begin{aligned} 8. \quad 2m + 7n &= 4 \\ m - 2n &= -20 \end{aligned}$$

$$\begin{aligned} 9. \quad 2r - s &= 1 \\ 3r + 2s &= 19 \end{aligned}$$

$$\begin{aligned} 10. \quad 3a + 5b &= 33 \\ 5a + 7b &= 51 \end{aligned}$$

$$\begin{aligned} 11. \quad 2m - 4n &= -1 \\ 3n - 4m &= -5 \end{aligned}$$

$$\begin{aligned} 12. \quad 4x + 3y &= 6 \\ 8x - y &= -9 \end{aligned}$$

13. **GEOMETRY** The two sides of an angle are contained in lines whose equations are  $4x + y = -4$  and  $2x - 3y = -9$ . Find the coordinates of the vertex of the angle.

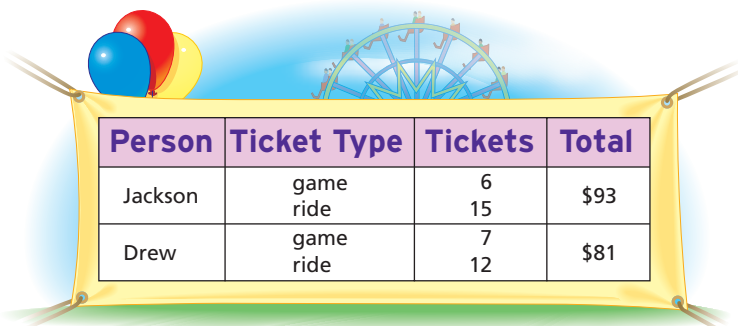
14. **GEOMETRY** Two sides of a parallelogram are contained in the lines whose equations are  $2.3x + 1.2y = 2.1$  and  $4.1x - 0.5y = 14.3$ . Find the coordinates of a vertex of the parallelogram.

**STATE FAIR** For Exercises 15 and 16, use the following information.

Jackson and Drew each purchased some game and ride tickets.

15. Write a system of two equations using the given information.

16. Find the price for each type of ticket.



Person	Ticket Type	Tickets	Total
Jackson	game ride	6 15	\$93
Drew	game ride	7 12	\$81

17. **RINGTONES** Ella's cell phone provider sells standard and premium ringtones. One month, Ella bought 2 standard and 2 premium ringtones for \$8.96. The next month Ella paid \$9.46 for 1 standard and 3 premium ringtones. What are the prices for standard and premium ringtones?



### Real-World Link

Video games are becoming increasingly popular among adults. In fact, more than 5% of adults play video games 2 or more times per week.

Source: U.S. Census Bureau



Use Cramer's Rule to solve each system of equations.

18.  $x + y + z = 6$

$$2x + y - 4z = -15$$

$$5x - 3y + z = -10$$

20.  $r - 2s - 5t = -1$

$$r + 2s - 2t = 5$$

$$4r + s + t = -1$$

22.  $4x + 2y - 3z = -32$

$$-x - 3y + z = 54$$

$$2y + 8z = 78$$

24.  $0.5r - s = -1$

$$0.75r + 0.5s = -0.25$$

26.  $\frac{1}{3}r + \frac{2}{5}s = 5$

$$\frac{2}{3}r - \frac{1}{2}s = -3$$

19.  $a - 2b + c = 7$

$$6a + 2b - 2c = 4$$

$$4a + 6b + 4c = 14$$

21.  $3a + c = 23$

$$4a + 7b - 2c = -22$$

$$8a - b - c = 34$$

23.  $2r + 25s = 40$

$$10r + 12s + 6t = -2$$

$$36r - 25s + 50t = -10$$

25.  $1.5m - 0.7n = 0.5$

$$2.2m - 0.6n = -7.4$$

27.  $\frac{3}{4}x + \frac{1}{2}y = \frac{11}{12}$

$$\frac{1}{2}x - \frac{1}{4}y = \frac{1}{8}$$

28. **ARCADE GAMES** Marcus and Cody purchased game cards to play virtual games at the arcade. Marcus used 47 points from his game card to drive the race car simulator and the snowboard simulator four times each. Cody used 48.25 points from his game card to drive the race car five times and the snowboard three times. How many points does each game charge per play?

29. **PRICING** The Harvest Nut Company sells made-to-order trail mixes. Sam's favorite mix contains peanuts, raisins, and carob-coated pretzels. Peanuts sell for \$3.20 per pound, raisins are \$2.40 per pound, and the carob-coated pretzels are \$4.00 per pound. Sam bought a 5-pound mixture for \$16.80 that contained twice as many pounds of carob-coated pretzels as raisins. How many pounds of peanuts, raisins, and carob-coated pretzels did Sam buy?

**EXTRA PRACTICE**  
See pages 898, 929.

**Math online**  
Self-Check Quiz at  
[algebra2.com](http://algebra2.com)

### H.O.T. Problems

30. **OPEN ENDED** Write a system of equations that *cannot* be solved using Cramer's Rule.

31. **REASONING** Write a system of equations whose solution is

$$x = \frac{\begin{vmatrix} -6 & 5 \\ 30 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}, y = \frac{\begin{vmatrix} 3 & -6 \\ 4 & 30 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}$$

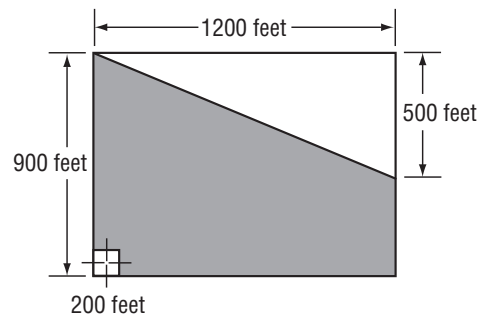
32. **CHALLENGE** In Cramer's Rule, if the value of the determinant is zero, what must be true of the graph of the system of equations represented by the determinant? Give examples to support your answer.

33. **Writing in Math** Use the information about two sides of the triangle on page 201 to explain how Cramer's Rule can be used to solve systems of equations. Include an explanation of how Cramer's rule uses determinants, and a situation where Cramer's rule would be easier to use to solve a system of equations than substitution or elimination.

- 34. ACT/SAT** Each year at Capital High School the students vote to choose the theme of that year's homecoming dance. The theme "A Night Under the Stars" received 225 votes, and "The Time of My Life" received 480 votes. If 40% of girls voted for "A Night Under the Stars", 75% of boys voted for "The Time of My Life", and all of the students voted, how many girls and boys are there at Capital High School?

- A 854 boys and 176 girls  
B 705 boys and 325 girls  
C 395 boys and 310 girls  
D 380 boys and 325 girls

- 35. REVIEW** What is the area of the shaded part of the rectangle below?



- F 440,000 ft<sup>2</sup>      H 640,000 ft<sup>2</sup>  
G 540,000 ft<sup>2</sup>      J 740,000 ft<sup>2</sup>

### Spiral Review

Find the value of each determinant. (Lesson 4-5)

36.  $\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix}$

37.  $\begin{vmatrix} 8 & 6 \\ 4 & 8 \end{vmatrix}$

38.  $\begin{vmatrix} -5 & 2 \\ 4 & 9 \end{vmatrix}$

For Exercises 39 and 40, use the following information. (Lesson 4-4)

Triangle  $ABC$  with vertices  $A(0, 2)$ ,  $B(-3, -1)$ , and  $C(-2, -4)$  is translated 1 unit right and 3 units up.

39. Write the translation matrix.  
40. Find the coordinates of  $\triangle A'B'C'$ . Then graph the preimage and the image.

Solve each system of equations by graphing. (Lesson 3-1)

41.  $y = 3x + 5$   
 $y = -2x - 5$

42.  $x + y = 7$   
 $\frac{1}{2}x - y = -1$

43.  $x - 2y = 10$   
 $2x - 4y = 12$

44. **BUSINESS** The Friendly Fix-It Company charges a base fee of \$45 for any in-home repair. In addition, the technician charges \$30 per hour. Write an equation for the cost  $c$  of an in-home repair of  $h$  hours. (Lesson 1-3)

### GET READY for the Next Lesson

**PREREQUISITE SKILL** Find each product, if possible. (Lesson 4-3)

45.  $\begin{bmatrix} 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$

46.  $\begin{bmatrix} 0 & 9 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -6 \\ 8 & 1 \end{bmatrix}$

47.  $\begin{bmatrix} 5 & -4 \\ 8 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

48.  $\begin{bmatrix} 7 & 11 & -5 \\ 3 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}$

**Main Ideas**

- Determine whether two matrices are inverses.
- Find the inverse of a  $2 \times 2$  matrix.

**New Vocabulary**

identity matrix

inverse

**GET READY for the Lesson**

With the rise of Internet shopping, ensuring the privacy of the user's personal information has become an important priority. Companies protect their computers by using codes. Cryptography is a method of preparing coded messages that can only be deciphered by using a "key."

The following technique is a simplified version of how cryptography works.

- First, assign a number to each letter of the alphabet.
- Convert your message into a matrix and multiply it by the coding matrix. The message is now unreadable to anyone who does not have the key to the code.
- To decode the message, the recipient of the coded message must multiply by the inverse of the coding matrix.



Code									
–	0	A	1	B	2	C	3	D	4
		E	5	F	6	G	7	H	8
I	9	J	10	K	11	L	12	M	13
		N	14	O	15	P	16	Q	17
R	18	S	19	T	20	U	21	V	22
		W	23	X	24	Y	25	Z	26

**Identity and Inverse Matrices** Recall that for real numbers, the multiplicative identity is 1. For matrices, the **identity matrix** is a square matrix that, when multiplied by another matrix, equals that same matrix.

 **$2 \times 2$  Identity Matrix**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

 **$3 \times 3$  Identity Matrix**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**KEY CONCEPT****Identity Matrix for Multiplication**

**Word** The identity matrix for multiplication  $I$  is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix  $A$  of the same dimension as  $I$ ,  $A \cdot I = I \cdot A = A$ .

**Symbols** If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$



Two  $n \times n$  matrices are **inverses** of each other if their product is the identity matrix. If matrix  $A$  has an inverse symbolized by  $A^{-1}$ , then  $A \cdot A^{-1} = A^{-1} \cdot A = I$ .

### EXAMPLE Verify Inverse Matrices

**I** Determine whether each pair of matrices are inverses of each other.

a.  $X = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$  and  $Y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

If  $X$  and  $Y$  are inverses, then  $X \cdot Y = Y \cdot X = I$ .

$$X \cdot Y = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 1 - 2 & 1 + \frac{1}{2} \\ -\frac{1}{2} + (-4) & -\frac{1}{2} + 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 1\frac{1}{2} \\ -4\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{Matrix multiplication}$$

Since  $X \cdot Y \neq I$ , they are *not* inverses.

b.  $P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$  and  $Q = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

If  $P$  and  $Q$  are inverses, then  $P \cdot Q = Q \cdot P = I$ .

$$P \cdot Q = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 3 - 2 & -6 + 6 \\ 1 - 1 & -2 + 3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Matrix multiplication}$$

$$Q \cdot P = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 3 - 2 & 4 - 4 \\ -\frac{3}{2} + \frac{3}{2} & -2 + 3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Matrix multiplication}$$

Since  $P \cdot Q = Q \cdot P = I$ ,  $P$  and  $Q$  are inverses.

### Study Tip

#### Verifying Inverses

Since multiplication of matrices is not commutative, it is necessary to check the product in both orders.

### CHECK Your Progress

1.  $X = \begin{bmatrix} 4 & -1 \\ 2 & -2 \end{bmatrix}$  and  $Y = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$





**Find Inverse Matrices** Some matrices do not have an inverse. You can determine whether a matrix has an inverse by using the determinant.

### KEY CONCEPT

### Inverse of a $2 \times 2$ Matrix

The inverse of matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is  $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ , where  $ad - bc \neq 0$ .

Notice that  $ad - bc$  is the value of  $\det A$ . Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

### EXAMPLE

### Find the Inverse of a Matrix

**2** Find the inverse of each matrix, if it exists.

a.  $R = \begin{bmatrix} -4 & -3 \\ 8 & 6 \end{bmatrix}$

First find the determinant to see if the matrix has an inverse.

$$\begin{vmatrix} -4 & -3 \\ 8 & 6 \end{vmatrix} = -24 - (-24) = 0$$

Since the determinant equals 0,  $R^{-1}$  does not exist.

b.  $P = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Find the determinant.

$$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 \text{ or } 1$$

Since the determinant does not equal 0,  $P^{-1}$  exists.

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Definition of inverse}$$

$$= \frac{1}{3(2) - 1(5)} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad a = 3, b = 1, c = 5, d = 2$$

$$= 1 \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad \text{Simplify.}$$

$$= \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad \text{Simplify.}$$

**CHECK** Find the product of the matrices. If the product is  $I$ , then they are inverses.

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 - 5 & 2 - 2 \\ -15 + 15 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

### CHECK Your Progress

2A.  $\begin{bmatrix} -3 & 7 \\ 1 & -4 \end{bmatrix}$

2B.  $\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$



Personal Tutor at [algebra2.com](http://algebra2.com)



### Real-World Link

The Enigma was a German coding machine used in World War II. Its code was considered to be unbreakable. However, the code was eventually solved by a group of Polish mathematicians.

Source: [bletchleypark.org.uk](http://bletchleypark.org.uk)

### Study Tip

#### Messages

If there is an odd number of letters to be coded, add a 0 at the end of the message.



Matrices can be used to code messages by placing the message in a  $n \times 2$  matrix.



### Real-World EXAMPLE

3

- a. **CRYPTOGRAPHY** Use the table at the beginning of the lesson to assign a number to each letter in the message GO\_TONIGHT.

Then code the message with the matrix  $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ .

Convert the message to numbers using the table.

G O \_ T O N I G H T

7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

Write the message in matrix form. Arrange the numbers in a matrix with 2 columns and as many rows as are needed. Then multiply the message matrix  $B$  by the coding matrix  $A$ .

$$BA = \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 14 + 60 & 7 + 45 \\ 0 + 80 & 0 + 60 \\ 30 + 56 & 15 + 42 \\ 18 + 28 & 9 + 21 \\ 16 + 80 & 8 + 60 \end{bmatrix} \quad \text{Multiply the matrices.}$$

$$= \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \quad \text{Write an equation.}$$

The coded message is 74 | 52 | 80 | 60 | 86 | 57 | 46 | 30 | 96 | 68.

- b. Use the inverse matrix  $A^{-1}$  to decode the message in Example 3a.

First find the inverse matrix of  $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ .

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Definition of inverse}$$

$$= \frac{1}{2(3) - (1)(4)} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \quad a = 2, b = 1, c = 4, d = 3$$

$$= \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \quad \text{Simplify.}$$

$$= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} \quad \text{Simplify.}$$

(continued on the next page)



Next, decode the message by multiplying the coded matrix  $C$  by  $A^{-1}$ .

$$CA^{-1} = \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 111 - 104 & -37 + 52 \\ 120 - 120 & -40 + 60 \\ 129 - 114 & -43 + 57 \\ 69 - 60 & -23 + 30 \\ 144 - 136 & -48 + 68 \end{bmatrix} \quad \text{Multiply the matrices.}$$

$$= \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix} \quad \text{Simplify.}$$

Use the table again to convert the numbers to letters. You can now read the message.

7|15|0|20|15|14|9|7|8|20

G O \_ T O N I G H T



### CHECK Your Progress

3. Use the table at the beginning of the lesson to assign a number to each letter in the message SECRET\_CODE. Then code the message with the matrix  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ . Use the inverse matrix  $A^{-1}$  to decode the message.



### CHECK Your Understanding

**Example 1**  
(p. 209)

Determine whether each pair of matrices are inverses of each other.

1.  $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$

2.  $X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, Y = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

3.  $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

4.  $F = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, G = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

**Example 2**  
(p. 210)

Find the inverse of each matrix, if it exists.

5.  $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

6.  $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$

7.  $\begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$

**Example 3**  
(pp. 211–212)

8. **CRYPTOGRAPHY** Code a message using your own coding matrix. Give your message and the matrix to a friend to decode. (*Hint:* Use a coding matrix whose determinant is 1 and that has all positive elements.)



# Exercises

HOMEWORK	HELP
For Exercises	See Examples
9–12	1
13–21	2
22–24	3

Determine whether each pair of matrices are inverses of each other.

9.  $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

10.  $R = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, S = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$

11.  $A = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -\frac{5}{2} & -3 \end{bmatrix}$

12.  $X = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, Y = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Find the inverse of each matrix, if it exists.

13.  $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

14.  $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

15.  $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$

16.  $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$

17.  $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$

18.  $\begin{bmatrix} -3 & 7 \\ 2 & -6 \end{bmatrix}$

19.  $\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}$

20.  $\begin{bmatrix} -2 & 0 \\ 5 & 6 \end{bmatrix}$

21.  $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

**CRYPTOGRAPHY** For Exercises 22–24, use the alphabet table at the right.

Your friend sent you messages that were coded with the coding matrix  $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ . Use the inverse of matrix  $C$  to decode each message.

22. 50 | 36 | 51 | 29 | 18 | 18 | 26 | 13 | 33 | 26 | 44 | 22 | 48 | 33 | 59 | 34 | 61 | 35 | 4 | 2

23. 59 | 33 | 8 | 8 | 39 | 21 | 7 | 7 | 56 | 37 | 25 | 16 | 4 | 2

24. 59 | 34 | 49 | 31 | 40 | 20 | 16 | 14 | 21 | 15 | 25 | 25 | 36 | 24 | 32 | 16

CODE		
A 26	J 17	S 8
B 25	K 16	T 7
C 24	L 15	U 6
D 23	M 14	V 5
E 22	N 13	W 4
F 21	O 12	X 3
G 20	P 11	Y 2
H 19	Q 10	Z 1
I 18	R 9	_ 0

25. **RESEARCH** Use the Internet or other reference to find examples of codes used throughout history. Explain how messages were coded.

Determine whether each statement is *true* or *false*.

26. Only square matrices have multiplicative identities.

27. Only square matrices have multiplicative inverses.

28. Some square matrices do not have multiplicative inverses.

29. Some square matrices do not have multiplicative identities.

Determine whether each pair of matrices are inverses of each other.

30.  $C = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} \frac{2}{7} & \frac{5}{7} \\ \frac{1}{7} & -\frac{1}{7} \end{bmatrix}$

31.  $J = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, K = \begin{bmatrix} -\frac{5}{4} & \frac{1}{4} & \frac{7}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

**EXTRA PRACTICE**  
See pages 899, 929.  
**Math online**  
Self-Check Quiz at [algebra2.com](http://algebra2.com)

Find the inverse of each matrix, if it exists.

32.  $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$

33.  $\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}$

34.  $\begin{bmatrix} \frac{3}{10} & \frac{5}{8} \\ \frac{1}{5} & \frac{3}{4} \end{bmatrix}$

35. **GEOMETRY** Compare the matrix used to reflect a figure over the  $x$ -axis to the matrix used to reflect a figure over the  $y$ -axis.
- Are they inverses?
  - Does your answer make sense based on the geometry? Use a drawing to support your answer.
36. **GEOMETRY** The matrix used to rotate a figure  $270^\circ$  counterclockwise about the origin is  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . Compare this matrix with the matrix used to rotate a figure  $90^\circ$  counterclockwise about the origin.
- Are they inverses?
  - Does your answer make sense? Use a drawing to support your answer.

**GEOMETRY** For Exercises 37–41, use the figure at the right.

37. Write the vertex matrix  $A$  for the rectangle.

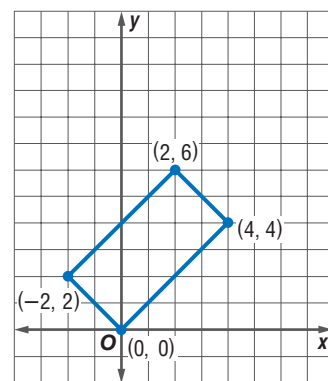
38. Use matrix multiplication to find  $BA$  if

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

39. Graph the vertices of the transformed rectangle. Describe the transformation.

40. Make a conjecture about what transformation  $B^{-1}$  describes on a coordinate plane.

41. Find  $B^{-1}$  and multiply it by  $BA$ . Make a drawing to verify your conjecture.



**Graphing Calculator**

**INVERSE FUNCTION** The  $[X^{-1}]$  key on a TI-83/84 Plus graphing calculator is used to find the inverse of a matrix. If you get a **SINGULAR MATRIX** error on the screen, then the matrix has no inverse. Find the inverse of each matrix.

42.  $\begin{bmatrix} -11 & 9 \\ 6 & -5 \end{bmatrix}$

43.  $\begin{bmatrix} 12 & 4 \\ 15 & 5 \end{bmatrix}$

44.  $\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 5 & 2 \end{bmatrix}$



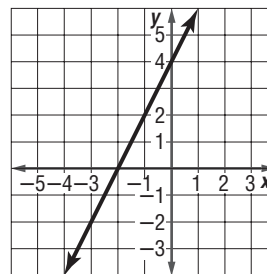
**H.O.T. Problems**

45. **REASONING** Explain how to find the inverse of a  $2 \times 2$  matrix.
46. **OPEN ENDED** Create a square matrix that does not have an inverse. Explain how you know it has no inverse.
47. **CHALLENGE** For which values of  $a$ ,  $b$ ,  $c$ , and  $d$  will  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1}$ ?
48. **Writing in Math** Use the information about cryptography on page 208 to explain how inverse matrices are used in cryptography. Explain why the inverse matrix works in decoding a message, and describe the conditions you must consider when writing a message in matrix form.

49. **ACT/SAT** The message MEET\_ME\_TOMORROW is converted into numbers (0 = space, A = 1, B = 2, etc.) and encoded using a numeric key. After the message is encoded it becomes 31 | -11 | 30 | 50 | 13 | 39 | 10 | -10 | 55 | 5 | 41 | 19 | 54 | 18 | 53 | 39. Which key was used to encode this message?

A  $\begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$  C  $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$   
 B  $\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$  D  $\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$

50. **REVIEW** Line  $q$  is shown below. Which equation best represents a line parallel to line  $q$ ?



F  $y = x + 2$  H  $y = 2x - 3$   
 G  $y = x + 5$  J  $y = -2x + 2$

## Spiral Review

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

51.  $3x + 2y = -2$  52.  $2x + 5y = 35$  53.  $4x - 3z = -23$   
 $x - 3y = 14$   $7x - 4y = -28$   $-2x - 5y + z = -9$   
 $y - z = 3$

Evaluate each determinant. (Lesson 4-5)

54.  $\begin{vmatrix} 2 & 8 & -6 \\ 4 & 5 & 2 \\ -3 & -6 & -1 \end{vmatrix}$  55.  $\begin{vmatrix} -3 & -3 & 1 \\ -9 & -2 & 3 \\ 5 & -2 & -1 \end{vmatrix}$  56.  $\begin{vmatrix} 5 & -7 & 3 \\ -1 & -2 & -9 \\ 5 & -7 & 3 \end{vmatrix}$

Find each product, if possible. (Lesson 4-3)

57.  $\begin{bmatrix} 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}$  58.  $\begin{bmatrix} 7 & 4 \\ -1 & 2 \\ -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \end{bmatrix}$  59.  $\begin{bmatrix} 4 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 5 & 6 \end{bmatrix}$

Solve each system of equations. (Lesson 3-2)

60.  $3x + 5y = 2$  61.  $6x + 2y = 22$  62.  $3x - 2y = -2$   
 $2x - y = -3$   $3x + 7y = 41$   $4x + 7y = 65$

Find the slope of the line that passes through each pair of points. (Lesson 2-3)

63. (2, 5), (6, 9) 64. (1, 0), (-2, 9) 65. (-5, 4), (-3, -6)  
 66. (-2, 2), (-5, 1) 67. (0, 3), (-2, -2) 68. (-8, 9), (0, 6)

69. **OCEANOGRAPHY** The bottom of the Mariana Trench in the Pacific Ocean is 6.8 miles below sea level. Water pressure in the ocean is represented by the function  $f(x) = 1.15x$ , where  $x$  is the depth in miles and  $f(x)$  is the pressure in tons per square inch. Find the pressure in the Mariana Trench. (Lesson 2-1)

## GET READY for the Next Lesson

Solve each equation. (Lesson 1-3)

70.  $3k + 8 = 5$  71.  $12 = -5h + 2$  72.  $7z - 4 = 5z + 8$   
 73.  $\frac{x}{2} + 5 = 7$  74.  $\frac{3+n}{6} = -4$  75.  $6 = \frac{s-8}{-7}$



# Using Matrices to Solve Systems of Equations

## Main Ideas

- Write matrix equations for systems of equations.
- Solve systems of equations using matrix equations.

## New Vocabulary

matrix equation

## GET READY for the Lesson

An ecologist is studying two species of birds that compete for food and territory. He estimates that a particular region with an area of 14.25 acres (approximately 69,000 square yards) can supply 20,000 pounds of food for the birds.

Species A needs 140 pounds of food and has a territory of 500 square yards per nesting pair. Species B needs 120 pounds of food and has a territory of 400 square yards per nesting pair. The biologist can use this information to find the number of birds of each species that the area can support.



**Write Matrix Equations** The situation above can be represented using a system of equations that can be solved using matrices. Let's examine a similar situation. Consider the system of equations below. You can write this system with matrices by using the left and right sides of the equations.

$$\begin{aligned} 5x + 7y &= 11 \\ 3x + 8y &= 18 \end{aligned} \rightarrow \begin{bmatrix} 5x + 7y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix}$$

Write the matrix on the left as the product of the coefficient matrix and the variable matrix.

$$\begin{array}{ccc} A & \cdot & X = B \\ \begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix} & \cdot & \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix} \\ \text{coefficient} & & \text{variable} & & \text{constant} \\ \text{matrix} & & \text{matrix} & & \text{matrix} \end{array}$$

The system of equations is now expressed as a **matrix equation**.

## EXAMPLE Two-Variable Matrix Equation

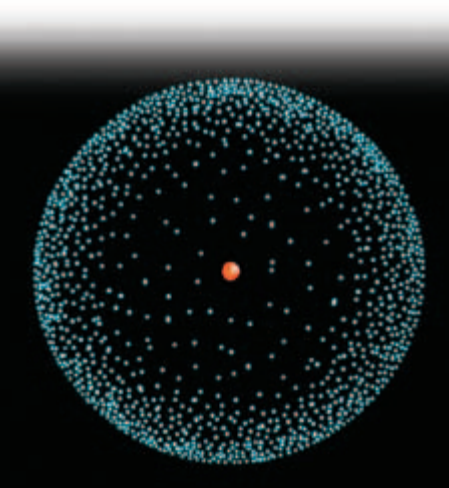
- 1 Write a matrix equation for the system of equations.

$$5x - 6y = -47$$

$$3x + 2y = -17$$

Determine the coefficient, variable, and constant matrices.

$$\begin{aligned} 5x - 6y &= -47 \\ 3x + 2y &= -17 \end{aligned} \rightarrow \begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$



### Real-World Link

Atomic mass units (amu) are relative units of weight because they were compared to the weight of a hydrogen atom. So a molecule of nitrogen, whose weight is 14.0 amu, weighs 14 times as much as a hydrogen atom.

Source: [www.sizes.com](http://www.sizes.com)

Write the matrix equation.

$$\begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$



### CHECK Your Progress

1.  $2x + 4y = 7$   
 $3x - y = 6$



### Real-World EXAMPLE



**CHEMISTRY** The molecular formula for glucose is  $C_6H_{12}O_6$ , which represents that a molecule of glucose has 6 carbon (C) atoms, 12 hydrogen (H) atoms, and 6 oxygen (O) atoms. One molecule of glucose weighs 180 atomic mass units (amu), and one oxygen atom weighs 16 amu. The formulas and weights for glucose and sucrose are listed below.

Sugar	Formula	Atomic Weight (amu)
glucose	$C_6H_{12}O_6$	180
sucrose	$C_{12}H_{22}O_{11}$	342

- a. Write a system of equations that represents the weight of each atom.

Let  $c$  represent the weight of a carbon atom.

Let  $h$  represent the weight of a hydrogen atom.

Glucose:  $6c + 12h + 6(16) = 180$  Equation for glucose  
 $6c + 12h + 96 = 180$  Simplify.  
 $6c + 12h = 84$  Subtract 96 from each side.

Sucrose:  $12c + 22h + 11(16) = 342$  Equation for sucrose  
 $12c + 22h + 176 = 342$  Simplify.  
 $12c + 22h = 166$  Subtract 176 from each side.

- b. Write a matrix equation for the system of equations.

Determine the coefficient, variable, and constant matrices. Then write the matrix equation.

$$\begin{array}{l} 6c + 12h = 84 \\ 12c + 22h = 166 \end{array} \rightarrow \begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c \\ h \end{bmatrix} = \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$

$$\begin{array}{l} A \cdot X = B \\ \begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c \\ h \end{bmatrix} = \begin{bmatrix} 84 \\ 166 \end{bmatrix} \end{array} \quad \text{You will solve this matrix equation in Exercise 3.}$$



### CHECK Your Progress

2. The formula for propane is  $C_3H_8$ , and its atomic weight is 44 amu. Butane is  $C_4H_{10}$ , and its atomic weight is 58 amu. Write a system of equations for the weight of each. Then write a matrix equation for the system of equations.



Extra Examples at [algebra2.com](http://algebra2.com)



## Study Tip

### Solving Using Inverses

Notice that  $A^{-1}$  is on the left on both sides of the equation. It is important to multiply both sides of the matrix equation with the inverse in the same order since matrix multiplication is not commutative.

**Solve Systems of Equations** A matrix equation in the form  $AX = B$ , where  $A$  is a coefficient matrix,  $X$  is a variable matrix, and  $B$  is a constant matrix, can be solved in a similar manner as a linear equation of the form  $ax = b$ .

$$ax = b$$

Write the equation.

$$AX = B$$

$$\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b$$

Multiply each side by the inverse of the coefficient, if it exists.

$$A^{-1}AX = A^{-1}B$$

$$1x = \left(\frac{1}{a}\right)b$$

$$\left(\frac{1}{a}\right)a = 1, A^{-1}A = I$$

$$IX = A^{-1}B$$

$$x = \left(\frac{1}{a}\right)b$$

$$1x = x, IX = X$$

$$X = A^{-1}B$$

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.

## EXAMPLE

### Solve Systems of Equations



Use a matrix equation to solve each system of equations.

a.  $6x + 2y = 11$

$$3x - 8y = 1$$

The matrix equation is  $\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$ , when  $A = \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix}$ ,

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} 11 \\ 1 \end{bmatrix}.$$

**Step 1** Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-48 - 6} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \text{ or } -\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix}$$

**Step 2** Multiply each side of the matrix equation by the inverse matrix.

$$-\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 1 \end{bmatrix} \quad \text{Multiply each side by } A^{-1}.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54} \begin{bmatrix} -90 \\ -27 \end{bmatrix} \quad \text{Multiply matrices.}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The solution is  $\left(\frac{5}{3}, \frac{1}{2}\right)$ . Check this solution in the original equation.

b.  $6a - 9b = -18$

$$8a - 12b = 24$$

The matrix equation is  $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$ , when  $A = \begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$ ,

$$X = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ and } B = \begin{bmatrix} -18 \\ 24 \end{bmatrix}.$$

## Study Tip

### Identity Matrix

The identity matrix on the left verifies that the inverse matrix has been calculated correctly.

## Review Vocabulary

**Inconsistent System of Equations:** a system of equations that does not have a solution (Lesson 3-1)

Find the inverse of the coefficient matrix.

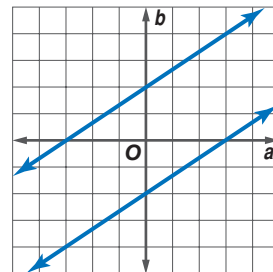
$$A^{-1} = \frac{1}{-72 + 72} \begin{bmatrix} -12 & 9 \\ -8 & 6 \end{bmatrix}$$

The determinant of the coefficient matrix

$$\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix} \text{ is } 0, \text{ so } A^{-1} \text{ does not exist.}$$

There is no unique solution of this system.

Graph the system of equations. Since the lines are parallel, this system has no solution. Therefore, the system is inconsistent.



## CHECK Your Progress

**3A.**  $-2x + 3y = -7$

$$4x - 8y = 16$$

**3B.**  $2x - 4y = -24$

$$3x - 6y = -12$$

Online Personal Tutor at [algebra2.com](http://algebra2.com)



To solve a system of equations with three variables, you can use the  $3 \times 3$  identity matrix. However, finding the inverse of a  $3 \times 3$  matrix may be tedious. Graphing calculators and computers offer fast and accurate calculations.

## GRAPHING CALCULATOR LAB

### Systems of Three Equations in Three Variables

You can use a graphing calculator and a matrix equation to solve systems of equations. Consider the system of equations below.

$$3x - 2y + z = 0$$

$$2x + 3y - z = 17$$

$$5x - y + 4z = -7$$

### THINK AND DISCUSS

1. Write a matrix equation for the system of equations.
2. Enter the coefficient matrix as matrix  $A$  and the constant matrix as matrix  $B$ . Find the product of  $A^{-1}$  and  $B$ . Recall that the  $\boxed{x^{-1}}$  key is used to find  $A^{-1}$ .
3. How is the result related to the solution?

## CHECK Your Understanding

**Example 1**  
(pp. 216–217)

Write a matrix equation for each system of equations.

**1.**  $x - y = -3$

$$x + 3y = 5$$

**2.**  $2g + 3h = 8$

$$-4g - 7h = -5$$

**Example 2**  
(p. 217)

- 3. CHEMISTRY** Refer to Example 2 on page 217. Solve the system of equations to find the weight of a carbon, hydrogen, and oxygen atom.

**Example 3**  
(pp. 218–219)

Use a matrix equation to solve each system of equations.

4.  $5x - 3y = -30$   
 $8x + 5y = 1$

6.  $3x + 6y = 11$   
 $2x + 4y = 7$

5.  $5s + 4t = 12$   
 $4s - 3t = -1.25$

7.  $3x + 4y = 3$   
 $6x + 8y = 5$

## Exercises

HOMEWORK	HELP
For Exercises	See Examples
8–11	1
12, 13	2
14–23	3, 4

Write a matrix equation for each system of equations.

8.  $3x - y = 0$   
 $x + 2y = -21$

10.  $5a - 6b = -47$   
 $3a + 2b = -17$

9.  $4x - 7y = 2$   
 $3x + 5y = 9$

11.  $3m - 7n = -43$   
 $6m + 5n = -10$

12. **MONEY** Mykia had 25 quarters and dimes. The total value of all the coins was \$4.00. How many quarters and dimes did Mykia have?

13. **PILOT TRAINING** Flight instruction costs \$105 per hour, and the simulator costs \$45 per hour. Hai-Ling spent 4 more hours in airplane training than in the simulator. If Hai-Ling spent \$3870, how much time did he spend training in an airplane and in a simulator?

Use a matrix equation to solve each system of equations.

14.  $p - 2q = 1$   
 $p + 5q = 22$

16.  $-2x + 4y = 3$   
 $2x - 4y = 5$

18.  $5a + 9b = -28$   
 $2a - b = -2$

20.  $4m - 7n = -63$   
 $3m + 2n = 18$

22.  $x + 2y = 8$   
 $3x + 2y = 6$

15.  $3x - 9y = 12$   
 $-2x + 6y = 9$

17.  $6r + s = 9$   
 $3r = -2s$

19.  $6x - 10y = 7$   
 $3x - 5y = 8$

21.  $8x - 3y = 19.5$   
 $2.5x + 7y = 18$

23.  $4x - 3y = 5$   
 $2x + 9y = 6$

24. **NUMBER THEORY** Find two numbers whose sum is 75 and the second number is 15 less than twice the first.

25. **CHEMISTRY** Refer to Check Your Progress 2 on page 217. Solve the system of equations to find the weights of a carbon and a hydrogen atom.

26. **SPORTING GOODS** Use three rows from the table of sporting goods sales and write a matrix. Then use the matrix to find the cost of each type of ball.

Day	Baseballs	Basketballs	Footballs	Sales (\$)
Monday	10	3	6	97
Tuesday	13	1	4	83
Wednesday	8	5	2	79
Thursday	15	2	7	116
Friday	9	0	8	84

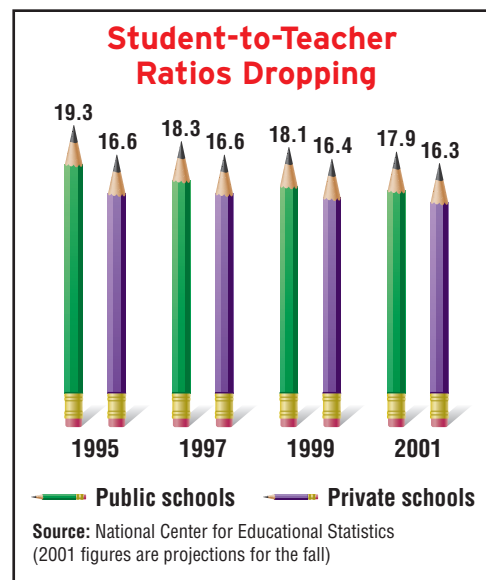
**EXTRA PRACTICE**  
See pages 899, 929.

**Math online**  
Self-Check Quiz at [algebra2.com](http://algebra2.com)



- 27. SCHOOLS** The graphic shows that student-to-teacher ratios are dropping in both public and private schools. If these rates of change remain constant, predict when the student-to-teacher ratios for private and public schools will be the same.

- 28. CHEMISTRY** Cara is preparing an acid solution. She needs 200 milliliters of 48% concentration solution. Cara has 60% and 40% concentration solutions in her lab. How many milliliters of 40% acid solution should be mixed with 60% acid solution to make the required amount of 48% acid solution?



### Graphing Calculator

Use a graphing calculator to solve each system of equations using inverse matrices.

**29.**  $2a - b + 4c = 6$   
 $a + 5b - 2c = -6$   
 $3a - 2b + 6c = 8$

**30.**  $3x - 5y + 2z = 22$   
 $2x + 3y - z = -9$   
 $4x + 3y + 3z = 1$

**31.**  $2q + r + s = 2$   
 $-q - r + 2s = 7$   
 $-3q + 2r + 3s = 7$

### H.O.T. Problems

- 32. REASONING** Write the matrix equation  $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$  as a system of linear equations.

- 33. OPEN ENDED** Write a system of equations that does not have a unique solution.

- 34. FIND THE ERROR** Tommy and Laura are solving a system of equations.

They find that  $A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$ ,  $B = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$ , and  $X = \begin{bmatrix} x \\ y \end{bmatrix}$ . Who is correct?

Explain your reasoning.

*Tommy*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

*Laura*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 31 \end{bmatrix}$$

- 35. CHALLENGE** What can you conclude about the solution set of a system of equations if the coefficient matrix does not have an inverse?

- 36. Writing in Math** Use the information about ecology found on page 216 to explain how matrices can be used to find the number of species of birds that an area can support. Demonstrate a system of equations that can be used to find the number of each species the region can support, and a solution of the problem using matrices.



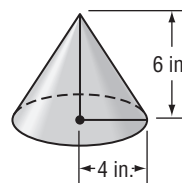
37. **ACT/SAT** The Yogurt Shoppe sells cones in three sizes: small, \$0.89; medium, \$1.19; and large, \$1.39. One day Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold \$58.98 in cones, how many medium cones did he sell?

A 11                      C 24  
B 17                      D 36

38. **ACT/SAT** What is the solution to the system of equations  $6a + 8b = 5$  and  $10a - 12b = 2$ ?

F  $\left(\frac{3}{4}, \frac{1}{2}\right)$                       H  $\left(\frac{1}{2}, \frac{3}{4}\right)$   
G  $\left(\frac{1}{2}, -\frac{1}{2}\right)$                       J  $\left(\frac{1}{2}, \frac{1}{4}\right)$

39. **REVIEW** A right circular cone has radius 4 inches and height 6 inches.



What is the lateral area of the cone?  
(Lateral area of cone =  $\pi r \ell$ , where  $\ell$  = slant height)?

A  $24\pi$  sq in.  
B  $2\sqrt{13}\pi$  sq in.  
C  $2\sqrt{52}\pi$  sq in.  
D  $8\sqrt{13}\pi$  sq in.

## Spiral Review

Find the inverse of each matrix, if it exists. (Lesson 4-7)

40.  $\begin{bmatrix} 4 & 4 \\ 2 & 3 \end{bmatrix}$

41.  $\begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$

42.  $\begin{bmatrix} -3 & -6 \\ 5 & 10 \end{bmatrix}$

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

43.  $6x + 7y = 10$   
 $3x - 4y = 20$

44.  $6a + 7b = -10.15$   
 $9.2a - 6b = 69.944$

45.  $\frac{x}{2} - \frac{2y}{3} = 2\frac{1}{3}$   
 $3x + 4y = -50$

46. **ECOLOGY** If you recycle a  $3\frac{1}{2}$ -foot stack of newspapers, one less 20-foot loblolly pine tree will be needed for paper. Use a prediction equation to determine how many feet of loblolly pine trees will *not* be needed for paper if you recycle a pile of newspapers 20 feet tall. (Lesson 2-5)

## Cross-Curricular Project

### Algebra and Consumer Science

**What Does it Take to Buy a House?** It is time to complete your project. Use the information and data you have gathered about home buying and selling to prepare a portfolio or Web page. Be sure to include your tables, graphs, and calculations in the presentation. You may also wish to include additional data, information, or pictures.



Cross-Curricular Project at [algebra2.com](http://algebra2.com)

# Graphing Calculator Lab

## Augmented Matrices

Using a TI-83/84 Plus, you can solve a system of linear equations using the **MATRIX** function. An **augmented matrix** contains the coefficient matrix with an extra column containing the constant terms. The reduced row echelon function of a graphing calculator reduces the augmented matrix so that the solution of the system of equations can be easily determined.

### ACTIVITY

Write an augmented matrix for the system of equations. Then solve the system by using the reduced row echelon form on the graphing calculator.

$$3x + y + 3z = 2$$

$$2x + y + 2z = 1$$

$$4x + 2y + 5z = 5$$

**Step 1** Write the augmented matrix and enter it into a calculator.

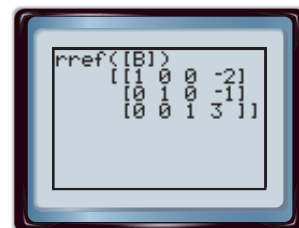
$$\text{The augmented matrix } B = \left[ \begin{array}{ccc|c} 3 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 2 & 5 & 5 \end{array} \right]$$

**KEYSTROKES:** Review matrices on page 172.

**Step 2** Find the reduced row echelon form (rref) using the graphing calculator.

**KEYSTROKES:** **2nd** **[MATRIX]** **►** **ALPHA** **[B]** **2nd** **[MATRIX]** **2** **)** **ENTER**

Study the reduced echelon matrix. The first three columns are the same as a  $3 \times 3$  identity matrix. The first row represents  $x = -2$ , the second row represents  $y = -1$ , and the third row represents  $z = 3$ . The solution is  $(-2, -1, 3)$ .



### EXERCISES

Write an augmented matrix for each system of equations. Then solve with a graphing calculator. Round to the nearest hundredth.

1.  $x - 3y = 5$

$$2x + y = 1$$

4.  $-x + 3y = 10$

$$4x + 2y = 16$$

7.  $3x - y = 0$

$$2x - 3y = 1$$

2.  $15x + 11y = 36$

$$4x - 3y = -26$$

5.  $8x - 7y = 45.1$

$$2x + 5y = -8.3$$

8.  $3x - 2y + z = -2$

$$x - y + 3z = 5$$

$$-x + y + z = -1$$

3.  $2x - y = 5$

$$2x - 3y = 1$$

6.  $0.5x + 0.7y = 5.5$

$$3x - 2.5y = -0.5$$

9.  $x - y + z = 2$

$$x - z = 1$$

$$y + 2z = 0$$





## FOLDABLES

Study Organizer

## GET READY to Study

Be sure the following  
Key Concepts are noted  
in your Foldable.

○	A-1 Introduction
○	A-2 Operations
○	A-3 Multiplying
○	A-4 Transformations
○	A-5 Determinants
○	A-6 Cramer's Rule
○	A-7 Identity
○	A-8 Using Matrices

## Key Concepts

### Matrices (Lesson 4-1)

- A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns.
- Equal matrices have the same dimensions and corresponding elements are equal.

### Operations (Lessons 4-2, 4-3)

- Matrices can be added or subtracted if they have the same dimensions. Add or subtract corresponding elements.
- To multiply a matrix by a scalar  $k$ , multiply each element in the matrix by  $k$ .
- Two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.
- Use matrix addition and a translation matrix to find the coordinates of a translated figure.
- Use scalar multiplication to perform dilations.

### Transformations (Lesson 4-4)

- To rotate a figure counterclockwise about the origin, multiply the vertex matrix on the left by a rotation matrix.

### Identity and Inverse Matrices (Lesson 4-7)

- An identity matrix is a square matrix with ones on the diagonal and zeros in the other positions.
- Two matrices are inverses of each other if their product is the identity matrix.

### Matrix Equations (Lesson 4-8)

- To solve a matrix equation, find the inverse of the coefficient matrix. Then multiply each side of the equation by the inverse matrix.

## Key Vocabulary

Cramer's Rule (p. 201)	inverse (p. 209)
determinant (p. 194)	matrix (p. 162)
dilation (p. 187)	matrix equation (p. 216)
dimension (p. 163)	reflection (p. 188)
element (p. 163)	rotation (p. 188)
equal matrices (p. 164)	scalar multiplication (p. 171)
identity matrix (p. 208)	translation (p. 185)

## Vocabulary Check

Choose the correct term from the list above to complete each sentence.

- The matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is a(n) \_\_\_\_\_ for multiplication.
- \_\_\_\_\_ is the process of multiplying a matrix by a constant.
- A(n) \_\_\_\_\_ is when a figure is moved around a center point.
- The \_\_\_\_\_ of  $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$  is  $-1$ .
- A(n) \_\_\_\_\_ is the product of the coefficient matrix and the variable matrix equal to the constant matrix.
- The \_\_\_\_\_ of a matrix tell how many rows and columns are in the matrix.
- A(n) \_\_\_\_\_ is a rectangular array of constants or variables.
- Each value in a matrix is called an \_\_\_\_\_.
- If the product of two matrices is the identity matrix, they are \_\_\_\_\_.
- \_\_\_\_\_ can be used to solve a system of equations.
- A(n) \_\_\_\_\_ is when a geometric figure is enlarged or reduced.
- A(n) \_\_\_\_\_ occurs when a figure is slid from one location to another on the coordinate plane.

## Lesson-by-Lesson Review

### 4-1

#### Introduction to Matrices (pp. 162–167)

Solve each equation.

$$13. \begin{bmatrix} 2y - x \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ 4y - 1 \end{bmatrix}$$

$$14. \begin{bmatrix} 7x \\ x + y \end{bmatrix} = \begin{bmatrix} 5 + 2y \\ 11 \end{bmatrix}$$

$$15. \begin{bmatrix} 3x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$16. \begin{bmatrix} 2x - y \\ 6x - y \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \end{bmatrix}$$

17. **FAMILY** Three sisters, Tionna, Diana, and Caroline each have 3 children. Tionna's children are 17, 20, and 23 years old. Diana's children are 12, 19, and 22 years old. Caroline's children are 6, 7, and 11 years old. Write a matrix of the children's ages. Which element represents the youngest child?

**Example 1** Solve  $\begin{bmatrix} 2x \\ y \end{bmatrix} = \begin{bmatrix} 32 + 6y \\ 7 - x \end{bmatrix}$ .

Write two linear equations.

$$2x = 32 + 6y$$

$$y = 7 - x$$

Solve the system of equations.

$$2x = 32 + 6y \quad \text{First equation}$$

$$2x = 32 + 6(7 - x) \quad \text{Substitute } 7 - x \text{ for } y.$$

$$2x = 32 + 42 - 6x \quad \text{Distributive Property}$$

$$8x = 74 \quad \text{Add } 6x \text{ to each side.}$$

$$x = 9.25 \quad \text{Divide each side by 8.}$$

To find the value of  $y$ , substitute 9.25 for  $x$  in either equation.

$$y = 7 - x \quad \text{Second equation}$$

$$= 7 - 9.25 \quad \text{Substitute } 9.25 \text{ for } x.$$

$$= -2.25 \quad \text{Simplify.}$$

The solution is  $(9.25, -2.25)$ .

### 4-2

#### Operations with Matrices (pp. 169–176)

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

$$18. \begin{bmatrix} -4 & 3 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 3 & -8 \end{bmatrix}$$

$$19. [0.2 \quad 1.3 \quad -0.4] - [2 \quad 1.7 \quad 2.6]$$

$$20. \begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 0 & 4 \\ -16 & 8 \end{bmatrix}$$

$$21. \begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix} - 2 \begin{bmatrix} -2 & 3 & 5 \\ -3 & -1 & 2 \end{bmatrix}$$

$$22. \begin{bmatrix} 90 & 70 & 85 \\ 72 & 53 & 97 \\ 84 & 61 & 79 \end{bmatrix} - \begin{bmatrix} 93 & 77 & 91 \\ 83 & 52 & 92 \\ 83 & 64 & 89 \end{bmatrix}$$

**Example 2** Find  $A - B$  if  $A = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix}$ .

$$A - B = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix} - \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix} \quad \text{Matrix subtraction}$$

$$= \begin{bmatrix} 3 - (-4) & 8 - 6 \\ -5 - 1 & 2 - 9 \end{bmatrix} \quad \text{Subtract.}$$

$$= \begin{bmatrix} 7 & 2 \\ -6 & -7 \end{bmatrix} \quad \text{Simplify.}$$

## 4-3

## Multiplying Matrices (pp. 177–184)

Find each product, if possible.

$$23. \begin{bmatrix} 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -4 \end{bmatrix} \quad 24. \begin{bmatrix} 8 & -3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & -5 \end{bmatrix}$$

$$25. \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 5 \\ 3 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

26. **SHOPPING** Mark went shopping and bought two shirts, three pairs of pants, one belt, and two pairs of shoes. The following matrix shows the prices for each item respectively.

$$\begin{bmatrix} \$20.15 & \$32 & \$15 & \$25.99 \end{bmatrix}$$

Use matrix multiplication to find the total amount of money Mark spent while shopping.

**Example 3** Find  $XY$  if  $X = \begin{bmatrix} 6 & 4 \end{bmatrix}$  and

$$Y = \begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix}.$$

$$XY = \begin{bmatrix} 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix} \quad \text{Write an equation.}$$

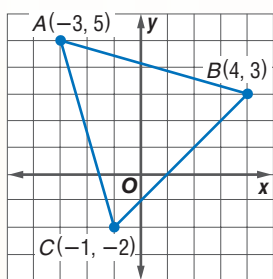
$$= [6(2) + 4(-3) \quad 6(5) + 4(0)] \quad \text{Multiply columns by rows.}$$

$$= [0 \quad 30] \quad \text{Simplify.}$$

## 4-4

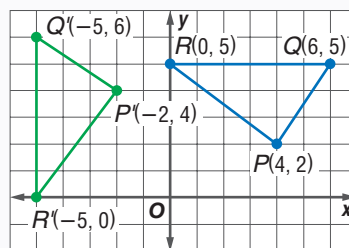
## Transformations with Matrices (pp. 185–192)

For Exercises 27–30, use the figure to find the coordinates of the image after each transformation.



27. translation 4 units right and 5 units down
28. dilation by a scale factor of 2
29. reflection over the  $y$ -axis
30. rotation of  $180^\circ$
31. **MAPS** Kala is drawing a map of her neighborhood. Her house is represented by quadrilateral  $ABCD$  with  $A(2, 2)$ ,  $B(6, 2)$ ,  $C(6, 6)$ , and  $D(2, 6)$ . Kala wants to use the same coordinates to make a map one half the size. What will the new coordinates of her house be?

**Example 4** Find the coordinates of the vertices of the image of  $\triangle PQR$  with  $P(4, 2)$ ,  $Q(6, 5)$ , and  $R(0, 5)$  after a rotation of  $90^\circ$  counterclockwise about the origin.



Write the ordered pairs in a vertex matrix. Then multiply by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 & 0 \\ 2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -5 \\ 4 & 6 & 0 \end{bmatrix}$$

The coordinates of the vertices of  $\triangle P'Q'R'$  are  $P'(-2, 4)$ ,  $Q'(-5, 6)$ , and  $R'(-5, 0)$ .



**4-5 Determinants** (pp. 194–200)

Find the value of each determinant.

32.  $\begin{vmatrix} 4 & 11 \\ -7 & 8 \end{vmatrix}$

33.  $\begin{vmatrix} 6 & -7 \\ 5 & 3 \end{vmatrix}$

34.  $\begin{vmatrix} 12 & 8 \\ 9 & 6 \end{vmatrix}$

35.  $\begin{vmatrix} 2 & -3 & 1 \\ 0 & 7 & 8 \\ 2 & 1 & 3 \end{vmatrix}$

36.  $\begin{vmatrix} 7 & -4 & 5 \\ 1 & 3 & -6 \\ 5 & -1 & -2 \end{vmatrix}$

37.  $\begin{vmatrix} 6 & 3 & -2 \\ -4 & 2 & 5 \\ -3 & -1 & 0 \end{vmatrix}$

38. **GEOMETRY** Alex wants to find the area of a triangle. He draws the triangle on a coordinate plane and finds that it has vertices at (2, 1), (3, 4) and (1, 4). Find the area of the triangle.

**Example 5** Evaluate  $\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix}$ .

$$\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix} = 3(2) - (-4)(6) \quad \text{Definition of determinant}$$

$$= 6 - (-24) \text{ or } 30 \quad \text{Simplify.}$$

**Example 6** Evaluate  $\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$  using expansion by minors.

$$\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3 \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 5 \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix}$$

$$= 3(-4 - (-1)) - 1(2 - 0) + 5(-1 - 0)$$

$$= -9 - 2 - 5 \text{ or } -16$$

**4-6 Cramer's Rule** (pp. 201–207)

Use Cramer's Rule to solve each system of equations.

39.  $9a - b = 1$   
 $3a + 2b = 12$

40.  $x + 5y = 14$   
 $-2x + 6y = 4$

41.  $4f + 5g = -2$   
 $-3f - 7g = 8$

42.  $-6m + n = -13$   
 $11m - 6n = 3$

43.  $6x - 7z = 13$   
 $8y + 2z = 14$   
 $7x + z = 6$

44.  $2a - b - 3c = -20$   
 $4a + 2b + c = 6$   
 $2a + b - c = -6$

45. **ENTERTAINMENT** Selena paid \$25.25 to play three games of miniature golf and two rides on go-karts. Selena paid \$25.75 for four games of miniature golf and one ride on the go-karts. Use Cramer's Rule to find out how much each activity costs.

**Example 7** Use Cramer's Rule to solve  $5a - 3b = 7$  and  $3a + 9b = -3$ .

$$a = \frac{\begin{vmatrix} 7 & -3 \\ -3 & 9 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}} \quad \text{Cramer's Rule} \quad b = \frac{\begin{vmatrix} 5 & 7 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}}$$

$$= \frac{63 - 9}{45 + 9} \quad \text{Evaluate each determinant.} \quad = \frac{-15 - 21}{45 + 9}$$

$$= \frac{54}{54} \text{ or } 1 \quad \text{Simplify.} \quad = \frac{-36}{54} \text{ or } -\frac{2}{3}$$

The solution is  $\left(1, -\frac{2}{3}\right)$ .



## Study Guide and Review

## 4-7

## Identity and Inverse Matrices (pp. 208–215)

Find the inverse of each matrix, if it exists.

46.  $\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix}$

47.  $\begin{bmatrix} 8 & 6 \\ 9 & 7 \end{bmatrix}$

48.  $\begin{bmatrix} 0 & 2 \\ 5 & -4 \end{bmatrix}$

49.  $\begin{bmatrix} 6 & -1 & 0 \\ 5 & 8 & -2 \end{bmatrix}$

50. **CRYPTOGRAPHY** Martin wrote a coded message to his friend using a coding matrix,  $C = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ . What is Martin's message if the matrix he gave his friend

was  $\begin{bmatrix} 26 & 12 \\ 80 & 80 \\ 75 & 25 \\ 24 & 38 \\ 94 & 98 \\ 32 & 24 \\ 53 & 101 \end{bmatrix}$  (Hint: Assume that the letters are labeled 1–26 with A = 1 and \_ = 0.)

**Example 8** Find the inverse of

$$S = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}.$$

First evaluate the determinant.

$$\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3 - (-8) \text{ or } 11$$

Then use the formula for the inverse matrix.

$$S^{-1} = \frac{1}{11} \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$

## 4-8

## Using Matrices to Solve Systems of Equations (pp. 216–222)

Solve each matrix equation or system of equations by using inverse matrices.

51.  $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$

52.  $\begin{bmatrix} 4 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

53.  $3x + 8 = -y$       54.  $3x - 5y = -13$   
 $4x - 2y = -14$        $4x + 3y = 2$

55. **SHOES** Joan is preparing a dye solution for her shoes. For the right color she needs 1500 milliliters of a 63% concentration solution. The store has only 75% and 50% concentration solutions. How many milliliters of 50% dye solution should be mixed with 75% dye solution to make the necessary amount of 63% dye solution?

**Example 9** Solve  $\begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \end{bmatrix}$ .

**Step 1** Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-12 - 16} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \text{ or } -\frac{1}{28} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix}$$

**Step 2** Multiply each side by the inverse matrix.

$$\begin{aligned} -\frac{1}{28} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\ = -\frac{1}{28} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 13 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{28} \begin{bmatrix} -140 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Solve each equation.

1.  $\begin{bmatrix} 3x + 1 \\ 2y \end{bmatrix} = \begin{bmatrix} 10 \\ 4 + y \end{bmatrix}$

2.  $\begin{bmatrix} 2x & y + 1 \\ 13 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -7 \\ 13 & z - 8 \end{bmatrix}$

Perform the indicated operations. If the matrix does not exist, write *impossible*.

3.  $\begin{bmatrix} 2 & -4 & 1 \\ 3 & 8 & -2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 7 \end{bmatrix}$

4.  $\begin{bmatrix} 1 & 2 \\ -4 & 3 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

5.  $\begin{bmatrix} 1 & 6 & 7 \\ 1 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 3 \\ -1 & -2 \\ 2 & 5 \end{bmatrix}$

Find the value of each determinant.

6.  $\begin{vmatrix} -1 & 4 \\ -6 & 3 \end{vmatrix}$

7.  $\begin{vmatrix} -2 & 0 & 5 \\ -3 & 4 & 0 \\ 1 & 3 & -1 \end{vmatrix}$

Find the inverse of each matrix, if it exists.

8.  $\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}$

9.  $\begin{bmatrix} -6 & -3 \\ 8 & 4 \end{bmatrix}$

Solve each matrix equation or system of equations by using inverse matrices.

10.  $\begin{bmatrix} 5 & 7 \\ -9 & 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 41 \\ -105 \end{bmatrix}$

11.  $\begin{bmatrix} -2 & 3 \\ 11 & -7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \end{bmatrix}$

12.  $5a + 2b = -49$

$2a + 9b = 5$

13.  $4c + 9d = 6$

$13c - 11d = -61$

14. **ACCOUNTING** A small business' bank account is charged a service fee for each electronic credit and electronic debit transaction. Their transactions and charges for two recent months are listed in the table.

Month	Electronic Credits	Electronic Debits	Cost
January	28	18	\$7.22
February	25	31	\$7.79

Use a system of equations to find the fee for each electronic credit and electronic debit transaction.

For Exercises 15–17, use  $\triangle ABC$  whose vertices have coordinates  $A(6, 3)$ ,  $B(1, 5)$ , and  $C(-1, 4)$ .

15. Use a determinant to find the area of  $\triangle ABC$ .
16. Translate  $\triangle ABC$  so that the coordinates of  $B'$  are  $(3, 1)$ . What are the coordinates of  $A'$  and  $C'$ ?
17. Find the coordinates of the vertices of a triangle that is a dilation of  $\triangle ABC$  with a perimeter five times that of  $\triangle ABC$ .

18. **MULTIPLE CHOICE** Lupe is preparing boxes of assorted chocolates. Chocolate-covered peanuts cost \$7 per pound. Chocolate-covered caramels cost \$6.50 per pound. The boxes of assorted candies contain five more pounds of peanut candies than caramel candies. If the total amount sold was \$575, how many pounds of each candy were needed to make the boxes?

A 40 lb peanut, 45 lb caramel

B 40 lb caramel, 45 lb peanut

C 40 lb peanut, 35 lb caramel

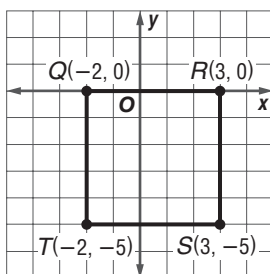
D 40 lb caramel, 35 lb peanut

# Standardized Test Practice

Cumulative, Chapters 1–4

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Figure  $QRST$  is shown on the coordinate plane.



Which transformation creates an image with a vertex at the origin?

- A Reflect figure  $QRST$  across the line  $y = -1$ .  
 B Reflect figure  $QRST$  across the line  $x = -3$ .  
 C Rotate figure  $QRST$  180 degrees around  $R$ .  
 D Translate figure  $QRST$  to the left 3 units and up 5 units.
2. The algebraic form of a linear function is  $d = 35t$ , where  $d$  is the distance in miles and  $t$  is the time in hours. Which one of the following choices identifies the same linear function?
- F For every 6 hours that a car is driven, it travels about 4 miles.  
 G For every 6 hours that a car is driven, it travels about 210 miles.

H

$t$	$d$
0	0
2	17.5
4	8.75
6	5.83

J

$t$	$d$
0	0
70	2
140	4
210	6

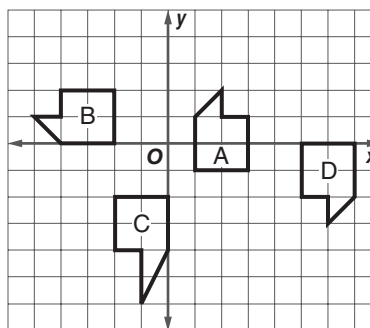
3. **GRIDDABLE** What is the value of  $a$  in the matrix equation below?

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 21 \\ 9 \end{bmatrix}$$

## TEST-TAKING TIP

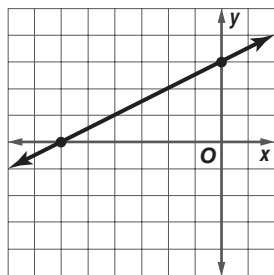
**Question 3** When answering questions, read carefully and make sure that you know exactly what the question is asking you to find. For example, if you find the value of  $b$  in question 3, you have not solved the problem. You need to find the value of  $a$ .

4. Pedro is creating a scale drawing of a car. He finds that the height of the car in the drawing is  $\frac{1}{32}$  of the actual height of the car  $x$ . Which equation best represents this relationship?
- A  $y = x - \frac{1}{32}$       C  $y = \frac{1}{32}x$   
 B  $y = -\frac{1}{32}x$       D  $y = x + \frac{1}{32}$
5. Which pair of polygons is congruent?



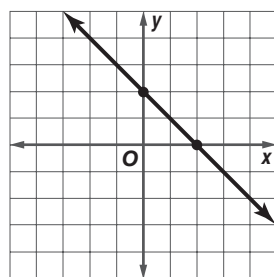
- F Polygon A and Polygon B  
 G Polygon B and Polygon C  
 H Polygon A and Polygon C  
 J Polygon C and Polygon D
6. For Marla's vacation, it will cost her \$100 to drive her car plus between \$0.50 to \$0.75 per mile. If she will drive her car for 400 miles, what is a reasonable conclusion about  $c$ , the total cost to drive her car on the vacation?
- A  $300 < c < 400$       C  $100 < c < 400$   
 B  $300 < c \leq 400$       D  $300 \leq c \leq 400$

7. What are the slope and  $y$ -intercept of the equation of the line graphed below?



- F  $m = 4; b = \frac{2}{3}$       H  $m = \frac{1}{2}; b = 3$   
G  $m = 4; b = \frac{3}{2}$       J  $m = \frac{1}{2}; b = 4$

8. The graph of a line is shown below.

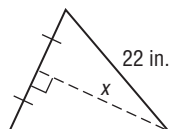


If the slope of this line is multiplied by 2 and the  $y$ -intercept increases by 1 unit, which linear equation represents these changes?

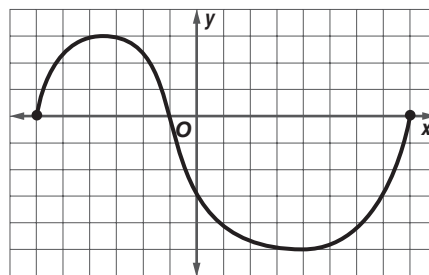
- A  $y = -\frac{1}{2}x + 1$       C  $y = -4x + 3$   
B  $y = -2x + 1$       D  $y = -2x + 3$

9. Given the equilateral triangle below, what is the approximate measure of  $x$ ?

- F 19.1 in.  
G 22.0 in.  
H 24.6 in.  
J 31.1 in.



10. What is the domain of the function shown on the graph?



- A  $\{x \mid -5 \leq x \leq 3\}$       C  $\{x \mid -5 < x < 3\}$   
B  $\{x \mid -6 \leq x \leq 8\}$       D  $\{x \mid -6 < x < 8\}$

### Pre-AP

Record your answers on a sheet of paper.  
Show your work.

11. The Colonial High School Yearbook Staff is selling yearbooks and chrome picture frames engraved with the year. The number of yearbooks and frames sold to members of each grade is shown in the table.

Sales for Each Class		
Grade	Yearbooks	Frames
9th	423	256
10th	464	278
11th	546	344
12th	575	497

- a. Find the difference in the sales of yearbooks and frames made to the 10<sup>th</sup> and 11<sup>th</sup> grade classes.  
b. Find the total numbers of yearbooks and frames sold.  
c. A yearbook costs \$48, and a frame costs \$18. Find the sales of yearbooks and frames for each class.

NEED EXTRA HELP?											
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11
Go to Lesson or Page...	4-4	2-4	4-8	4-2	4-5	3-3	2-3	2-3	879	2-1	1-3