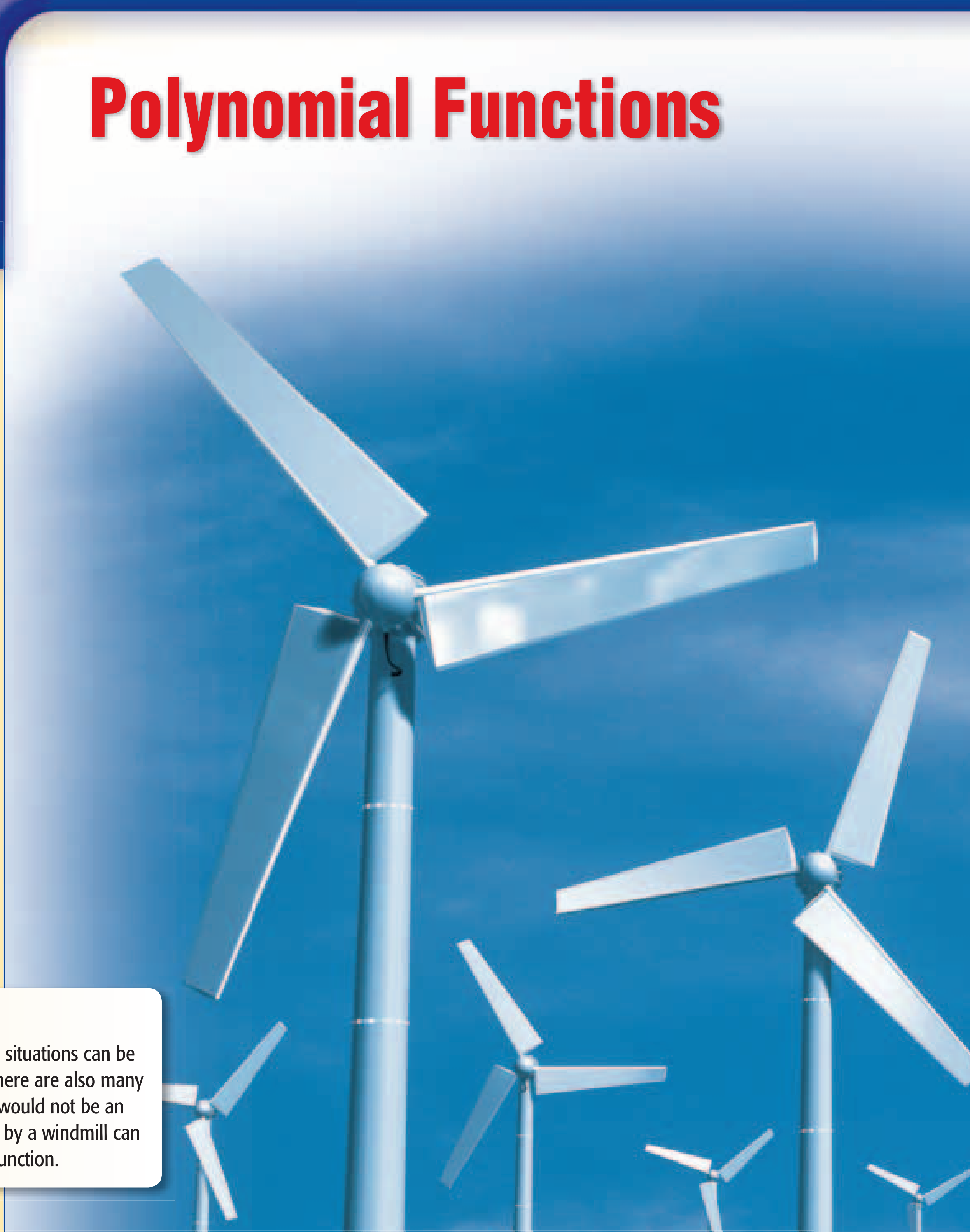
CHAPTER 6

Polynomial Functions



BIG Ideas

- Add, subtract, multiply, divide, and factor polynomials.
- Analyze and graph polynomial functions.
- Evaluate polynomial functions and solve polynomial equations.
- Find factors and zeroes of polynomial functions.

Key Vocabulary

polynomial function (p. 332)
scientific notation (p. 315)
synthetic division (p. 327)
synthetic substitution (p. 356)

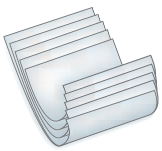
Real-World Link

Power Generation Many real-world situations can be modeled using linear equations. But there are also many situations for which a linear equation would not be an accurate model. The power generated by a windmill can be best modeled using a polynomial function.

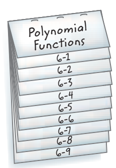
FOLDABLES Study Organizer

Polynomial Functions Make this Foldable to help you organize your notes. Begin with five sheets of grid paper.

1. **Stack** sheets of paper with edges $\frac{3}{4}$-inch apart. Fold up the bottom edges to create equal tabs.



2. **Staple** along the fold. Label the tabs with lesson numbers.



Guy Grenier/Masterfile

GET READY for Chapter 6

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

Math Online Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Rewrite each difference as a sum (Prerequisite Skill)

1. $2 - 7$
2. $-6 - 11$
3. $x - y$
4. $8 - 2x$
5. $2xy - 6yz$
6. $6a^2b - 12b^2c$
7. **CANDY** Janet has \$4. She buys x candy bars for \$0.50 each. Rewrite the amount of money she has left as a sum. (Prerequisite Skill)

Use the Distributive Property to rewrite each expression without parentheses. (Lesson 1-2)

8. $-2(4x^3 + x - 3)$
9. $-1(x + 2)$
10. $-1(x - 3)$
11. $-3(2x^4 - 5x^2 - 2)$
12. $-\frac{1}{2}(3a + 2)$
13. $-\frac{2}{3}(2 + 6z)$

SCHOOL SHOPPING For Exercises 14 and 15, use the following information.
Students, ages 12 to 17, plan on spending an average of \$113 on clothing for school. The students plan on spending 36% of their money at specialty stores and 19% at department stores. (Lesson 1-2)

14. Write an expression to represent the amount that the average student spends shopping for clothes at specialty and department stores.
15. Evaluate the expression from Exercise 14 by using the Distributive Property.

Solve each equation. (Lesson 5-6)

16. $x^2 - 17x + 60 = 0$
17. $14x^2 + 23x + 3 = 0$
18. $2x^2 + 5x + 1 = 0$
19. $3x^2 - 5x + 2 = 0$

QUICKReview

EXAMPLE 1

Rewrite $a - b - c$ as a sum.

$a - b - c$ Write the expression.

$= a + (-b) + (-c)$ Rewrite by adding $(-b)$ and $(-c)$.

EXAMPLE 2

Use the Distributive Property to rewrite $-x(y - z + y)$ without parentheses.

$-x(y - z + y)$ Original expression

$= -x(y) + (-x)(-z) + (-x)(y)$ Distributive Property

$= -xy + xz - xy$ Simplify.

EXAMPLE 3

Solve $4x^2 - 6x - 5 = 0$.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula

$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(4)(-5)}}{2(4)}$ Substitute.

$x = \dfrac{6 \pm \sqrt{116}}{8}$ Simplify.

$x = \dfrac{6 \pm 2\sqrt{29}}{8}$ or $x = \dfrac{3 \pm \sqrt{29}}{4}$ $\sqrt{116} = \sqrt{4 \cdot 29}$ or $2\sqrt{29}$

The exact solutions are $\dfrac{3 + \sqrt{29}}{4}$ and $\dfrac{3 - \sqrt{29}}{4}$.
The approximate solutions are 2.1 and −0.6.

6-1 Properties of Exponents

Main Ideas

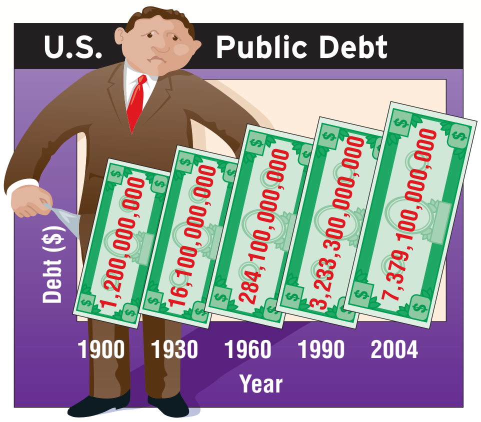
- Use properties of exponents to multiply and divide monomials.
- Use expressions written in scientific notation.

New Vocabulary

simplify
standard notation
scientific notation
dimensional analysis

GET READY for the Lesson

Economists often deal with very large numbers. For example, the table shows the U.S. public debt for several years. Such numbers, written in standard notation, are difficult to work with because they contain so many digits. Scientific notation uses powers of ten to make very large or very small numbers more manageable.



Source: Bureau of the Public Debt

Multiply and Divide Monomials To **simplify** an expression containing powers means to rewrite the expression without parentheses or negative exponents. Negative exponents are a way of expressing the multiplicative inverse of a number. For example, $\frac{1}{x^2}$ can be written as x^{-2}. Note that an expression such as x^{-2} is not a monomial. *Why?*

KEY CONCEPT — *Negative Exponents*

Words For any real number $a \neq 0$ and any integer n, $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.

Examples $2^{-3} = \frac{1}{2^3}$ and $\frac{1}{b^{-8}} = b^8$

Study Tip

Look Back

You can review **monomials** in Lesson 1-1.

EXAMPLE 1 Simplify Expressions with Multiplication

Simplify each expression. Assume that no variable equals 0.

a. $(3x^3y^2)(-4x^2y^4)$

$(3x^3y^2)(-4x^2y^4)$

$= (3 \cdot x \cdot x \cdot x \cdot y \cdot y) \cdot (-4 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y)$ Definition of exponents

$= 3(-4) \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y$ Commutative Property

$= -12x^5y^6$ Definition of exponents

b. $(a^{-3})(a^2b^4)(c^{-1})$

$$(a^{-3})(a^2b^4)(c^{-1}) = \left(\frac{1}{a^3}\right)(a^2b^4)\left(\frac{1}{c}\right)$$ Definition of negative exponents

$$= \left(\frac{1}{a \cdot a \cdot a}\right)(a \cdot a \cdot b \cdot b \cdot b \cdot b)\left(\frac{1}{c}\right)$$ Definition of exponents

$$= \left(\frac{1}{\cancel{a} \cdot \cancel{a} \cdot a}\right)(\cancel{a} \cdot \cancel{a} \cdot b \cdot b \cdot b \cdot b)\left(\frac{1}{c}\right)$$ Cancel out common factors.

$$= \frac{b^4}{ac}$$ Definition of exponents and fractions

CHECK Your Progress

1A. $(-5x^4y^3)(-3xy^5)$

1B. $(2x^{-3}y^3)(-7x^5y^{-6})$

Example 1 suggests the following property of exponents.

KEY CONCEPT — ***Product of Powers***

Words	For any real number a and integers m and n, $a^m \cdot a^n = a^{m+n}$.
Examples	$4^2 \cdot 4^9 = 4^{11}$ and $b^3 \cdot b^5 = b^8$

To multiply powers of the same variable, add the exponents. Knowing this, it seems reasonable to expect that when dividing powers, you would subtract exponents. Consider $\frac{x^9}{x^5}$.

$$\frac{x^9}{x^5} = \frac{\overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot x \cdot x \cdot x \cdot x}{\underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}}}$$ Remember that $x \neq 0$.

$$= x \cdot x \cdot x \cdot x$$ Simplify.

$$= x^4$$ Definition of exponents

It appears that our conjecture is true. To divide powers of the same base, you subtract exponents.

KEY CONCEPT — ***Quotient of Powers***

Words	For any real number $a \neq 0$, and any integers m and n, $\frac{a^m}{a^n} = a^{m-n}$.
Examples	$\frac{5^3}{5} = 5^{3-1}$ or 5^2 and $\frac{x^7}{x^3} = x^{7-3}$ or x^4

Study Tip

Check

You can check your answer using the definition of exponents.

$$\frac{p^3}{p^8} = \frac{\overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}}}{\underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot p \cdot p \cdot p \cdot p \cdot p}$$

or $\frac{1}{p^5}$

EXAMPLE Simplify Expressions with Division

2 **Simplify $\frac{p^3}{p^8}$. Assume that $p \neq 0$.**

$$\frac{p^3}{p^8} = p^{3-8}$$ Subtract exponents.

$$= p^{-5} \text{ or } \frac{1}{p^5}$$ Remember that a simplified expression cannot contain negative exponents.

CHECK Your Progress

Simplify each expression. Assume that no variable equals 0.

2A. $\frac{y^{12}}{y^4}$

2B. $\frac{15c^5d^3}{-3c^2d^7}$

You can use the Quotient of Powers property and the definition of exponents to simplify $\frac{y^4}{y^4}$, if $y \neq 0$.

Method 1

$$\frac{y^4}{y^4} = y^{4-4} \quad \text{Quotient of Powers}$$
$$= y^0 \quad \text{Subtract.}$$

Method 2

$$\frac{y^4}{y^4} = \frac{\overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}}}{\underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}}} \quad \text{Definition of exponents}$$
$$= 1 \quad \text{Divide.}$$

In order to make the results of these two methods consistent, we define $y^0 = 1$, where $y \neq 0$. In other words, any nonzero number raised to the zero power is equal to 1. *Notice that 0^0 is undefined.*

The properties we have presented can be used to verify the properties of powers that are listed below.

KEY CONCEPT — *Properties of Powers*

Words Suppose a and b are real numbers and m and n are integers. Then the following properties hold.

Property	Examples
Power of a Power: $(a^m)^n = a^{mn}$	$(a^2)^3 = a^6$
Power of a Product: $(ab)^m = a^m b^m$	$(xy)^2 = x^2y^2$
Power of a Quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$ and	$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$
$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ or $\frac{b^n}{a^n}$, $a \neq 0$, $b \neq 0$	$\left(\frac{x}{y}\right)^{-4} = \frac{y^4}{x^4}$
Zero Power: $a^0 = 1$, $a \neq 0$	$2^0 = 1$

Study Tip

Simplified Expressions

A monomial expression is in simplified form when:

- there are no powers of powers,
- each base appears exactly once,
- all fractions are in simplest form, and
- there are no negative exponents.

EXAMPLE 3 Simplify Expressions with Powers

Simplify each expression.

a. $(a^3)^6$

$(a^3)^6 = a^{3(6)}$ Power of a power

$= a^{18}$ Simplify.

b. $\left(\frac{-3x}{y}\right)^4$

$\left(\frac{-3x}{y}\right)^4 = \frac{(-3x)^4}{y^4}$ Power of a quotient

$= \frac{(-3)^4x^4}{y^4}$ Power of a product

$= \frac{81x^4}{y^4}$ $(-3)^4 = 81$

CHECK Your Progress

3A. $(-2p^3s^2)^5$ **3B.** $\left(\frac{a}{4}\right)^{-3}$

With complicated expressions, you often have a choice of which way to start simplifying.

EXAMPLE 4 Simplify Expressions Using Several Properties

Simplify $\left(\frac{-2x^{3n}}{x^{2n}y^3}\right)^4$.

Method 1

Raise the numerator and denominator to the fourth power before simplifying.

$$\left(\frac{-2x^{3n}}{x^{2n}y^3}\right)^4 = \frac{(-2x^{3n})^4}{(x^{2n}y^3)^4}$$

$$= \frac{(-2)^4(x^{3n})^4}{(x^{2n})^4\,(y^3)^4}$$

$$= \frac{16x^{12n}}{x^{8n}\,y^{12}}$$

$$= \frac{16x^{12n-8n}}{y^{12}} \text{ or } \frac{16x^{4n}}{y^{12}}$$

Method 2

Simplify the fraction before raising to the fourth power.

$$\left(\frac{-2x^{3n}}{x^{2n}y^3}\right)^4 = \left(\frac{-2x^{3n-2n}}{y^3}\right)^4$$

$$= \left(\frac{-2x^{n}}{y^3}\right)^4$$

$$= \frac{16x^{4n}}{y^{12}}$$

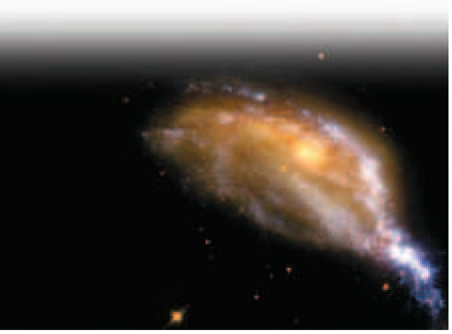
CHECK Your Progress

4A. $\left(\frac{3x^2y}{2xy^4}\right)^3$

4B. $\left(\frac{-3x^{-5}y^{-2n}}{5x^{-6}}\right)^4$

Personal Tutor at algebra2.com

Scientific Notation The form that you usually write numbers in is **standard notation.** A number is in **scientific notation** when it is in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer. Real-world problems using numbers in scientific notation often involve units of measure. Performing operations with units is known as **dimensional analysis.**



Real-World Link

Light travels at a speed of about 3.00×10^8 m/s. The distance that light travels in a year is called a *light-year.*

Source: www.britannica.com

Real-World EXAMPLE

5 **ASTRONOMY** **After the Sun, the next-closest star to Earth is Alpha Centauri C, which is about 4×10^{16} meters away. How long does it take light from Alpha Centauri C to reach Earth? Use the information at the left.**

Begin with the formula $d = rt$, where d is distance, r is rate, and t is time.

$t = \frac{d}{r}$ Solve the formula for time.

$= \frac{4 \times 10^{16}\text{ m}}{3.00 \times 10^8\text{ m/s}}$ ← Distance from Alpha Centauri C to Earth

$= \frac{4}{3.00} \cdot \frac{10^{16}}{10^8} \cdot \frac{\text{m}}{\text{m/s}}$ **Estimate:** The result should be slightly greater than $\frac{10^{16}}{10^8}$ or 10^8.

$\approx 1.33 \times 10^8$ s $\frac{4}{3.00} \approx 1.33$, $\frac{10^{16}}{10^8} = 10^{16-8}$ or 10^8, $\frac{\text{m}}{\text{m/s}} = \text{m} \cdot \frac{\text{s}}{\text{m}} = \text{s}$

It takes about 1.33×10^8 seconds or 4.2 years for light from Alpha Centauri C to reach Earth.

CHECK Your Progress

5. The density D of an object in grams per milliliter is found by dividing the mass m of the substance by the volume V of the object. A sample of platinum has a mass of 8.4×10^{-2} kilogram and a volume of 4×10^{-6} cubic meter. Use this information to calculate the density of platinum.

CHECK Your Understanding

Simplify. Assume that no variable equals 0.

Examples 1, 2 (pp. 312–313)

1. $(-3x^2y^3)(5x^5y^6)$ **2.** $\dfrac{30y^4}{-5y^2}$ **3.** $\dfrac{-2a^3b^6}{18a^2b^2}$

Example 3 (p. 314)

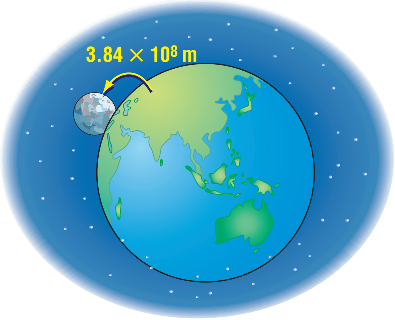
4. $(2b)^4$ **5.** $\left(\dfrac{1}{w^4z^2}\right)^3$ **6.** $\left(\dfrac{cd}{3}\right)^{-2}$

Example 4 (p. 315)

7. $(n^3)^3(n^{-3})^3$ **8.** $\dfrac{81p^6q^5}{(3p^2q)^2}$ **9.** $\left(\dfrac{-6x^6}{3x^3}\right)^{-2}$

Example 5 (p. 315)

10. **ASTRONOMY** Refer to Example 5 on page 315. The average distance from Earth to the Moon is about 3.84×10^8 meters. How long would it take a radio signal traveling at the speed of light to cover that distance?



Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–14	1
15–18	2
16–19	3
23–26	4
27, 28	5

Simplify. Assume that no variable equals 0.

11. $\left(\frac{1}{3}a^8b^2\right)(2a^2b^2)$ **12.** $(5cd^2)(-c^4d)$ **13.** $(7x^3y^{-5})(4xy^3)$

14. $(-3b^3c)(7b^2c^2)$ **15.** $\dfrac{a^2n^6}{an^5}$ **16.** $\dfrac{-y^5z^7}{y^2z^5}$

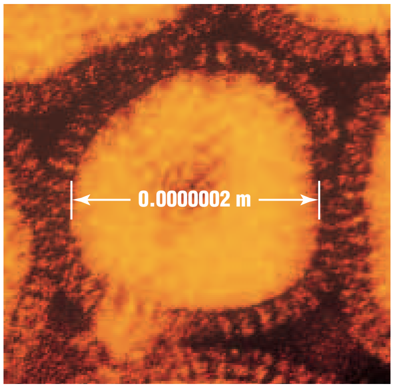
17. $\dfrac{-5x^3y^3z^4}{20x^3y^7z^4}$ **18.** $\dfrac{3a^5b^3c^3}{9a^3b^7c}$ **19.** $(n^4)^4$

20. $(z^2)^5$ **21.** $(2x)^4$ **22.** $(-2c)^3$

23. $(a^3b^3)(ab)^{-2}$ **24.** $(-2r^2s)^3(3rs^2)$

25. $\dfrac{2c^3d(3c^2d^5)}{30c^4d^2}$ **26.** $\dfrac{-12m^4n^8\,(m^3n^2)}{36m^3n}$

27. **BIOLOGY** Use the diagram at the right to write the diameter of a typical flu virus in scientific notation. Then estimate the area of a typical flu virus. (*Hint:* Treat the virus as a circle.)



28. **POPULATION** The population of Earth is about 6.445×10^9. The land surface area of Earth is 1.483×10^8 km^2. What is the population density for the land surface area of Earth?

Simplify. Assume that no variable equals 0.

29. $2x^2(6y^3)(2x^2y)$ **30.** $3a(5a^2b)(6ab^3)$ **31.** $\dfrac{30a^{-2}b^{-6}}{60a^{-6}b^{-8}}$

32. $\dfrac{12x^{-3}y^{-2}z^{-8}}{30x^{-6}y^{-4}z^{-1}}$ **33.** $\left(\dfrac{x}{y^{-1}}\right)^{-2}$ **34.** $\left(\dfrac{v}{w^{-2}}\right)^{-3}$

35. $\left(\dfrac{8a^3b^2}{16a^2b^3}\right)^4$ **36.** $\left(\dfrac{6x^2y^4}{3x^4y^3}\right)^3$ **37.** $\left(\dfrac{4x^{-3}y^2}{xy^{-5}}\right)^{-2}$

38. If $2^{r+5} = 2^{2r-1}$, what is the value of r?

39. What value of r makes $y^{28} = y^{3r} \cdot y^7$ true?

EXTRA PRACTICE
See pages 902, 931.
Math Online
Self-Check Quiz at
algebra2.com

40. INCOME In 2003, the population of Texas was about 2.21×10^7. The personal income for the state that year was about 6.43×10^{11} dollars. What was the average personal income?

41. RESEARCH Use the Internet or other source to find the masses of Earth and the Sun. About how many times as large as Earth is the Sun?

H.O.T. Problems

42. OPEN ENDED Write an example that illustrates a property of powers. Then use multiplication or division to explain why it is true.

43. FIND THE ERROR Alejandra and Kyle both simplified $\frac{2a^2b}{(-2ab^3)^{-2}}$. Who is correct? Explain your reasoning.

Alejandra

$$\frac{2a^2b}{(-2ab^3)^{-2}} = (2a^2b)(-2ab^3)^2$$
$$= (2a^2b)(-2)^2a^2(b^3)^2$$
$$= (2a^2b)2^2a^2b^6$$
$$= 8a^4b^7$$

Kyle

$$\frac{2a^2b}{(-2ab^3)^{-2}} = \frac{2a^2b}{(-2)^2a(b^3)^{-2}}$$
$$= \frac{2a^2b}{4ab^{-6}}$$
$$= \frac{2a^2bb^6}{4a}$$
$$= \frac{ab^7}{2}$$

44. REASONING Determine whether $x^y \cdot x^z = x^{yz}$ is *sometimes*, *always*, or *never* true. Explain your reasoning.

45. CHALLENGE Determine which is greater, 100^{10} or 10^{100}. Explain.

46. *Writing in Math* Use the information on page 312 to explain why scientific notation is useful in economics. Include the 2004 national debt of \$7,379,100,000,000 and the U.S. population of 293,700,000, both written in words and in scientific notation, and an explanation of how to find the amount of debt per person with the result written in scientific notation and in standard notation.

STANDARDIZED TEST PRACTICE

47. ACT/SAT Which expression is equal to $\frac{(2x^2)^3}{12x^4}$?

A $\frac{x}{2}$

B $\frac{2x}{3}$

C $\frac{1}{2x^2}$

D $\frac{2x^2}{3}$

48. REVIEW Four students worked the same math problem. Each student's work is shown below.

Student F

$$x^2x^{-5} = \frac{x^2}{x^5} = \frac{1}{x^3}, x \neq 0$$

Student G

$$x^2x^{-5} = \frac{x^2}{x^{-5}} = x^7, x \neq 0$$

Student H

$$x^2x^{-5} = \frac{x^2}{x^{-5}} = x^{-7}, x \neq 0$$

Student J

$$x^2x^{-5} = \frac{x^2}{x^5} = x^3, x \neq 0$$

Which is a completely correct solution?

F Student F

G Student G

H Student H

J Student J

Spiral Review

Solve each inequality algebraically. (Lesson 5-8)

49. $x^2 - 8x + 12 < 0$ **50.** $x^2 + 2x - 86 \geq -23$ **51.** $15x^2 + 4x + 12 \leq 0$

Graph each function. (Lesson 5-7)

52. $y = -2(x - 2)^2 + 3$ **53.** $y = \frac{1}{3}(x + 5)^2 - 1$ **54.** $y = \frac{1}{2}x^2 + x + \frac{3}{2}$

Evaluate each determinant. (Lesson 4-3)

55. $\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}$ **56.** $\begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 4 \\ -3 & 0 & 2 \end{vmatrix}$

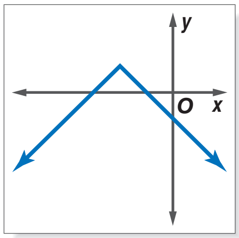
Solve each system of equations. (Lesson 3-5)

57. $x + y = 5$
$x + y + z = 4$
$2x - y + 2z = -1$

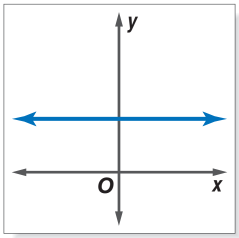
58. $a + b + c = 6$
$2a - b + 3c = 16$
$a + 3b - 2c = -6$

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. (Lesson 2-6)

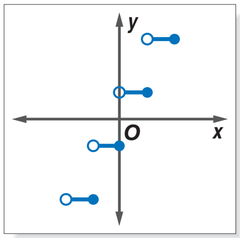
59.



60.



61.



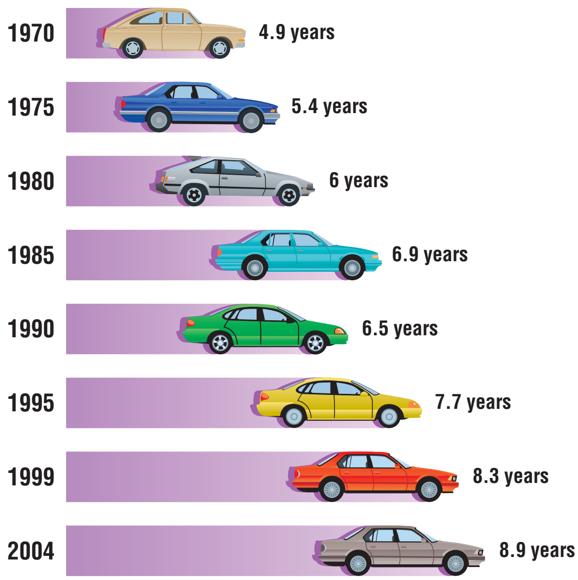
TRANSPORTATION **For Exercises 62–64, refer to the graph at the right.** (Lesson 2-5)

62. Make a scatter plot of the data, where the horizontal axis is the number of years since 1975.

63. Write a prediction equation.

64. Predict the median age of vehicles on the road in 2015.

Median Age of Vehicles



Source: Transportation Department

Solve each equation. (Lesson 1-3)

65. $2x + 11 = 25$

66. $-12 - 5x = 3$

GET READY for the Next Lesson

PREREQUISITE SKILL **Use the Distributive Property to find each product.** (Lesson 1-2)

67. $2(x + y)$ **68.** $3(x - z)$ **69.** $4(x + 2)$

70. $-2(3x - 5)$ **71.** $-5(x - 2y)$ **72.** $-3(-y + 5)$

READING MATH

Dimensional Analysis

Real-world problems often involve units of measure. Performing operations with units is called **dimensional analysis.** You can use dimensional analysis to convert units or to perform calculations.

Example **A car's gas tank holds 14 gallons of gasoline and the car gets 16 miles per gallon. How many miles can be driven on a full tank of gasoline?**

You want to find the number of miles that can be driven on 1 tank of gasoline, or the number of *miles per tank*. You know that there are 14 gallons per tank and 16 miles per gallon. Translate these into fractions that you can multiply.

$$\frac{14 \text{ gal}}{1 \text{ tank}} \cdot \frac{16 \text{ mi}}{1 \text{ gal}} = \frac{14 \cancel{\text{gal}}}{1 \text{ tank}} \cdot \frac{16 \text{ mi}}{1 \cancel{\text{gal}}} \qquad \text{The units } \textit{gallons} \text{ cancel out.}$$

$$= (14)(16) \text{ mi/tank} \qquad \text{Simplify.}$$

$$= 224 \text{ mi/tank} \qquad \text{Multiply.}$$

So, 224 miles can be driven on a full tank of gasoline. This answer is reasonable because the final units are mi/tank, not mi/gal, gal/mi, or mi.

Reading to Learn

Solve each problem using dimensional analysis. Include the appropriate units with your answer.

1. How many miles will a person run during a 5-kilometer race? (*Hint:* 1 km ≈ 0.62 mi)
2. A zebra can run 40 miles per hour. How far can a zebra run in 3 minutes?
3. A cyclist traveled 43.2 miles at an average speed of 12 miles per hour. How long did the cyclist ride?
4. The average student is in class 315 minutes/day. How many hours per day is this?
5. If you are going 50 miles per hour, how many feet per second are you traveling?
6. The equation $d = \frac{1}{2}(9.8 \text{ m/s}^2)(3.5 \text{ s})^2$ represents the distance d that a ball falls 3.5 seconds after it is dropped from a tower. Find the distance.
7. Explain what the following statement means. *Dimensional analysis tells you what to multiply or divide.*
8. Explain how dimensional analysis can be useful in checking the reasonableness of your answer.

6-2 Operations with Polynomials

Main Ideas

- Add and subtract polynomials.
- Multiply polynomials.

New Vocabulary

degree of a polynomial

GET READY for the Lesson

Shenequa has narrowed her choice for which college to attend. She is most interested in Coastal Carolina University, where the current year's tuition is \$3430. Shenequa assumes that tuition will increase at a rate of 6% per year. You can use polynomials to represent the increasing tuition costs.

College Choices

College	Tuition
Allegheny College	\$26,650
University of Maryland	\$7821
Coastal Carolina University	\$3430

Add and Subtract Polynomials If r represents the rate of increase of tuition, then the tuition for the second year will be $3430(1 + r)$. For the third year, it will be $3430(1 + r)^2$, or $3430r^2 + 6860r + 3430$ in expanded form. The **degree of a polynomial** is the degree of the monomial with the greatest degree. For example, the degree of this polynomial is 2.

Study Tip

Look Back
You can review **polynomials** in Lesson 1-1.

EXAMPLE Degree of a Polynomial

1 **Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.**

a. $\frac{1}{6}x^3y^5 - 9x^4$

This expression is a polynomial because each term is a monomial. The degree of the first term is $3 + 5$ or 8, and the degree of the second term is 4. The degree of the polynomial is 8.

b. $x + \sqrt{x} + 5$

This expression is not a polynomial because $\sqrt{x}$ is not a monomial.

c. $x^{-2} + 3x^{-1} - 4$

This expression is not a polynomial because x^{-2} and x^{-1} are not monomials. $x^{-2} = \frac{1}{x^2}$ and $x^{-1} = \frac{1}{x}$. Monomials cannot contain variables in the denominator.

CHECK Your Progress

1A. $\frac{x}{y} + 3x^2$ **1B.** $x^5y + 9x^4y^3 - 2xy$

To *simplify* a polynomial means to perform the operations indicated and combine like terms.

Study Tip

Alternate Methods

Notice that Example 2a uses a horizontal method and Example 2b uses a vertical method to simplify. Either method will yield a correct solution.

EXAMPLE Simplify Polynomials

2 **Simplify each expression.**

a. $(3x^2 - 2x + 3) - (x^2 + 4x - 2)$

Remove parentheses and group like terms together.

$(3x^2 - 2x + 3) - (x^2 + 4x - 2)$

$= 3x^2 - 2x + 3 - x^2 - 4x + 2$ — Distribute the -1.

$= (3x^2 - x^2) + (-2x - 4x) + (3 + 2)$ — Group like terms.

$= 2x^2 - 6x + 5$ — Combine like terms.

b. $(5x^2 - 4x + 1) + (-3x^2 + x - 3)$

Align like terms vertically and add.

$$\begin{array}{r} 5x^2 - 4x + 1 \\ (+)\ -3x^2 + \ \ x - 3 \\ \hline 2x^2 - 3x - 2 \end{array}$$

CHECK Your Progress

2A. $(-x^2 - 3x + 4) - (x^2 + 2x + 5)$ **2B.** $(3x^2 - 6) + (-x + 1)$

Multiply Polynomials You can use the Distributive Property to multiply polynomials.

EXAMPLE Simplify Using the Distributive Property

3 **Find $2x(7x^2 - 3x + 5)$.**

$2x(7x^2 - 3x + 5) = 2x(7x^2) + 2x(-3x) + 2x(5)$ — Distributive Property

$= 14x^3 - 6x^2 + 10x$ — Multiply the monomials.

CHECK Your Progress **Find each product.**

3A. $\frac{4}{3}x^2(6x^2 + 9x - 12)$ **3B.** $-2a(-3a^2 - 11a + 20)$

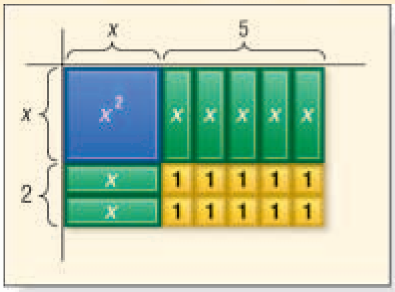
You can use algebra tiles to model the product of two binomials.

ALGEBRA LAB

Multiplying Binomials

Use algebra tiles to find the product of $x + 5$ and $x + 2$.

- Draw a 90° angle on your paper.
- Use an x tile and a 1 tile to mark off a length equal to $x + 5$ along the top.
- Use the tiles to mark off a length equal to $x + 2$ along the side.
- Draw lines to show the grid formed.
- Fill in the lines with the appropriate tiles to show the area product. The model shows the polynomial $x^2 + 7x + 10$.



The area of the rectangle is the product of its length and width. So, $(x + 5)(x + 2) = x^2 + 7x + 10$.

EXAMPLE 4 Multiply Polynomials

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4 Find $(n^2 + 6n - 2)(n + 4)$.

Method 1 Horizontally

$(n^2 + 6n - 2)(n + 4)$

$= n^2(n + 4) + 6n(n + 4) + (-2)(n + 4)$ Distributive Property

$= n^2 \cdot n + n^2 \cdot 4 + 6n \cdot n + 6n \cdot 4 + (-2) \cdot n + (-2) \cdot 4$ Distributive Property

$= n^3 + 4n^2 + 6n^2 + 24n - 2n - 8$ Multiply monomials.

$= n^3 + 10n^2 + 22n - 8$ Combine like terms.

Method 2 Vertically

$$\begin{array}{r} n^2 + \ \ 6n - 2 \\ (\times) \qquad\quad n + 4 \\ \hline 4n^2 + 24n - 8 \\ n^3 + \ \ 6n^2 - \ \ 2n \qquad \\ \hline n^3 + 10n^2 + 22n - 8 \end{array}$$

In Motion
Concepts in Motion
Animation
algebra2.com

CHECK Your Progress Find each product.

4A. $(x^2 + 4x + 16)(x - 4)$ **4B.** $(2x^2 - 4x + 5)(3x - 1)$

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CHECK Your Understanding

Example 1 (p. 320)

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

1. $2a + 5b$ **2.** $\frac{1}{3}x^3 - 9y$ **3.** $\frac{mw^2 - 3}{nz^3 + 1}$

Simplify.

Examples 2–4 (pp. 321–322)

4. $(2a + 3b) + (8a - 5b)$ **5.** $(x^2 - 4x + 3) - (4x^2 + 3x - 5)$

6. $2x(3y + 9)$ **7.** $2p^2q(5pq - 3p^3q^2 + 4pq^4)$

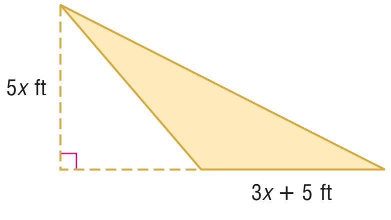
8. $(y - 10)(y + 7)$ **9.** $(x + 6)(x + 3)$

10. $(2z - 1)(2z + 1)$ **11.** $(2m - 3n)^2$

12. $(x + 1)(x^2 - 2x + 3)$ **13.** $(2x - 1)(x^2 - 4x + 4)$

Example 4 (p. 322)

14. GEOMETRY Find the area of the triangle.



Exercises

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

15. $3z^2 - 5z + 11$ **16.** $x^3 - 9$ **17.** $\frac{6xy}{z} - \frac{3c}{d}$

18. $\sqrt{m - 5}$ **19.** $5x^2y^4 + x\sqrt{3}$ **20.** $\frac{4}{3}y^2 + \frac{5}{6}y^7$

HOMEWORK HELP	
For Exercises	See Examples
15–20	1
21–24	2
25–28	3
29–36	4

Simplify.

21. $(3x^2 - x + 2) + (x^2 + 4x - 9)$

22. $(5y + 3y^2) + (-8y - 6y^2)$

23. $(9r^2 + 6r + 16) - (8r^2 + 7r + 10)$

24. $(7m^2 + 5m - 9) + (3m^2 - 6)$

25. $4b(cb - zd)$

26. $4a(3a^2 + b)$

27. $-5ab^2(-3a^2b + 6a^3b - 3a^4b^4)$

28. $2xy(3xy^3 - 4xy + 2y^4)$

29. $(p + 6)(p - 4)$

30. $(a + 6)(a + 3)$

31. $(b + 5)(b - 5)$

32. $(6 - z)(6 + z)$

33. $(3x + 8)(2x + 6)$

34. $(4y - 6)(2y + 7)$

35. $(3b - c)^3$

36. $(x^2 + xy + y^2)(x - y)$

37. PERSONAL FINANCE Toshiro has $850 to invest. He can invest in a savings account that has an annual interest rate of 1.7%, and he can invest in a money market account that pays about 3.5% per year. Write a polynomial to represent the amount of interest he will earn in 1 year if he invests x dollars in the savings account and the rest in the money market account.

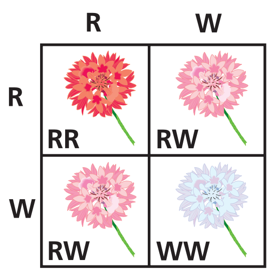
E-SALES For Exercises 38 and 39, use the following information.
A small online retailer estimates that the cost, in dollars, associated with selling x units of a particular product is given by the expression $0.001x^2 + 5x + 500$. The revenue from selling x units is given by $10x$.

38. Write a polynomial to represent the profit generated by the product.

39. Find the profit from sales of 1850 units.

40. Simplify $(c^2 - 6cd - 2d^2) + (7c^2 - cd + 8d^2) - (-c^2 + 5cd - d^2)$.

41. Find the product of $x^2 + 6x - 5$ and $-3x + 2$.



Real-World Link

Genetics

The possible genes of parents and offspring can be summarized in a *Punnett square*, such as the one above.

Source: *Biology: The Dynamics of Life*

Simplify.

42. $(4x^2 - 3y^2 + 5xy) - (8xy + 3y^2)$

43. $(10x^2 - 3xy + 4y^2) - (3x^2 + 5xy)$

44. $\frac{3}{4}x^2(8x + 12y - 16xy^2)$

45. $\frac{1}{2}a^3(4a - 6b + 8ab^4)$

46. $d^{-3}(d^5 - 2d^3 + d^{-1})$

47. $x^{-3}y^2(yx^4 + y^{-1}x^3 + y^{-2}x^2)$

48. $(a^3 - b)(a^3 + b)$

49. $(m^2 - 5)(2m^2 + 3)$

50. $(x - 3y)^2$

51. $(1 + 4c)^2$

52. GENETICS Suppose R and W represent two genes that a plant can inherit from its parents. The terms of the expansion of $(R + W)^2$ represent the possible pairings of the genes in the offspring. Write $(R + W)^2$ as a polynomial.

H.O.T. Problems

53. OPEN ENDED Write a polynomial of degree 5 that has three terms.

54. Which One Doesn't Belong? Identify the expression that does not belong with the other three. Explain your reasoning.

$3xy + 6x^2$	$\frac{5}{x^2}$	$x + 5$	$5b + 11c - 9ad^2$

55. CHALLENGE What is the degree of the product of a polynomial of degree 8 and a polynomial of degree 6? Include an example to support your answer.

56. *Writing in Math* Use the information about tuition increases to explain how polynomials can be applied to financial situations. Include an explanation of how a polynomial can be applied to a situation with a fixed percent rate of increase and an explanation of how to use an expression and the 6% rate of increase to estimate Shenequa's tuition in the fourth year.

EXTRA PRACTICE
See pages 902, 931.

Math Online
Self-Check Quiz at algebra2.com

STANDARDIZED TEST PRACTICE

57. ACT/SAT Which polynomial has degree 3?

A $x^3 + x^2 - 2x^4$

B $-2x^2 - 3x + 4$

C $x^2 + x + 12^3$

D $1 + x + x^3$

58. REVIEW
$(-4x^2 + 2x + 3) - 3(2x^2 - 5x + 1) =$

F $2x^2$

G $-10x^2$

H $-10x^2 + 17x$

J $2x^2 + 17x$

Spiral Review

Simplify. Assume that no variable equals 0. (Lesson 6-1)

59. $(-4d^2)^3$ **60.** $5rt^2(2rt)^2$ **61.** $\frac{x^2yz^4}{xy^3z^2}$ **62.** $\left(\frac{3ab^2}{6a^2b}\right)^2$

Graph each inequality. (Lesson 5-8)

63. $y > x^2 - 4x + 6$ **64.** $y \leq -x^2 + 6x - 3$ **65.** $y < x^2 - 2x$

Determine whether each function has a maximum or a minimum value. Then find the maximum or minimum value of each function. (Lesson 5-1)

66. $f(x) = x^2 - 8x + 3$ **67.** $f(x) = -3x^2 - 18x + 5$ **68.** $f(x) = -7 + 4x^2$

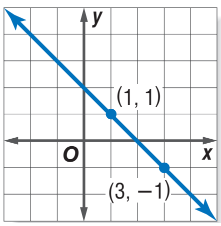
Use matrices A, B, C, and D to find the following. (Lesson 4-2)

$$A = \begin{bmatrix} -4 & 4 \\ 2 & -3 \\ 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 6 & -2 \end{bmatrix} \quad C = \begin{bmatrix} -4 & -5 \\ -3 & 1 \\ 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ -3 & 4 \end{bmatrix}$$

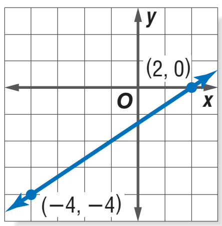
69. $A + D$ **70.** $B - C$ **71.** $3B - 2A$

Write an equation in slope-intercept form for each graph. (Lesson 2-4)

72.



73.



74. In 1990, 2,573,225 people attended St. Louis Cardinals home games. In 2004, the attendance was 3,048,427. What was the average annual rate of increase in attendance?

GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify. Assume that no variable equals 0.** (Lesson 6-1)

75. $\frac{x^3}{x}$ **76.** $\frac{4y^5}{2y^2}$ **77.** $\frac{x^2y^3}{xy}$ **78.** $\frac{9a^3b}{3ab}$

6-3 Dividing Polynomials

Main Ideas

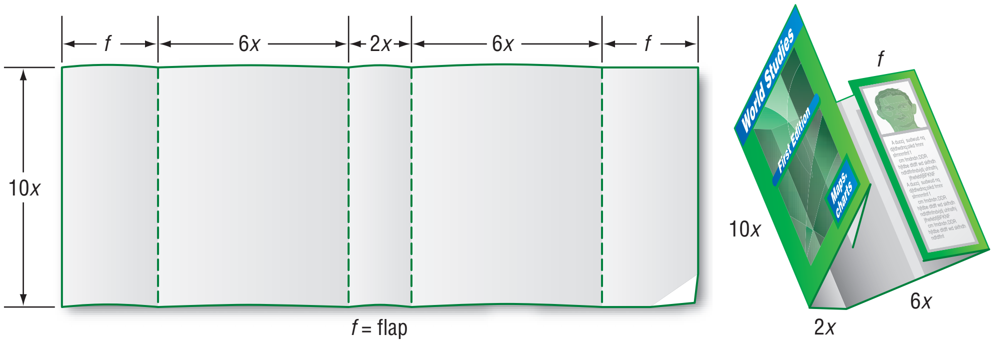
- Divide polynomials using long division.
- Divide polynomials using synthetic division.

New Vocabulary

synthetic division

GET READY for the Lesson

Arianna needed $140x^2 + 60x$ square inches of paper to make a book jacket $10x$ inches tall. In figuring the area she needed, she allowed for a front and back flap. If the spine of the book jacket is $2x$ inches, and the front and back of the book jacket are $6x$ inches, how wide are the front and back flaps? You can use a quotient of polynomials to help you find the answer.



Use Long Division In Lesson 6-1, you learned to divide monomials. You can divide a polynomial by a monomial by using those same skills.

EXAMPLE 1 Divide a Polynomial by a Monomial

Simplify $\dfrac{4x^3y^2 + 8xy^2 - 12x^2y^3}{4xy}$.

$$\frac{4x^3y^2 + 8xy^2 - 12x^2y^3}{4xy} = \frac{4x^3y^2}{4xy} + \frac{8xy^2}{4xy} - \frac{12x^2y^3}{4xy} \quad \text{Sum of quotients}$$

$$= \frac{4}{4} \cdot x^{3-1}y^{2-1} + \frac{8}{4} \cdot x^{1-1}y^{2-1} - \frac{12}{4} \cdot x^{2-1}y^{3-1} \quad \text{Divide.}$$

$$= x^2y + 2y - 3xy^2 \quad x^{1-1} = x^0 \text{ or } 1$$

CHECK Your Progress **Simplify.**

1A. $\dfrac{9x^2y^3 - 15xy^2 + 12xy^3}{3xy^2}$

1B. $\dfrac{16a^5b^3 + 12a^3b^4 - 20ab^5}{4ab^3}$

1C. $(20c^4d^2f - 16cf + 4cdf)(4cdf)^{-1}$

1D. $(18x^2y + 27x^3y^2z)(3xy)^{-2}$

You can use a process similar to long division to divide a polynomial by a polynomial with more than one term. The process is known as the *division algorithm.* When doing the division, remember that you can only add or subtract like terms.

EXAMPLE Division Algorithm

2 **Use long division to find $(z^2 + 2z - 24) \div (z - 4)$.**

$$\begin{array}{r|l}
& z \\
z - 4 & \overline{z^2 + 2z - 24} \\
& \underline{(-)z^2 - 4z} \quad z(z-4) = z^2 - 4z \\
& \quad 6z - 24 \quad 2z - (-4z) = 6z
\end{array}
\qquad
\begin{array}{r|l}
& z + 6 \\
z - 4 & \overline{z^2 + 2z - 24} \\
& \underline{(-)z^2 - 4z} \\
& \quad 6z - 24 \\
& \quad \underline{(-)6z - 24} \\
& \qquad 0
\end{array}$$

The quotient is $z + 6$. The remainder is 0.

CHECK Your Progress

Use long division to find each quotient.

2A. $(x^2 + 7x - 30) \div (x - 3)$ **2B.** $(x^2 - 13x + 12) \div (x - 1)$

Just as with the division of whole numbers, the division of two polynomials may result in a quotient with a remainder. Remember that $9 \div 4 = 2 + \text{R}1$ and is often written as $2\frac{1}{4}$. The result of a division of polynomials with a remainder can be written in a similar manner.

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STANDARDIZED TEST EXAMPLE Quotient with Remainder

3 Which expression is equal to $(t^2 + 3t - 9)(5 - t)^{-1}$?

A $t + 8 - \frac{31}{5 - t}$

B $-t - 8$

C $-t - 8 + \frac{31}{5 - t}$

D $-t - 8 - \frac{31}{5 - t}$

Read the Test Item

Since the second factor has an exponent of -1, this is a division problem.

$(t^2 + 3t - 9)(5 - t)^{-1} = \frac{t^2 + 3t - 9}{5 - t}$

Solve the Test Item

$$\begin{array}{r|ll}
& -t - 8 & \\
-t + 5 & \overline{t^2 + 3t - 9} & \text{For ease in dividing, rewrite } 5 - t \text{ as } -t + 5. \\
& \underline{(-)t^2 - 5t} & -t(-t + 5) = t^2 - 5t \\
& \quad 8t - 9 & 3t - (-5t) = 8t \\
& \quad \underline{(-)8t - 40} & -8(-t + 5) = 8t - 40 \\
& \qquad 31 & \text{Subtract. } -9 - (-40) = 31
\end{array}$$

The quotient is $-t - 8$, and the remainder is 31. Therefore, $(t^2 + 3t - 9)(5 - t)^{-1} = -t - 8 + \frac{31}{5 - t}$. The answer is C.

Test-Taking Tip

You may be able to eliminate some of the answer choices by substituting the same value for t in the original expression and the answer choices and evaluating.

CHECK Your Progress

3. Which expression is equal to $(r^2 + 5r + 7)(1 - r)^{-1}$?

F $-r - 6 + \frac{13}{1 - r}$ **G** $r + 6$ **H** $r - 6 + \frac{13}{1 - r}$ **J** $r + 6 - \frac{13}{1 - r}$

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Use Synthetic Division Synthetic division is a simpler process for dividing a polynomial by a binomial. Suppose you want to divide $5x^3 - 13x^2 + 10x - 8$ by $x - 2$ using long division. Compare the coefficients in this division with those in Example 4.

$$
\begin{array}{r}
5x^2 - 3x + 4 \\
x - 2\,\overline{)\,5x^3 - 13x^2 + 10x - 8} \\
\underline{(-)5x^3 - 10x^2} \qquad\qquad \\
-3x^2 + 10x \qquad \\
\underline{(-)-3x^2 + \ \ 6x} \qquad \\
4x - 8 \\
\underline{(-)4x - 8} \\
0
\end{array}
$$

EXAMPLE 4 Synthetic Division

Use synthetic division to find $(5x^3 - 13x^2 + 10x - 8) \div (x - 2)$.

Step 1 Write the terms of the dividend so that the degrees of the terms are in descending order. Then write just the coefficients as shown at the right.

$$
\begin{array}{cccc}
5x^3 & -13x^2 & +10x & -8 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
5 & -13 & 10 & -8
\end{array}
$$

Step 2 Write the constant r of the divisor $x - r$ to the left. In this case, $r = 2$. Bring the first coefficient, 5, down.

$$
\begin{array}{r|rrrr}
2 & 5 & -13 & 10 & -8 \\
\hline
 & 5 & & &
\end{array}
$$

Step 3 Multiply the first coefficient by r: $2 \cdot 5 = 10$. Write the product under the second coefficient. Then add the product and the second coefficient: $-13 + 10 = -3$.

$$
\begin{array}{r|rrrr}
2 & 5 & -13 & 10 & -8 \\
 & & 10 & & \\
\hline
 & 5 & -3 & &
\end{array}
$$

Step 4 Multiply the sum, -3, by r: $2(-3) = -6$. Write the product under the next coefficient and add: $10 + (-6) = 4$.

$$
\begin{array}{r|rrrr}
2 & 5 & -13 & 10 & -8 \\
 & & 10 & -6 & \\
\hline
 & 5 & -3 & 4 &
\end{array}
$$

Step 5 Multiply the sum, 4, by r: $2 \cdot 4 = 8$. Write the product under the next coefficient and add: $-8 + 8 = 0$. The remainder is 0.

$$
\begin{array}{r|rrrr}
2 & 5 & -13 & 10 & -8 \\
 & & 10 & -6 & 8 \\
\hline
 & 5 & -3 & 4 & 0
\end{array}
$$

The numbers along the bottom row are the coefficients of the quotient. Start with the power of x that is one less than the degree of the dividend. Thus, the quotient is $5x^2 - 3x + 4$.

CHECK Your Progress **Use synthetic division to find each quotient.**

4A. $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$ **4B.** $(3x^3 - 8x^2 + 11x - 14) \div (x - 2)$

To use synthetic division, the divisor must be of the form $x - r$. If the coefficient of x in a divisor is not 1, you can rewrite the division expression so that you can use synthetic division.

EXAMPLE 5 Divisor with First Coefficient Other than 1

Use synthetic division to find $(8x^4 - 4x^2 + x + 4) \div (2x + 1)$.

Use division to rewrite the divisor so it has a first coefficient of 1.

$$\frac{8x^4 - 4x^2 + x + 4}{2x + 1} = \frac{(8x^4 - 4x^2 + x + 4) \div 2}{(2x + 1) \div 2} \qquad \text{Divide numerator and denominator by 2.}$$

$$= \frac{4x^4 - 2x^2 + \frac{1}{2}x + 2}{x + \frac{1}{2}} \qquad \text{Simplify the numerator and denominator.}$$

(continued on the next page)

Since the numerator does not have an x^3-term, use a coefficient of 0 for x^3.
$x - r = x + \frac{1}{2}$, so $r = -\frac{1}{2}$.

$$\begin{array}{r|rrrrr}
-\frac{1}{2} & 4 & 0 & -2 & \frac{1}{2} & 2 \\
 & & -2 & 1 & \frac{1}{2} & -\frac{1}{2} \\
\hline
 & 4 & -2 & -1 & 1 & \frac{3}{2}
\end{array}$$

The result is $4x^3 - 2x^2 - x + 1 + \dfrac{\frac{3}{2}}{x + \frac{1}{2}}$. Now simplify the fraction.

$$\frac{\frac{3}{2}}{x + \frac{1}{2}} = \frac{3}{2} \div \left(x + \frac{1}{2}\right) \quad \text{Rewrite as a division expression.}$$

$$= \frac{3}{2} \div \frac{2x + 1}{2} \quad x + \frac{1}{2} = \frac{2x}{2} + \frac{1}{2} = \frac{2x + 1}{2}$$

$$= \frac{3}{2} \cdot \frac{2}{2x + 1} \quad \text{Multiply by the reciprocal.}$$

$$= \frac{3}{2x + 1}$$

The solution is $4x^3 - 2x^2 - x + 1 + \dfrac{3}{2x + 1}$.

CHECK Divide using long division.

$$\begin{array}{r}
4x^3 - 2x^2 - x + 1 \\
2x + 1 \overline{\smash{\big)}\, 8x^4 + 0x^3 - 4x^2 + x + 4} \\
\underline{(-)8x^4 + 4x^3} \qquad\qquad\qquad \\
-4x^3 - 4x^2 \qquad\quad \\
\underline{(-)-4x^3 - 2x^2} \qquad\quad \\
-2x^2 + x \quad \\
\underline{(-)-2x^2 - x} \quad \\
2x + 4 \\
\underline{(-)2x + 1} \\
3
\end{array}$$

The result is $4x^3 - 2x^2 - x + 1 + \dfrac{3}{2x + 1}$. ✓

CHECK Your Progress **Use synthetic division to find each quotient.**

5A. $(3x^4 - 5x^3 + x^2 + 7x) \div (3x + 1)$ **5B.** $(8y^5 - 2y^4 - 16y^2 + 4) \div (4y - 1)$

CHECK Your Understanding

Example 1 (p. 325)

Simplify.

1. $\dfrac{6xy^2 - 3xy + 2x^2y}{xy}$

2. $(5ab^2 - 4ab + 7a^2b)(ab)^{-1}$

3. BAKING The number of cookies produced in a factory each day can be estimated by $C(w) = -w^2 + 16w + 1000$, where w is the number of workers and C is the number of cookies produced. Divide to find the average number of cookies produced per worker.

Examples 2, 4 (pp. 326–327)

Simplify.

4. $(x^2 - 10x - 24) \div (x + 2)$

5. $(3a^4 - 6a^3 - 2a^2 + a - 6) \div (a + 1)$

6. $(z^5 - 3z^2 - 20) \div (z - 2)$

7. $(x^3 + y^3) \div (x + y)$

8. $\dfrac{x^3 + 13x^2 - 12x - 8}{x + 2}$

9. $(b^4 - 2b^3 + b^2 - 3b + 4)(b - 2)^{-1}$

Example 3 (p. 326)

10. STANDARDIZED TEST PRACTICE Which expression is equal to $(x^2 - 4x + 6)(x - 3)^{-1}$?

A $x - 1$ **B** $x - 1 + \frac{3}{x - 3}$ **C** $x - 1 - \frac{3}{x - 3}$ **D** $-x + 1 - \frac{3}{x - 3}$

Example 5 (pp. 327–328)

Simplify.

11. $(12y^2 + 36y + 15) \div (6y + 3)$

12. $\frac{9b^2 + 9b - 10}{3b - 2}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
13–16	1
17–22	2, 4
23–28	3, 4
29–34	2, 3, 5

Simplify.

13. $\frac{9a^3b^2 - 18a^2b^3}{3a^2b}$

14. $\frac{5xy^2 - 6y^3 + 3x^2y^3}{xy}$

15. $(28c^3d - 42cd^2 + 56cd^3) \div (14cd)$

16. $(a^3b^2 - a^2b + 2a)(-ab)^{-1}$

17. $(x^3 - 4x^2) \div (x - 4)$

18. $(x^3 - 27) \div (x - 3)$

19. $(b^3 + 8b^2 - 20b) \div (b - 2)$

20. $(g^2 + 8g + 15)(g + 3)^{-1}$

21. $\frac{y^3 + 3y^2 - 5y - 4}{y + 4}$

22. $\frac{m^3 + 3m^2 - 7m - 21}{m + 3}$

23. $(t^5 - 3t^2 - 20)(t - 2)^{-1}$

24. $(y^5 + 32)(y + 2)^{-1}$

25. $(2c^3 - 3c^2 + 3c - 4) \div (c - 2)$

26. $(2b^3 + b^2 - 2b + 3)(b + 1)^{-1}$

27. $\frac{x^5 - 7x^3 + x + 1}{x + 3}$

28. $\frac{3c^5 + 5c^4 + c + 5}{c + 2}$

29. $\frac{4x^3 + 5x^2 - 3x + 1}{4x + 1}$

30. $\frac{x^3 - 3x^2 + x - 3}{x^2 + 1}$

31. $(6t^3 + 5t^2 + 9) \div (2t + 3)$

32. $\frac{x^4 + x^2 - 3x + 5}{x^2 + 2}$

33. $\frac{2x^4 + 3x^3 - 2x^2 - 3x - 6}{2x + 3}$

34. $\frac{6x^4 + 5x^3 + x^2 - 3x + 1}{3x + 1}$

35. ENTERTAINMENT A magician gives these instructions to a volunteer.
- Choose a number and multiply it by 4.
- Then add the sum of your number and 15 to the product you found.
- Now divide by the sum of your number and 3.

What number will the volunteer always have at the end? Explain.

Real-World Career

Cost Analyst

Cost analysts study and write reports about the factors involved in the cost of production.

Math Online

For more information, go to algebra2.com.

BUSINESS For Exercises 36 and 37, use the following information.

The number of sports magazines sold can be estimated by $n = \frac{3500a^2}{a^2 + 100}$, where a is the amount of money spent on advertising in hundreds of dollars and n is the number of subscriptions sold.

36. Perform the division indicated by $\frac{3500a^2}{a^2 + 100}$.

37. About how many subscriptions will be sold if \$1500 is spent on advertising?

PHYSICS For Exercises 38–40, suppose an object moves in a straight line so that, after t seconds, it is $t^3 + t^2 + 6t$ feet from its starting point.

38. Find the distance the object travels between the times $t = 2$ and $t = x$, where $x > 2$.

39. How much time elapses between $t = 2$ and $t = x$?

40. Find a simplified expression for the average speed of the object between times $t = 2$ and $t = x$.

H.O.T. Problems

EXTRA PRACTICE
See pages 903, 931.
Math Online
Self-Check Quiz at
algebra2.com

41. OPEN ENDED Write a quotient of two polynomials such that the remainder is 5.

42. REASONING Review any of the division problems in this lesson. What is the relationship between the degrees of the dividend, the divisor, and the quotient?

43. FIND THE ERROR Shelly and Jorge are dividing $x^3 - 2x^2 + x - 3$ by $x - 4$. Who is correct? Explain your reasoning.

Shelly

$$\begin{array}{r|rrrr} 4 & 1 & -2 & 1 & -3 \\ & & 4 & -24 & 100 \\ \hline & 1 & -6 & 25 & \vert\, -103 \end{array}$$

Jorge

$$\begin{array}{r|rrrr} 4 & 1 & -2 & 1 & -3 \\ & & 4 & 8 & 36 \\ \hline & 1 & 2 & 9 & \vert\, 33 \end{array}$$

44. CHALLENGE Suppose the result of dividing one polynomial by another is $r^2 - 6r + 9 - \frac{1}{r - 3}$. What two polynomials might have been divided?

45. *Writing in Math* Use the information on page 325 to explain how you can use division of polynomials in manufacturing. Include the dimensions of the piece of paper that the publisher needs, the formula from geometry that applies to this situation, and an explanation of how to use division of polynomials to find the width of the flap.

STANDARDIZED TEST PRACTICE

46. ACT/SAT What is the remainder when $x^3 - 7x + 5$ is divided by $x + 3$?

A -11 **C** 1

B -1 **D** 11

47. REVIEW If $i = \sqrt{-1}$, then $5i(7i) =$

F 70 **H** -35

G 35 **J** -70

Spiral Review

Simplify. (Lesson 6-2)

48. $(2x^2 - 3x + 5) - (3x^2 + x - 9)$

49. $y^2z(y^2z^3 - yz^2 + 3)$

50. $(y + 5)(y - 3)$

51. $(a - b)^2$

52. ASTRONOMY Earth is an average of 1.5×10^{11} meters from the Sun. Light travels at 3×10^8 meters per second. About how long does it take sunlight to reach Earth? (Lesson 6-1)

GET READY for the Next Lesson

PREREQUISITE SKILL Given $f(x) = x^2 - 5x + 6$, find each value. (Lesson 2-1)

53. $f(-2)$

54. $f(2)$

55. $f(2a)$

56. $f(a + 1)$

6-4 Polynomial Functions

Main Ideas

- Evaluate polynomial functions.
- Identify general shapes of graphs of polynomial functions.

New Vocabulary

polynomial in one variable
leading coefficient
polynomial function
end behavior

GET READY for the Lesson

A cross section of a honeycomb has a pattern with one hexagon surrounded by six more hexagons. Surrounding these is a third ring of 12 hexagons, and so on. The total number of hexagons in a honeycomb can be modeled by the function $f(r) = 3r^2 - 3r + 1$, where r is the number of rings and $f(r)$ is the number of hexagons.

Polynomial Functions The expression $3r^2 - 3r + 1$ is a **polynomial in one variable** since it only contains one variable, r.

KEY CONCEPT — Polynomial in One Variable

Words A polynomial of degree n in one variable x is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ represent real numbers, a_n is not zero, and n represents a nonnegative integer.

Example $3x^5 + 2x^4 - 5x^3 + x^2 + 1$
$n = 5, a_5 = 3, a_4 = 2, a_3 = -5, a_2 = 1, a_1 = 0$, and $a_0 = 1$

The degree of a polynomial in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

Polynomial	Expression	Degree	Leading Coefficient
Constant	9	0	9
Linear	$x - 2$	1	1
Quadratic	$3x^2 + 4x - 5$	2	3
Cubic	$4x^3 - 6$	3	4
General	$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$	n	a_n

EXAMPLE 1 Find Degrees and Leading Coefficients

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. $7x^4 + 5x^2 + x - 9$

This is a polynomial in one variable.

The degree is 4, and the leading coefficient is 7.

(continued on the next page)

b. $8x^2 + 3xy - 2y^2$

This is not a polynomial in one variable. It contains two variables, x and y.

CHECK Your Progress

1A. $7x^6 - 4x^3 + \frac{1}{x}$ **1B.** $\frac{1}{2}x^2 + 2x^3 - x^5$

A polynomial equation used to represent a function is called a **polynomial function**. For example, the equation $f(x) = 4x^2 - 5x + 2$ is a quadratic polynomial function, and the equation $p(x) = 2x^3 + 4x^2 - 5x + 7$ is a cubic polynomial function. Other polynomial functions can be defined by the following general rule.

KEY CONCEPT *Definition of a Polynomial Function*

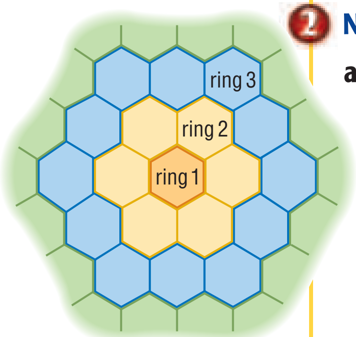
Words A polynomial function of degree n is a continuous function that can be described by an equation of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ represent real numbers, a_n is not zero, and n represents a nonnegative integer.

Example $f(x) = 4x^2 - 3x + 2$
$n = 2, a_2 = 4, a_1 = -3, a_0 = 2$

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Recall that $f(3)$ can be found by evaluating the function for $x = 3$.

Real-World EXAMPLE

2 **NATURE** **Refer to the application at the beginning of the lesson.**



Rings of a Honeycomb

a. Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 1$, 2, and 3.

Find the values of $f(1)$, $f(2)$, and $f(3)$.

$$\begin{aligned} f(r) &= 3r^2 - 3r + 1 \\ f(1) &= 3(1)^2 - 3(1) + 1 \\ &= 3 - 3 + 1 \text{ or } 1 \end{aligned} \qquad \begin{aligned} f(r) &= 3r^2 - 3r + 1 \\ f(2) &= 3(2)^2 - 3(2) + 1 \\ &= 12 - 6 + 1 \text{ or } 7 \end{aligned} \qquad \begin{aligned} f(r) &= 3r^2 - 3r + 1 \\ f(3) &= 3(3)^2 - 3(3) + 1 \\ &= 27 - 9 + 1 \text{ or } 19 \end{aligned}$$

You know the numbers of hexagons in the first three rings are 1, 6, and 12. So, the total number of hexagons with one ring is 1, two rings is 6 + 1 or 7, and three rings is 12 + 6 + 1 or 19. These match the functional values for $r = 1$, 2, and 3, respectively. That is 1, 7, and 19 are the range values corresponding to the domain values of 1, 2, and 3.

b. Find the total number of hexagons in a honeycomb with 12 rings.

$f(r) = 3r^2 - 3r + 1$ Original function

$f(12) = 3(12)^2 - 3(12) + 1$ Replace r with 12.

$= 432 - 36 + 1$ or 397 Simplify.

CHECK Your Progress

2A. Show that $f(r)$ gives the total number of hexagons when $r = 4$.

2B. Find the total number of hexagons in a honeycomb with 20 rings.

You can also evaluate functions for variables and algebraic expressions.

EXAMPLE Function Values of Variables

3 **Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.**

To evaluate $q(a + 1)$, replace x in $q(x)$ with $a + 1$.

$q(x) = x^2 + 3x + 4$	Original function
$q(a + 1) = (a + 1)^2 + 3(a + 1) + 4$	Replace x with $a + 1$.
$= a^2 + 2a + 1 + 3a + 3 + 4$	Simplify $(a + 1)^2$ and $3(a + 1)$.
$= a^2 + 5a + 8$	Simplify.

To evaluate $2q(a)$, replace x with a in $q(x)$, then multiply the expression by 2.

$q(x) = x^2 + 3x + 4$	Original function
$2q(a) = 2(a^2 + 3a + 4)$	Replace x with a.
$= 2a^2 + 6a + 8$	Distributive Property

Now evaluate $q(a + 1) - 2q(a)$.

$q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8)$	Replace $q(a + 1)$ and $2q(a)$.
$= a^2 + 5a + 8 - 2a^2 - 6a - 8$	
$= -a^2 - a$	Simplify.

CHECK Your Progress

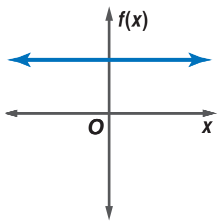
3A. Find $f(b^2)$ if $f(x) = 2x^2 + 3x - 1$.

3B. Find $2g(c + 2) + 3g(2c)$ if $g(x) = x^2 - 4$.

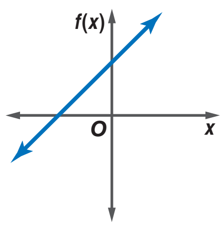
> **Study Tip**
>
> **Function Values**
>
> When finding function values of expressions, be sure to take note of where the coefficients occur. In Example 3, $2q(a)$ is 2 times the function value of a, not $q(2a)$, the function value of $2a$.

Graphs of Polynomial Functions The general shapes of the graphs of several polynomial functions are shown below. These graphs show the *maximum* number of times the graph of each type of polynomial may intersect the x-axis. Recall that the x-coordinate of the point at which the graph intersects the x-axis is called a *zero* of a function. How does the degree compare to the maximum number of real zeros?

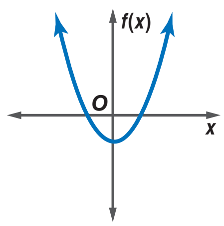
Constant function
Degree 0



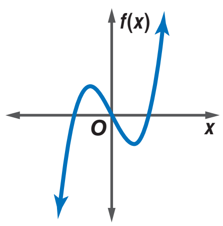
Linear function
Degree 1



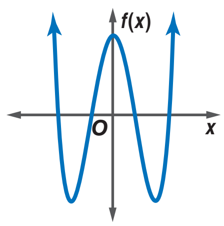
Quadratic function
Degree 2



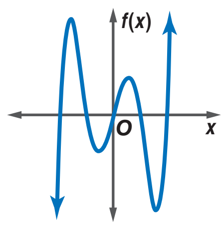
Cubic function
Degree 3



Quartic function
Degree 4



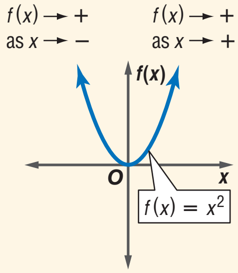
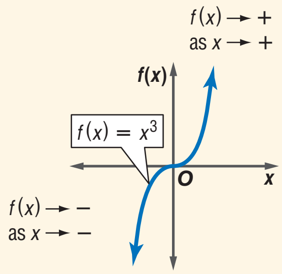
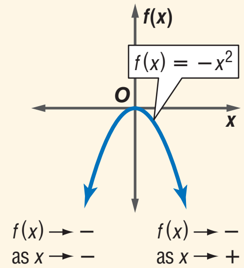
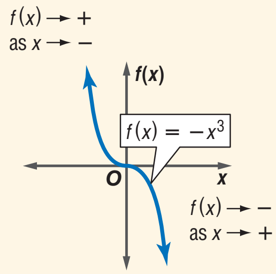
Quintic function
Degree 5



The **end behavior** is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). This is represented as $x \to +\infty$ and $x \to -\infty$, respectively. *$x \to +\infty$ is read x approaches positive infinity.* Notice the shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions. The degree and leading coefficient of a polynomial function determine the graph's end behavior.

Concepts in Motion
Animation
algebra2.com

CONCEPT SUMMARY — *End Behavior of a Polynomial Function*

Degree: even **Leading Coefficient:** positive **End Behavior:**	**Degree:** odd **Leading Coefficient:** positive **End Behavior:**	**Degree:** even **Leading Coefficient:** negative **End Behavior:**	**Degree:** odd **Leading Coefficient:** negative **End Behavior:**
			
Domain: all reals **Range:** all reals $\geq$ minimum	**Domain:** all reals **Range:** all reals	**Domain:** all reals **Range:** all reals $\leq$ maximum	**Domain:** all reals **Range:** all reals

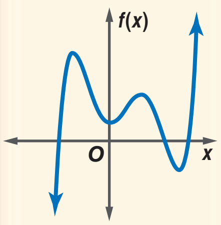
For any polynomial function, the domain is all real numbers. For any polynomial function of odd degree, the range is all real numbers. For polynomial functions of even degree, the range is all real numbers greater than or equal to some number or all real numbers less than or equal to some number; it is never all real numbers.

The graph of an even-degree function may or may not intersect the x-axis. If it intersects the x-axis in two places, the function has two real zeros. If it does not intersect the x-axis, the roots of the related equation are imaginary and cannot be determined from the graph. If the graph is tangent to the x-axis, as shown above, there are two zeros that are the same number. The graph of an odd-degree function always crosses the x-axis at least once, and thus the function always has at least one real zero.

Study Tip

Number of Zeros

The number of zeros of an odd-degree function may be less than the maximum by a multiple of 2. For example, the graph of a quintic function may only cross the x-axis 1, 3, or 5 times.



The same is true for an even-degree function. One exception is when the graph of $f(x)$ touches the x-axis.

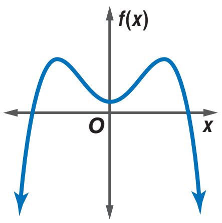
EXAMPLE 4 Graphs of Polynomial Functions

Personal Tutor

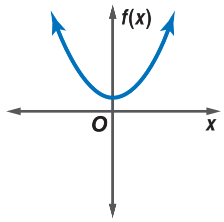
For each graph,

- **describe the end behavior,**
- **determine whether it represents an odd-degree or an even-degree polynomial function, and**
- **state the number of real zeros.**

a.



b.

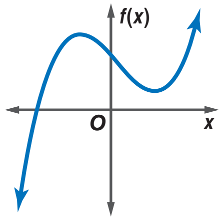


a. • $f(x) \to -\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
   • It is an even-degree polynomial function.
   • The graph intersects the x-axis at two points, so the function has two real zeros.

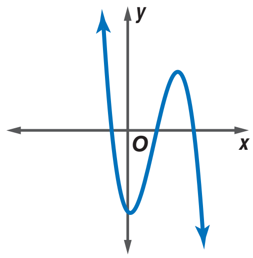
b. • $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to +\infty$ as $x \to -\infty$.
   • It is an even-degree polynomial function.
   • This graph does not intersect the x-axis, so the function has no real zeros.

CHECK Your Progress

4A.



4B.



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CHECK Your Understanding

Example 1 (pp. 331–332)

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. $5x^6 - 8x^2$
2. $2b + 4b^3 - 3b^5 - 7$

Example 2 (p. 332)

Find $p(3)$ and $p(-1)$ for each function.

3. $p(x) = -x^3 + x^2 - x$
4. $p(x) = x^4 - 3x^3 + 2x^2 - 5x + 1$

5. **BIOLOGY** The intensity of light emitted by a firefly can be determined by $L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3$, where t is temperature in degrees Celsius and $L(t)$ is light intensity in lumens. If the temperature is 30°C, find the light intensity.

Example 3 (p. 333)

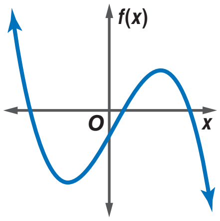
If $p(x) = 2x^3 + 6x - 12$ and $q(x) = 5x^2 + 4$, find each value.

6. $p(a^3)$
7. $5[q(2a)]$
8. $3p(a) - q(a + 1)$

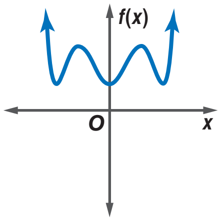
Example 4 (pp. 334–335)

For each graph,
a. describe the end behavior,
b. determine whether it represents an odd-degree or an even-degree polynomial function, and
c. state the number of real zeros.

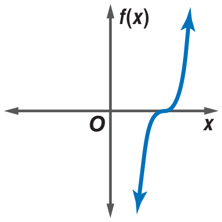
9.



10.



11.



Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–17	1
18–21, 34, 35	2
22–27	3
28–33	4

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

12. $7 - x$

13. $(a + 1)(a^2 - 4)$

14. $a^2 + 2ab + b^2$

15. $c^2 + c - \frac{1}{c}$

16. $6x^4 + 3x^2 + 4x - 8$

17. $7 + 3x^2 - 5x^3 + 6x^2 - 2x$

Find $p(4)$ and $p(-2)$ for each function.

18. $p(x) = 2 - x$

19. $p(x) = x^2 - 3x + 8$

20. $p(x) = 2x^3 - x^2 + 5x - 7$

21. $p(x) = x^5 - x^2$

If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

22. $r(3a)$

23. $4p(a)$

24. $p(a^2)$

25. $p(2a^3)$

26. $r(x + 1)$

27. $p(x^2 + 3)$

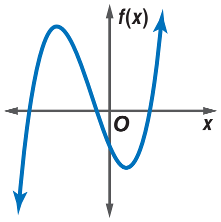
For each graph,

a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.

28.



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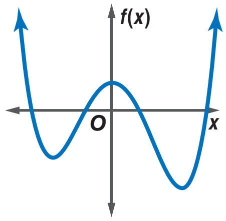

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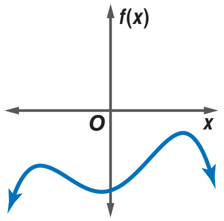

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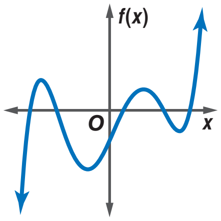

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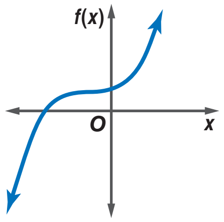

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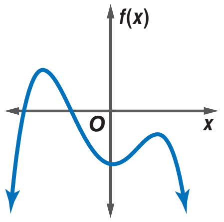


34. ENERGY The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function $P(s) = \frac{s^3}{1000}$, where s represents the speed of the wind in kilometers per hour. Find the units of power $P(s)$ generated by a windmill when the wind speed is 18 kilometers per hour.

35. PHYSICS For a moving object with mass m in kilograms, the kinetic energy KE in joules is given by the function $KE(v) = \frac{1}{2}mv^2$, where v represents the speed of the object in meters per second. Find the kinetic energy of an all-terrain vehicle with a mass of 171 kilograms moving at a speed of 11 meters/second.

Find $p(4)$ and $p(-2)$ for each function.

36. $p(x) = x^4 - 7x^3 + 8x - 6$

37. $p(x) = 7x^2 - 9x + 10$

38. $p(x) = \frac{1}{2}x^4 - 2x^2 + 4$

39. $p(x) = \frac{1}{8}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x + 5$

If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

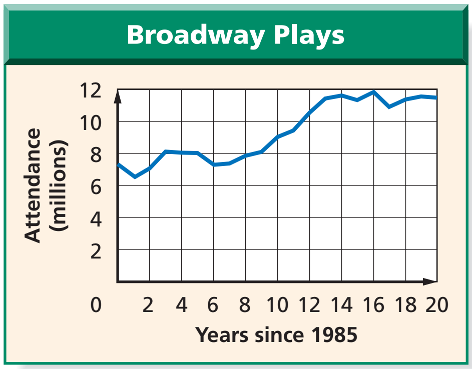
40. $2[p(x + 4)]$ **41.** $r(x + 1) - r(x^2)$ **42.** $3[p(x^2 - 1)] + 4p(x)$

Real-World Link

The Phantom of the Opera is the longest-running Broadway show in history.

Source: playbill.com

THEATER For Exercises 43–45, use the graph that models the attendance at Broadway plays (in millions) from 1985–2005.



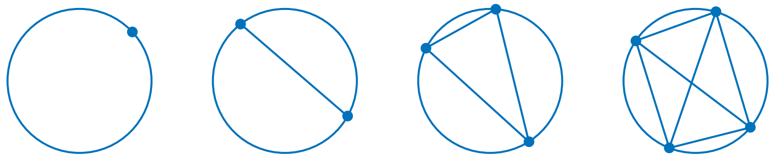
43. Is the graph an odd-degree or even-degree function?

44. Discuss the end behavior.

45. Do you think attendance at Broadway plays will increase or decrease after 2005? Explain your reasoning.

PATTERNS For Exercises 46–48, use the diagrams below that show the maximum number of regions formed by connecting points on a circle.

1 point, 1 region 2 points, 2 regions 3 points, 4 regions 4 points, 8 regions



46. The number of regions formed by connecting n points of a circle can be described by the function $f(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$. What is the degree of this polynomial function?

47. Find the number of regions formed by connecting 5 points of a circle. Draw a diagram to verify your solution.

48. How many points would you have to connect to form 99 regions?

EXTRA PRACTICE

See pages 903, 931.

Math Online

Self-Check Quiz at algebra2.com

H.O.T. Problems

49. REASONING Explain why a constant polynomial such as $f(x) = 4$ has degree 0 and a linear polynomial such as $f(x) = x + 5$ has degree 1.

50. OPEN ENDED Sketch the graph of an odd-degree polynomial function with a negative leading coefficient and three real roots.

51. REASONING Determine whether the following statement is *always*, *sometimes* or *never* true. Explain.

A polynomial function that has four real roots is a fourth-degree polynomial.

CHALLENGE For Exercises 52–55, use the following information.

The graph of the polynomial function $f(x) = ax(x - 4)(x + 1)$ goes through the point at $(5, 15)$.

52. Find the value of a.

53. For what value(s) of x will $f(x) = 0$?

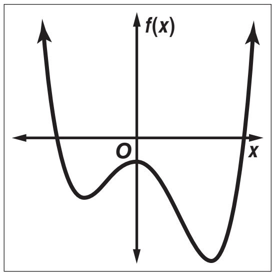
54. Simplify and rewrite the function as a cubic function.

55. Sketch the graph of the function.

56. ***Writing in Math*** Use the information on page 331 to explain where polynomial functions are found in nature. Include an explanation of how you could use the equation to find the number of hexagons in the tenth ring and any other examples of patterns found in nature that might be modeled by a polynomial equation.

STANDARDIZED TEST PRACTICE

57. ACT/SAT The figure at the right shows the graph of a polynomial function $f(x)$. Which of the following could be the degree of $f(x)$?



A 2
B 3
C 4
D 5

58. REVIEW Which polynomial represents $(4x^2 + 5x - 3)(2x - 7)$?

F $8x^3 - 18x^2 - 41x - 21$
G $8x^3 + 18x^2 + 29x - 21$
H $8x^3 - 18x^2 - 41x + 21$
J $8x^3 + 18x^2 - 29x + 21$

Spiral Review

Simplify. (Lesson 6-3)

59. $(t^3 - 3t + 2) \div (t + 2)$

60. $(y^2 + 4y + 3)(y + 1)^{-1}$

61. $\dfrac{x^3 - 3x^2 + 2x - 6}{x - 3}$

62. $\dfrac{3x^4 + x^3 - 8x^2 + 10x - 3}{3x - 2}$

63. BUSINESS Ms. Schifflet is writing a computer program to find the salaries of her employees after their annual raise. The percent of increase is represented by p. Marty's salary is \$23,450 now. Write a polynomial to represent Marty's salary in one year and another to represent Marty's salary after three years. Assume that the rate of increase will be the same for each of the three years. (Lesson 6-2)

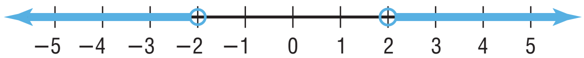
Solve each equation by completing the square. (Lesson 5-5)

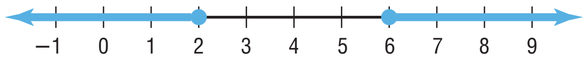
64. $x^2 - 8x - 2 = 0$

65. $x^2 + \frac{1}{3}x - \frac{35}{36} = 0$

Write an absolute value inequality for each graph. (Lesson 1-6)

66.


67.


68.


69.


Name the property illustrated by each statement. (Lesson 1-3)

70. If $3x = 4y$ and $4y = 15z$, then $3x = 15z$.

71. $5y(4a - 6b) = 20ay - 30by$

72. $2 + (3 + x) = (2 + 3) + x$

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each equation by making a table of values.** (Lesson 5-1)

73. $y = x^2 + 4$

74. $y = -x^2 + 6x - 5$

75. $y = \frac{1}{2}x^2 + 2x - 6$

6-5 Analyzing Graphs of Polynomial Functions

Main Ideas

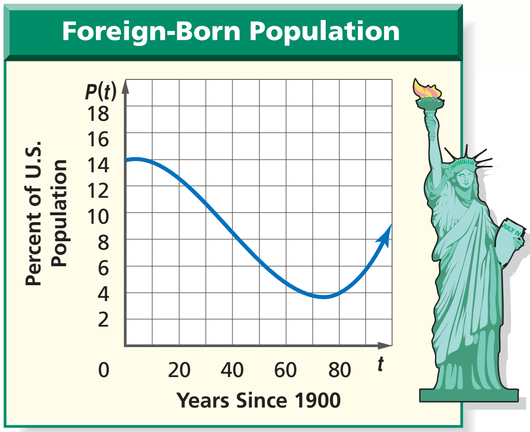
- Graph polynomial functions and locate their real zeros.
- Find the relative maxima and minima of polynomial functions.

New Vocabulary

Location Principle
relative maximum
relative minimum

GET READY for the Lesson

The percent of the United States population that was foreign-born since 1900 can be modeled by $P(t) = 0.00006t^3 - 0.007t^2 + 0.05t + 14$, where $t = 0$ in 1900. Notice that the graph is decreasing from $t = 5$ to $t = 75$ and then it begins to increase. The points at $t = 5$ and $t = 75$ are turning points in the graph.



Graph Polynomial Functions To graph a polynomial function, make a table of values to find several points and then connect them to make a smooth continuous curve. Knowing the end behavior of the graph will assist you in completing the sketch of the graph.

Study Tip

Graphing Polynomial Functions

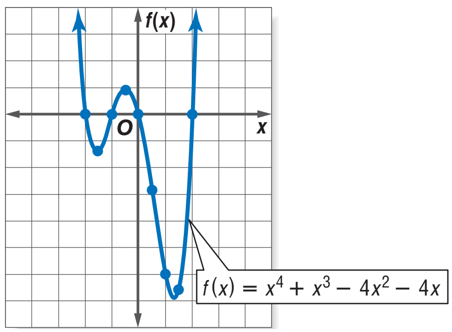
To graph polynomial functions it will often be necessary to include *x*-values that are not integers.

EXAMPLE 1 Graph a Polynomial Function

Graph $f(x) = x^4 + x^3 - 4x^2 - 4x$ by making a table of values.

x	$f(x)$
−2.5	≈ 8.4
−2.0	0.0
−1.5	≈ −1.3
−1.0	0.0
−0.5	≈ 0.9

x	$f(x)$
0.0	0.0
0.5	≈ −2.8
1.0	−6.0
1.5	≈ −6.6
2.0	0.0



This is an even-degree polynomial with a positive leading coefficient, so $f(x) \to +\infty$ as $x \to +\infty$, and $f(x) \to +\infty$ as $x \to -\infty$. Notice that the graph intersects the *x*-axis at four points, indicating there are four real zeros of this function.

CHECK Your Progress

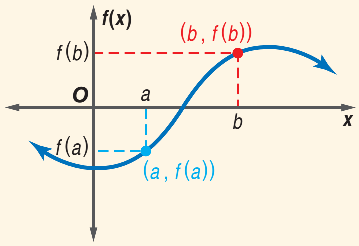
1. Graph $f(x) = x^4 - x^3 - x^2 + x$ by making a table of values.

In Example 1, the zeros occur at integral values that can be seen in the table used to plot the function. Notice that the values of the function before and after each zero are different in sign. In general, because it is a continuous function, the graph of a polynomial function will cross the x-axis somewhere between pairs of x-values at which the corresponding $f(x)$-values change signs. Since zeros of the function are located at x-intercepts, there is a zero between each pair of these x-values. This property for locating zeros is called the **Location Principle**.

KEY CONCEPT — *Location Principle*

Words Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.

Model

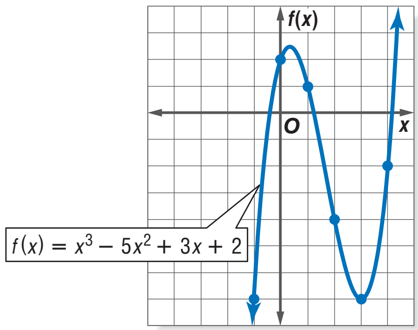


EXAMPLE 2 Locate Zeros of a Function

Determine consecutive integer values of x between which each real zero of the function $f(x) = x^3 - 5x^2 + 3x + 2$ is located. Then draw the graph.

Make a table of values. Since $f(x)$ is a third-degree polynomial function, it will have either 1, 2, or 3 real zeros. Look at the values of $f(x)$ to locate the zeros. Then use the points to sketch a graph of the function.

x	$f(x)$	
−2	−32	
−1	−7	change in sign
0	2	
1	1	change in sign
2	−4	
3	−7	
4	−2	change in sign
5	17	



The changes in sign indicate that there are zeros between $x = -1$ and $x = 0$, between $x = 1$ and $x = 2$, and between $x = 4$ and $x = 5$.

CHECK Your Progress

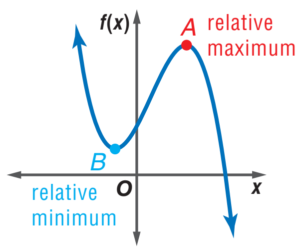
2. Determine consecutive integer values of x between which each real zero of the function $f(x) = x^3 + 4x^2 - 6x - 7$ is located. Then draw the graph.

Concepts in Motion
Animation
algebra2.com

Reading Math

Maximum and Minimum The plurals of *maximum* and *minimum* are *maxima* and *minima.*

Maximum and Minimum Points The graph at the right shows the shape of a general third-degree polynomial function.



Point A on the graph is a **relative maximum** of the cubic function since no other nearby points have a greater y-coordinate. Likewise, point B is a **relative minimum** since no other nearby points have a lesser y-coordinate. These points are often referred to as *turning points*. The graph of a polynomial function of degree n has at most $n - 1$ turning points.

EXAMPLE 3 Maximum and Minimum Points

3 Graph $f(x) = x^3 - 3x^2 + 5$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

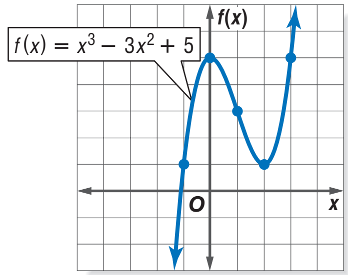
Make a table of values and graph the equation.

x	$f(x)$
−2	−15
−1	1
0	5
1	3
2	1
3	5

zero between $x = -2$ and $x = -1$

← indicates a relative maximum

← indicates a relative minimum



Look at the table of values and the graph.

- The values of $f(x)$ change signs between $x = -2$ and $x = -1$, indicating a zero of the function.
- The value of $f(x)$ at $x = 0$ is greater than the surrounding points, so it is a relative maximum.
- The value of $f(x)$ at $x = 2$ is less than the surrounding points, so it is a relative minimum.

CHECK Your Progress

3. Graph $f(x) = x^3 + 4x^2 - 3$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

The graph of a polynomial function can reveal trends in real-world data.

Real-World Link

Gasoline and diesel fuels are the most familiar transportation fuels in this country, but other energy sources are available, including ethanol, a grain alcohol that can be produced from corn or other crops.

Source: U.S. Environmental Protection Agency

Real-World EXAMPLE 4 Graph a Polynomial Model

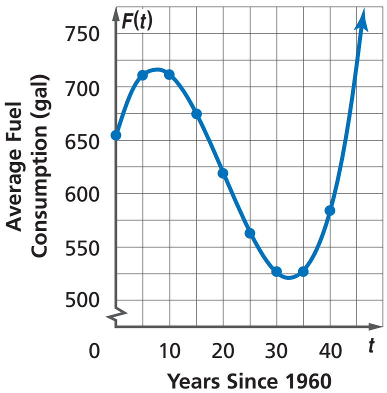
Personal Tutor

ENERGY The average fuel (in gallons) consumed by individual vehicles in the United States from 1960 to 2000 is modeled by the cubic equation $F(t) = 0.025t^3 - 1.5t^2 + 18.25t + 654$, where t is the number of years since 1960.

a. Graph the equation.

Make a table of values for the years 1960–2000. Plot the points and connect with a smooth curve. Finding and plotting the points for every fifth year gives a good approximation of the graph.

t	$F(t)$
0	654
5	710.88
10	711.5
15	674.63
20	619
25	563.38
30	526.5
35	527.13
40	584



(continued on the next page)

b. Describe the turning points of the graph and its end behavior.

There is a relative maximum between 1965 and 1970 and a relative minimum between 1990 and 1995. For the end behavior, as t increases, $F(t)$ increases.

c. What trends in fuel consumption does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

Average fuel consumption hit a maximum point around 1970 and then started to decline until 1990. Since 1990, fuel consumption has risen and continues to rise. The trend may continue for some years, but it is unlikely that consumption will rise this quickly indefinitely. Fuel supplies will limit consumption.

CHECK Your Progress

4. The price of one share of stock of a company is given by the function $f(x) = 0.001x^4 - 0.03x^3 + 0.15x^2 + 1.01x + 18.96$, where x is the number of months since January 2006. Graph the equation. Describe the turning points of the graph and its end behavior. What trends in the stock price does the graph suggest? Is it reasonable to assume the trend will continue indefinitely?

Online Personal Tutor at algebra2.com

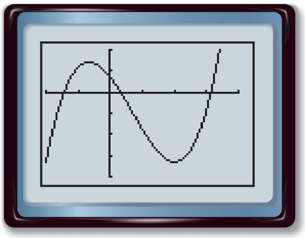
A graphing calculator can be helpful in finding the relative maximum and relative minimum of a function.

GRAPHING CALCULATOR LAB

Maximum and Minimum Points

You can use a TI-83/84 Plus to find the coordinates of relative maxima and relative minima. Enter the polynomial function in the Y= list and graph the function. Make sure that all the turning points are visible in the viewing window. Find the coordinates of the minimum and maximum points, respectively.

The graphing calculator screen at the right shows one relative maximum and one relative minimum of the function that is graphed.



KEYSTROKES: *Refer to page 243 to review finding maxima and minima.*

THINK AND DISCUSS

1. Graph $f(x) = x^3 - 3x^2 + 4$. Estimate the x-coordinates of the relative maximum and relative minimum points from the graph.
2. Use the maximum and minimum options from the CALC menu to find the exact coordinates of these points. You will need to use the arrow keys to select points to the left and to the right of the point.
3. Graph $f(x) = \frac{1}{2}x^4 - 4x^3 + 7x^2 - 8$. How many relative maximum and relative minimum points does the graph contain? What are the coordinates?

CHECK Your Understanding

Example 1 (p. 339)

Graph each polynomial function by making a table of values.

1. $f(x) = x^3 - x^2 - 4x + 4$ **2.** $f(x) = x^4 - 7x^2 + x + 5$

Example 2 (p. 340)

Determine the consecutive integer values of x between which each real zero of each function is located. Then draw the graph.

3. $f(x) = x^3 - x^2 + 1$ **4.** $f(x) = x^4 - 4x^2 + 2$

Example 3 (p. 341)

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. State the domain and range for each function.

5. $f(x) = x^3 + 2x^2 - 3x - 5$ **6.** $f(x) = x^4 - 8x^2 + 10$

Example 4 (pp. 341–342)

CABLE TV **For Exercises 7–10, use the following information.**
The number of cable TV systems after 1985 can be modeled by the function $C(t) = -43.2t^2 + 1343t + 790$, where t represents the number of years since 1985.

7. Graph this equation for the years 1985 to 2005.
8. Describe the turning points of the graph and its end behavior.
9. What is the domain of the function? Use the graph to estimate the range.
10. What trends in cable TV subscriptions does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

Exercises

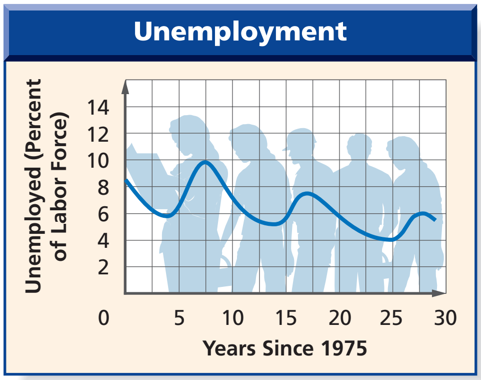
HOMEWORK HELP	
For Exercises	See Examples
11–18	1–3
19–25	4

For Exercises 11–18, complete each of the following.

a. Graph each function by making a table of values.
b. Determine the consecutive integer values of x between which each real zero is located.
c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

11. $f(x) = -x^3 - 4x^2$ **12.** $f(x) = x^3 - 2x^2 + 6$
13. $f(x) = x^3 - 3x^2 + 2$ **14.** $f(x) = x^3 + 5x^2 - 9$
15. $f(x) = -3x^3 + 20x^2 - 36x + 16$ **16.** $f(x) = x^3 - 4x^2 + 2x - 1$
17. $f(x) = x^4 - 8$ **18.** $f(x) = x^4 - 10x^2 + 9$

EMPLOYMENT **For Exercises 19–22, use the graph that models the unemployment rates from 1975–2004.**



19. In what year was the unemployment rate the highest? the lowest?
20. Describe the turning points and the end behavior of the graph.
21. If this graph was modeled by a polynomial equation, what is the least degree the equation could have?
22. Do you expect the unemployment rate to increase or decrease from 2005 to 2010? Explain your reasoning.

Real-World Link

As children develop, their sleeping needs change. Infants sleep about 16–18 hours a day. Toddlers usually sleep 10–12 hours at night and take one or two daytime naps. School-age children need 9–11 hours of sleep, and teens need at least 9 hours of sleep.

Source: www.kidshealth.org

HEALTH For Exercises 23–25, use the following information. During a regular respiratory cycle, the volume of air in liters in human lungs can be described by $V(t) = 0.173t + 0.152t^2 - 0.035t^3$, where t is the time in seconds.

23. Estimate the real zeros of the function by graphing.

24. About how long does a regular respiratory cycle last?

25. Estimate the time in seconds from the beginning of this respiratory cycle for the lungs to fill to their maximum volume of air.

For Exercises 26–31, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

26. $f(x) = -x^4 + 5x^2 - 2x - 1$

27. $f(x) = -x^4 + x^3 + 8x^2 - 3$

28. $f(x) = x^4 - 9x^3 + 25x^2 - 24x + 6$

29. $f(x) = 2x^4 - 4x^3 - 2x^2 + 3x - 5$

30. $f(x) = x^5 + 4x^4 - x^3 - 9x^2 + 3$

31. $f(x) = x^5 - 6x^4 + 4x^3 + 17x^2 - 5x - 6$

CHILD DEVELOPMENT For Exercises 32 and 33, use the following information. The average height (in inches) for boys ages 1 to 20 can be modeled by the equation $B(x) = -0.001x^4 + 0.04x^3 - 0.56x^2 + 5.5x + 25$, where x is the age (in years). The average height for girls ages 1 to 20 is modeled by the equation $G(x) = -0.0002x^4 + 0.006x^3 - 0.14x^2 + 3.7x + 26$.

32. Graph both equations by making a table of values. Use $x = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ as the domain. Round values to the nearest inch.

33. Compare the graphs. What do the graphs suggest about the growth rate for both boys and girls?

EXTRA PRACTICE

See pages 903, 931.

Math Online

Self-Check Quiz at algebra2.com

Use a graphing calculator to estimate the x-coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

34. $f(x) = x^3 + x^2 - 7x - 3$

35. $f(x) = -x^3 + 6x^2 - 6x - 5$

36. $f(x) = -x^4 + 3x^2 - 8$

37. $f(x) = 3x^4 - 7x^3 + 4x - 5$

H.O.T. Problems

38. OPEN ENDED Sketch a graph of a function that has one relative maximum point and two relative minimum points.

CHALLENGE For Exercises 39–41, sketch a graph of each polynomial.

39. even-degree polynomial function with one relative maximum and two relative minima

40. odd-degree polynomial function with one relative maximum and one relative minimum; the leading coefficient is negative

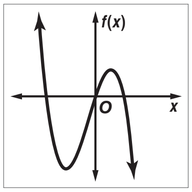
41. odd-degree polynomial function with three relative maxima and three relative minima; the leftmost points are negative

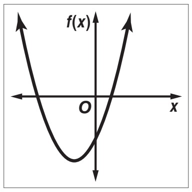
42. REASONING Explain the Location Principle and how to use it.

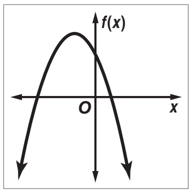
43. *Writing in Math* Use the information about foreign-born population on page 339 to explain how graphs of polynomial functions can be used to show trends in data. Include a description of the types of data that are best modeled by polynomial functions and an explanation of how you would determine when the percent of foreign-born citizens was at its highest and when the percent was at its lowest since 1900.

STANDARDIZED TEST PRACTICE

44. ACT/SAT Which of the following could be the graph of $f(x) = x^3 + x^2 - 3x$?

A


C


B


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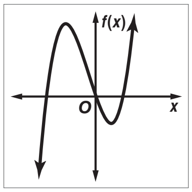


45. REVIEW Mandy went shopping. She spent two-fifths of her money in the first store. She spent three-fifths of what she had left in the next store. In the last store she visited, she spent three-fourths of the money she had left. When she finished shopping, Mandy had \$6. How much money in dollars did Mandy have when she started shopping?

F \$16 **H** \$100

G \$56 **J** \$106

Spiral Review

If $p(x) = 2x^2 - 5x + 4$ and $r(x) = 3x^3 - x^2 - 2$, find each value. (Lesson 6-4)

46. $r(2a)$ **47.** $5p(c)$ **48.** $p(2a^2)$

49. $r(x - 1)$ **50.** $p(x^2 + 4)$ **51.** $2[p(x^2 + 1)] - 3r(x - 1)$

Simplify. (Lesson 6-3)

52. $(4x^3 - 7x^2 + 3x - 2) \div (x - 2)$ **53.** $\dfrac{x^4 + 4x^3 - 4x^2 + 5x}{x - 5}$

Simplify. (Lesson 6-2)

54. $(3x^2 - 2xy + y^2) + (x^2 + 5xy - 4y^2)$ **55.** $(2x + 4)(7x - 1)$

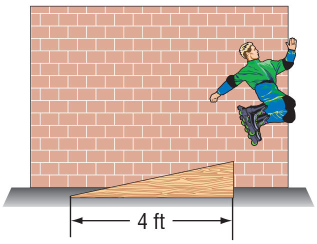
Solve each matrix equation or system of equations by using inverse matrices. (Lesson 4-8)

56. $\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}$ **57.** $\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

58. $3j + 2k = 8$
$j - 7k = 18$

59. $5y + 2z = 11$
$10y - 4z = -2$

60. SPORTS Bob and Minya want to build a ramp that they can use while rollerblading. If they want the ramp to have a slope of $\frac{1}{4}$, how tall should they make the ramp? (Lesson 2-3)



GET READY for the Next Lesson

PREREQUISITE SKILL **Find the greatest common factor of each set of numbers.**

61. 18, 27 **62.** 24, 84 **63.** 16, 28

64. 12, 27, 48 **65.** 12, 30, 54 **66.** 15, 30, 65

EXTEND 6-5

Graphing Calculator Lab

Modeling Data Using Polynomial Functions

You can use a TI-83/84 Plus graphing calculator to model data for which the curve of best fit is a polynomial function.

EXAMPLE

Concepts in Motion Interactive Lab algebra2.com

The table shows the distance a seismic wave can travel based on its distance from an earthquake's epicenter. Draw a scatter plot and a curve of best fit that relates distance to travel time. Then determine approximately how far from the epicenter the wave will be felt 8.5 minutes after the earthquake occurs.

Travel Time (min)	1	2	5	7	10	12	13
Distance (km)	400	800	2500	3900	6250	8400	10,000

Source: University of Arizona

Step 1 Enter the travel times in L1 and the distances in L2.

KEYSTROKES: *Refer to page 92 to review how to enter lists.*

Step 2 Graph the scatter plot.

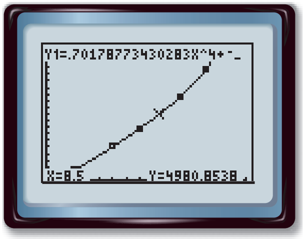
KEYSTROKES: *Refer to page 92 to review how to graph a scatter plot.*

Step 3 Compute and graph the equation for the curve of best fit. A quartic curve is the best fit for these data. You can verify this by comparing the R^2 values for each type of graph.

KEYSTROKES: STAT ▶ 7 2nd [L1] , 2nd [L2] ENTER Y= VARS 5 ▶ ▶ 1 GRAPH

Step 4 Use the [CALC] feature to find the value of the function for $x = 8.5$.

KEYSTROKES: 2nd [CALC] 1 8.5 ENTER



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After 8.5 minutes, you would expect the wave to be felt approximately 5000 kilometers away.

Exercises

For Exercises 1–3, use the table that shows how many minutes out of each eight-hour workday are used to pay one day's worth of taxes.

Year	Minutes
1940	83
1950	117
1960	130
1970	141
1980	145
1990	145
2000	160

Source: Tax Foundation

1. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of minutes to the number of years since 1930. Try LinReg, QuadReg, and CubicReg.
2. Write the equation for the curve that best fits the data.
3. Based on this equation, how many minutes should you expect to work each day in the year 2010 to pay one day's taxes?

Other Calculator Keystrokes at algebra2.com

For Exercises 4–7, use the table that shows the estimated number of alternative-fueled vehicles in use in the United States per year.

4. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of vehicles to the year. Try LinReg, QuadReg, and CubicReg. (*Hint:* Enter the *x*-values as years since 1994.)

5. Write the equation for the curve that best fits the data. Round to the nearest tenth.

6. Based on this equation before rounding, how many Alternative-Fueled Vehicles would you expect to be in use in the year 2008?

7. Find a curve of best fit that is quartic. Is it a better fit than the equation you wrote in Exercise 5? Explain.

Year	Estimated Alternative-Fueled Vehicles in Use in the United States
1995	333,049
1996	352,421
1997	367,526
1998	383,847
1999	411,525
2000	455,906
2001	490,019
2002	518,919

Source: eia.doe.gov

For Exercises 8–11, use the table that shows the distance from the Sun to the Earth for each month of the year.

8. Draw a scatter plot of the data. Then graph several curves of best fit that relate the distance to the month. Try LinReg, QuadReg, and CubicReg.

9. Write the equation for the curve that best fits the data.

10. Based on this equation, what is the distance from the Sun to the Earth halfway through September?

11. Would you use this model to find the distance from the Sun to Earth in subsequent years? Explain your reasoning.

Month	Distance
January	0.9840
February	0.9888
March	0.9962
April	1.0050
May	1.0122
June	1.0163
July	1.0161
August	1.0116
September	1.0039
October	0.9954
November	0.9878
December	0.9837

Source: astronomycafe.net

Extension

For Exercises 12–15, design and complete your own data analysis.

12. Write a question that could be answered by examining data. For example, you might estimate the number of people living in your town 5 years from now or predict the future cost of a car.

13. Collect and organize the data you need to answer the question you wrote. You may need to research your topic on the internet or conduct a survey to collect the data you need.

14. Make a scatter plot and find a regression equation for your data. Then use the regression equation to answer the question.

CHAPTER 6 Mid-Chapter Quiz

Lessons 6-1 through 6-5

Simplify. Assume that no variable equals 0. (Lesson 6-1)

1. $(-3x^2y)^3(2x)^2$ **2.** $\frac{a^6b^{-2}c}{a^3b^2c^4}$ **3.** $\left(\frac{x^2z}{xz^4}\right)^2$

4. CHEMISTRY One gram of water contains about 3.34×10^{22} molecules. About how many molecules are in 5×10^2 grams of water? (Lesson 6-1)

Simplify. (Lesson 6-2)

5. $(9x + 2y) - (7x - 3y)$ **6.** $(t + 2)(3t - 4)$

7. $(n + 2)(n^2 - 3n + 1)$ **8.** $4a(ab + 5a^2)$

9. MULTIPLE CHOICE The area of the base of a rectangular suitcase measures $3x^2 + 5x - 4$ square units. The height of the suitcase measures $2x$ units. Which polynomial expression represents the volume of the suitcase? (Lesson 6-2)

A $3x^3 + 5x^2 - 4x$

B $6x^2 + 10x - 8$

C $6x^3 + 10x^2 - 8x$

D $3x^3 + 10x^2 - 4$

Simplify. (Lesson 6-3)

10. $(m^3 - 4m^2 - 3m - 7) \div (m - 4)$

11. $\frac{2d^3 - d^2 - 9d + 9}{2d - 3}$

12. $(x^3 + x^2 - 13x - 28) \div (x - 4)$

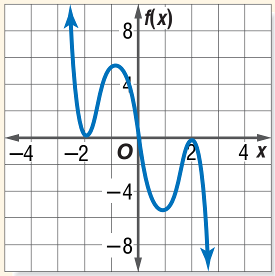
13. $\frac{3y^3 + 7y^2 - y - 5}{y + 2}$

14. WOODWORKING Arthur is building a rectangular table with an area of $3x^2 - 17x - 28$ square feet. If the length of the table is $3x + 4$ feet, what should the width of the rectangular table be? (Lesson 6-3)

15. PETS A pet food company estimates that it costs $0.02x^2 + 3x + 250$ dollars to produce x bags of dog food. Find an expression for the average cost per unit. (Lesson 6-3)

16. If $p(x) = 2x^3 - x$, find $p(a - 1)$. (Lesson 6-4)

17. Describe the end behavior of the graph. Then determine whether it represents an odd-degree or an even-degree polynomial function and state the number of real zeroes. (Lesson 6-4)



18. WIND CHILL The function $C(s) = 0.013s^2 - s - 7$ estimates the wind chill temperature $C(s)$ at 0°F for wind speeds s from 5 to 30 miles per hour. Estimate the wind chill temperature at 0°F if the wind speed is 27 miles per hour. (Lesson 6-4)

19. The formula $L(t) = \frac{8t^2}{\pi^2}$ represents the swing of a pendulum. L is the length of the pendulum in feet, and t is the time in seconds to swing back and forth. Find the length of a pendulum $L(t)$ that makes one swing in 2 seconds. (Lesson 6-4)

20. MULTIPLE CHOICE The function $f(x) = x^2 - 4x + 3$ has a relative minimum located at which of the following x-values? (Lesson 6-5)

F −2 H 3

G 2 J 4

21. Graph $y = x^3 + 2x^2 - 4x - 6$. Estimate the x-coordinates at which the relative maxima and relative minima occur. (Lesson 6-5)

22. MARKET PRICE Prices of oranges from January to August can be modeled by (1, 2.7), (2, 4.4), (3, 4.9), (4, 5.5), (5, 4.3), (6, 5.3), (7, 3.5), (8, 3.9). How many turning points would the graph of a polynomial function through these points have? Describe them. (Lesson 6-5)

6-6 Solving Polynomial Equations

Main Ideas

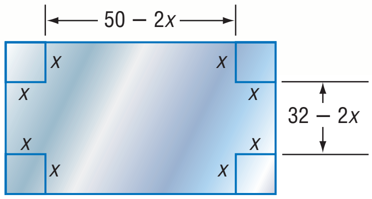
- Factor polynomials.
- Solve polynomial equations by factoring.

New Vocabulary

quadratic form

GET READY for the Lesson

The Taylor Manufacturing Company makes open metal boxes of various sizes. Each sheet of metal is 50 inches long and 32 inches wide. To make a box, a square is cut from each corner.



The volume of the box depends on the side length x of the cut squares. It is given by $V(x) = 4x^3 - 164x^2 + 1600x$. You can solve a polynomial equation to find the dimensions of the square to cut for a box with specific volume.

Factor Polynomials Whole numbers are factored using prime numbers. For example, $100 = 2 \cdot 2 \cdot 5 \cdot 5$. Many polynomials can also be factored. Their factors, however, are other polynomials. Polynomials that cannot be factored are called *prime*. One method for finding the dimensions of the square to cut to make a box involves factoring the polynomial that represents the volume.

The table below summarizes the most common factoring techniques used with polynomials. Some of these techniques were introduced in Lesson 5-3. The others will be presented in this lesson.

CONCEPT SUMMARY *Factoring Techniques*

Number of Terms	Factoring Technique	General Case
any number	Greatest Common Factor (GCF)	$a^3b^2 + 2a^2b - 4ab^2 = ab(a^2b + 2a - 4b)$
two	Difference of Two Squares Sum of Two Cubes Difference of Two Cubes	$a^2 - b^2 = (a + b)(a - b)$ $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
three	Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
	General Trinomials	$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
four or more	Grouping	$ax + bx + ay + by = x(a + b) + y(a + b)$ $= (a + b)(x + y)$

Whenever you factor a polynomial, always look for a common factor first. Then determine whether the resulting polynomial factor can be factored again using one or more of the methods listed above.

Study Tip

Checking You can check the result when factoring by finding the product.

EXAMPLE 1 GCF

Factor $6x^2y^2 - 2xy^2 + 6x^3y$.

$$6x^2y^2 - 2xy^2 + 6x^3y = (2 \cdot 3 \cdot x \cdot x \cdot y \cdot y) - (2 \cdot x \cdot y \cdot y) + (2 \cdot 3 \cdot x \cdot x \cdot x \cdot y)$$

$$= (2xy \cdot 3xy) - (2xy \cdot y) + (2xy \cdot 3x^2)$$

$$= 2xy(3xy - y + 3x^2)$$

The GCF is $2xy$. The remaining polynomial cannot be factored using the methods above.

CHECK Your Progress **Factor completely.**

1A. $18x^3y^4 + 12x^2y^3 - 6xy^2$ **1B.** $a^4b^4 + 3a^3b^4 + a^2b^3$

EXAMPLE 2 Grouping

Factor $a^3 - 4a^2 + 3a - 12$.

$a^3 - 4a^2 + 3a - 12 = (a^3 - 4a^2) + (3a - 12)$ Group to find a GCF.

$= a^2(a - 4) + 3(a - 4)$ Factor the GCF of each binomial.

$= (a - 4)(a^2 + 3)$ Distributive Property

CHECK Your Progress **Factor completely.**

2A. $x^2 + 3xy + 2xy^2 + 6y^3$ **2B.** $6a^3 - 9a^2b + 4ab - 6b^2$

Factoring by grouping is the only method that can be used to factor polynomials with four or more terms. For polynomials with two or three terms, it may be possible to factor the polynomial according to one of the patterns shown on page 349.

EXAMPLE 3 Two or Three Terms

Factor each polynomial.

a. $8x^3 - 24x^2 + 18x$

This trinomial does not fit any of the factoring patterns. First, factor out the GCF. Then the remaining trinomial is a perfect square trinomial.

$8x^3 - 24x^2 + 18x = 2x(4x^2 - 12x + 9)$ Factor out the GCF.

$= 2x(2x - 3)^2$ Perfect square trinomial

b. $m^6 - n^6$

This polynomial could be considered the difference of two squares or the difference of two cubes. The difference of two squares should always be done before the difference of two cubes. This will make the next step of the factorization easier.

$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$ Difference of two squares

$= (m + n)(m^2 - mn + n^2)(m - n)(m^2 + mn + n^2)$ Sum and difference of two cubes

CHECK Your Progress

3A. $3xy^2 - 48x$ **3B.** $c^3d^3 + 27$

Math Online Extra Examples at algebra2.com

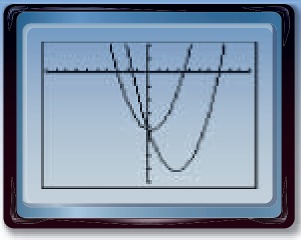
You can use a graphing calculator to check that the factored form of a polynomial is correct.

GRAPHING CALCULATOR LAB

Factoring Polynomials

Is the factored form of $2x^2 - 11x - 21$ equal to $(2x - 7)(x + 3)$? You can find out by graphing $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$. If the two graphs coincide, the factored form is probably correct.

- Enter $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$ on the **Y=** screen.
- Graph the functions. Since two different graphs appear, $2x^2 - 11x - 21 \neq (2x - 7)(x + 3)$.



$[-10, 10]$ scl: 1 by $[-40, 10]$ scl: 5

THINK AND DISCUSS

1. Determine if $x^2 + 5x - 6 = (x - 3)(x - 2)$ is a true statement. If not, write the correct factorization.
2. Does this method guarantee a way to check the factored form of a polynomial? Why or why not?

In some cases, you can rewrite a polynomial in x in the form $au^2 + bu + c$. For example, by letting $u = x^2$ the expression $x^4 - 16x^2 + 60$ can be written as $(x^2)^2 - 16(x^2) + 60$ or $u^2 - 16u + 60$. This new, but equivalent, expression is said to be in **quadratic form.**

KEY CONCEPT — *Quadratic Form*

An expression that is quadratic in form can be written as $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is some expression in x. The expression $au^2 + bu + c$ is called the quadratic form of the original expression.

EXAMPLE 4 Write Expressions in Quadratic Form

Write each expression in quadratic form, if possible.

a. $x^4 + 13x^2 + 36$

$x^4 + 13x^2 + 36 = (x^2)^2 + 13(x^2) + 36$ $\quad (x^2)^2 = x^4$

b. $12x^8 - x^2 + 10$

This cannot be written in quadratic form since $x^8 \neq (x^2)^2$.

CHECK Your Progress

4A. $16x^6 - 625$

4B. $9x^{10} - 15x^4 + 9$

Solve Equations Using Quadratic Form In Chapter 5, you learned to solve quadratic equations by factoring and using the Zero Product Property. You can extend these techniques to solve higher-degree polynomial equations.

Personal Tutor

EXAMPLE Solve Polynomial Equations

Study Tip

Substitution

To avoid confusion, you can substitute another variable for the expression in parentheses. For example, $x^4 - 13x^2 + 36 = 0$ could be written as $u^2 - 13u + 36 = 0$. Then once you have solved the equation for u, substitute x^2 for u and solve for x.

5 Solve each equation.

a. $x^4 - 13x^2 + 36 = 0$

$x^4 - 13x^2 + 36 = 0$	Original equation
$(x^2)^2 - 13(x^2) + 36 = 0$	Write the expression on the left in quadratic form.
$(x^2 - 9)(x^2 - 4) = 0$	Factor the trinomial.
$(x - 3)(x + 3)(x - 2)(x + 2) = 0$	Factor each difference of squares.

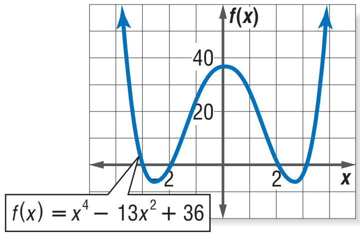
Use the Zero Product Property.

$x - 3 = 0$ or $x + 3 = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = 3$ $\quad x = -3$ $\quad x = 2$ $\quad x = -2$

The solutions are -3, -2, 2, and 3.

CHECK The graph of $f(x) = x^4 - 13x^2 + 36$ shows that the graph intersects the x-axis at -3, -2, 2, and 3.



b. $x^3 + 343 = 0$

$x^3 + 343 = 0$	Original equation
$(x)^3 + 7^3 = 0$	This is the sum of two cubes.
$(x + 7)[x^2 - x(7) + 7^2] = 0$	Sum of two cubes formula with $a = x$ and $b = 7$
$(x + 7)(x^2 - 7x + 49) = 0$	Simplify.
$(x + 7) = 0$ or $x^2 - 7x + 49 = 0$	Zero Product Property

The solution of the first equation is -7. The second equation can be solved by using the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

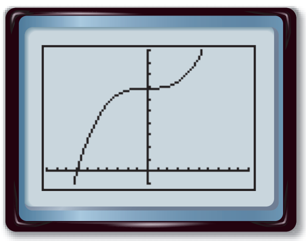
$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(49)}}{2(1)}$$ Replace a with 1, b with -7, and c with 49.

$$= \frac{7 \pm \sqrt{-147}}{2}$$ Simplify.

$$= \frac{7 \pm i\sqrt{147}}{2} \text{ or } \frac{7 \pm 7i\sqrt{3}}{2}$$ $\sqrt{147} \times \sqrt{-1} = 7i\sqrt{3}$

Thus, the solutions of the original equation are -7, $\frac{7 + 7i\sqrt{3}}{2}$, and $\frac{7 - 7i\sqrt{3}}{2}$.

CHECK The graph of $f(x) = x^3 + 343$ confirms the solution.



[−10, 10] sci: 1 by [−50, 500] sci: 50

CHECK Your Progress

5A. $x^4 - 29x^2 + 100 = 0$ **5B.** $x^3 + 8 = 0$

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Examples 1–3 (p. 350)

Factor completely. If the polynomial is not factorable, write *prime*.

1. $-12x^2 - 6x$

2. $a^2 + 5a + ab$

3. $21 - 7y + 3x - xy$

4. $y^2 + 4y + 2y + 8$

5. $z^2 - 4z - 12$

6. $3b^2 - 48$

7. $16w^2 - 169$

8. $h^3 + 8000$

Example 4 (p. 351)

Write each expression in quadratic form, if possible.

9. $5y^4 + 7y^3 - 8$

10. $84n^4 - 62n^2$

Example 5 (p. 352)

Solve each equation.

11. $x^4 - 50x^2 + 49 = 0$

12. $x^3 - 125 = 0$

13. **POOL** The Shelby University swimming pool is in the shape of a rectangular prism and has a volume of 28,000 cubic feet. The dimensions of the pool are x feet deep by $7x - 6$ feet wide by $9x - 2$ feet long. How deep is the pool?

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–17	1
18, 19	2
20–23	3
24–29	4
30–39	5

Factor completely. If the polynomial is not factorable, write *prime*.

14. $2xy^3 - 10x$

15. $6a^2b^2 + 18ab^3$

16. $12cd^3 - 8c^2d^2 + 10c^5d^3$

17. $3a^2bx + 15cx^2y + 25ad^3y$

18. $8yz - 6z - 12y + 9$

19. $3ax - 15a + x - 5$

20. $y^2 - 5y + 4$

21. $2b^2 + 13b - 7$

22. $z^3 + 125$

23. $t^3 - 8$

Write each expression in quadratic form, if possible.

24. $2x^4 + 6x^2 - 10$

25. $a^8 + 10a^2 - 16$

26. $11n^6 + 44n^3$

27. $7b^5 - 4b^3 + 2b$

28. $7x^{\frac{1}{9}} - 3x^{\frac{1}{3}} + 4$

29. $6x^{\frac{2}{5}} - 4x^{\frac{1}{5}} - 16$

Solve each equation.

30. $x^4 - 34x^2 + 225 = 0$

31. $x^4 - 15x^2 - 16 = 0$

32. $x^4 + 6x^2 - 27 = 0$

33. $x^3 + 64 = 0$

34. $27x^3 + 1 = 0$

35. $8x^3 - 27 = 0$

Real-World Career

Designer

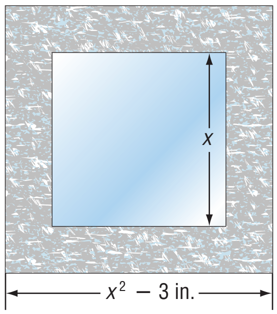
Designers combine practical knowledge with artistic ability to turn abstract ideas into formal designs.

Math Online

For more information, go to algebra2.com.

DESIGN For Exercises 36–38, use the following information.

Jill is designing a picture frame for an art project. She plans to have a square piece of glass in the center and surround it with a decorated ceramic frame, which will also be a square. The dimensions of the glass and frame are shown in the diagram at the right. Jill determines that she needs 27 square inches of material for the frame.



36. Write a polynomial equation that models the area of the frame.

37. What are the dimensions of the glass piece?

38. What are the dimensions of the frame?

Gregg Mancuso/Stock Boston

39. **GEOMETRY** The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism.

40. Find the factorization of $3x^2 + x - 2$.

41. What are the factors of $2y^2 + 9y + 4$?

Factor completely. If the polynomial is not factorable, write prime.

42. $3n^2 + 21n - 24$

43. $y^4 - z^2$

44. $16a^2 + 25b^2$

45. $3x^2 - 27y^2$

46. $x^4 - 81$

47. $3a^3 + 2a^2 - 5a + 9a^2b + 6ab - 15b$

PACKAGING For Exercises 48 and 49, use the following information.

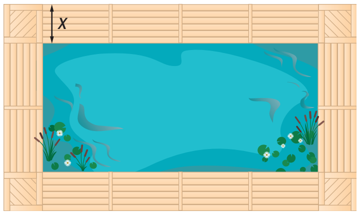
A computer manufacturer needs to change the dimensions of its foam packaging for a new model of computer. The width of the original piece is three times the height, and the length is equal to the height squared. The volume of the new piece can be represented by the equation $V(h) = 3h^4 + 11h^3 + 18h^2 + 44h + 24$, where h is the height of the original piece.

48. Factor the equation for the volume of the new piece to determine three expressions that represent the height, length, and width of the new piece.

49. How much did each dimension of the packaging increase for the new foam piece?

EXTRA PRACTICE
See pages 904, 931.
Math Online
Self-Check Quiz at algebra2.com

50. **LANDSCAPING** A boardwalk that is x feet wide is built around a rectangular pond. The pond is 30 feet wide and 40 feet long. The combined area of the pond and the boardwalk is 2000 square feet. What is the width of the boardwalk?



Graphing Calculator

CHECK FACTORING Use a graphing calculator to determine if each polynomial is factored correctly. Write *yes* or *no*. If the polynomial is not factored correctly, find the correct factorization.

51. $3x^2 + 5x + 2 \stackrel{?}{=} (3x + 2)(x + 1)$

52. $x^3 + 8 \stackrel{?}{=} (x + 2)(x^2 - x + 4)$

53. $2x^2 - 5x - 3 \stackrel{?}{=} (x - 1)(2x + 3)$

54. $3x^2 - 48 \stackrel{?}{=} 3(x + 4)(x - 4)$

H.O.T. Problems

55. **OPEN ENDED** Give an example of an equation that is not quadratic but can be written in quadratic form. Then write it in quadratic form.

56. **CHALLENGE** Factor $64p^{2n} + 16p^n + 1$.

57. **REASONING** Find a counterexample to the statement $a^2 + b^2 = (a + b)^2$.

58. **CHALLENGE** Explain how you would solve $|a - 3|^2 - 9|a - 3| = -8$. Then solve the equation.

59. ***Writing in Math*** Use the information on page 349 to explain how solving a polynomial equation can help you find dimensions. Explain how you could determine the dimensions of the cut square if the desired volume was 3600 cubic inches. Explain why there can be more than one square that can be cut to produce the same volume.

STANDARDIZED TEST PRACTICE

60. ACT/SAT Which is *not* a factor of $x^3 - x^2 - 2x$?

A x

B $x + 1$

C $x - 1$

D $x - 2$

61. ACT/SAT The measure of the largest angle of a triangle is 14 less than twice the measure of the smallest angle. The third angle is 2 more than the measure of the smallest angle. What is the measure of the smallest angle?

F 46 G 48 H 50 J 82

62. REVIEW $27x^3 + y^3 =$

A $(3x + y)(3x + y)(3x + y)$

B $(3x + y)(9x^2 - 3xy + y^2)$

C $(3x - y)(9x^2 + 3xy + y^2)$

D $(3x - y)(9x^2 + 9xy + y^2)$

Spiral Review

Graph each polynomial function. Estimate the *x*-coordinates at which the relative maxima and relative minima occur. (Lesson 6-5)

63. $f(x) = x^3 - 6x^2 + 4x + 3$

64. $f(x) = -x^4 + 2x^3 + 3x^2 - 7x + 4$

Find $p(7)$ and $p(-3)$ for each function. (Lesson 6-4)

65. $p(x) = x^2 - 5x + 3$

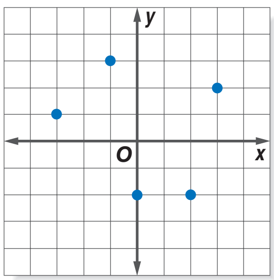
66. $p(x) = x^3 - 11x - 4$

67. $p(x) = \frac{2}{3}x^4 - 3x^3$

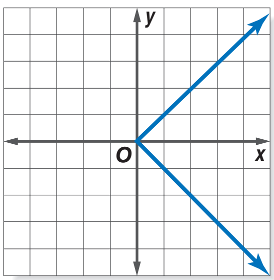
68. PHOTOGRAPHY The perimeter of a rectangular picture is 86 inches. Twice the width exceeds the length by 2 inches. What are the dimensions of the picture? (Lesson 3-2)

Determine whether each relation is a function. Write *yes* or *no*. (Lesson 2-1)

69.



70.



GET READY for the Next Lesson

PREREQUISITE SKILL **Find each quotient.** (Lesson 6-3)

71. $(x^3 + 4x^2 - 9x + 4) \div (x - 1)$

72. $(4x^3 - 8x^2 - 5x - 10) \div (x + 2)$

73. $(x^4 - 9x^2 - 2x + 6) \div (x - 3)$

74. $(x^4 + 3x^3 - 8x^2 + 5x - 6) \div (x + 1)$

6-7 The Remainder and Factor Theorems

Main Ideas

- Evaluate functions using synthetic substitution.
- Determine whether a binomial is a factor of a polynomial by using synthetic substitution.

New Vocabulary

synthetic substitution
depressed polynomial

GET READY for the Lesson

The number of international travelers to the United States since 1986 can be modeled by the equation $T(x) = 0.02x^3 - 0.6x^2 + 6x + 25.9$, where x is the number of years since 1986 and $T(x)$ is the number of travelers in millions. To estimate the number of travelers in 2006, you can evaluate the function by substituting 20 for x, or you can use synthetic substitution.

Synthetic Substitution Synthetic division can be used to find the value of a function. Consider the polynomial function $f(a) = 4a^2 - 3a + 6$. Divide the polynomial by $a - 2$.

Method 1 Long Division

$$
\begin{array}{r}
4a \;\; + 5 \\
a - 2 \overline{)\, 4a^2 - 3a + 6} \\
\underline{4a^2 - 8a} \quad\;\; \\
5a + \;\; 6 \\
\underline{5a - 10} \\
16
\end{array}
$$

Method 2 Synthetic Division

$$
\begin{array}{r|rrr}
2 & 4 & -3 & 6 \\
 & & 8 & 10 \\
\hline
 & 4 & 5 & | \; 16
\end{array}
$$

Compare the remainder of 16 to $f(2)$.

$$
\begin{aligned}
f(2) &= 4(2)^2 - 3(2) + 6 && \text{Replace } a \text{ with 2.} \\
&= 16 - 6 + 6 && \text{Multiply.} \\
&= 16 && \text{Simplify.}
\end{aligned}
$$

Notice that the value of $f(2)$ is the same as the remainder when the polynomial is divided by $a - 2$. This illustrates the **Remainder Theorem**.

KEY CONCEPT — *Remainder Theorem*

If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$, and

Dividend	equals	quotient	times	divisor	plus	remainder.
$f(x)$	$=$	$q(x)$	$\cdot$	$(x - a)$	$+$	$f(a)$,

where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.

When synthetic division is used to evaluate a function, it is called **synthetic substitution**. It is a convenient way of finding the value of a function, especially when the degree of the polynomial is greater than 2.

Boden/Ledingham/Masterfile

EXAMPLE Synthetic Substitution

1 **If $f(x) = 2x^4 - 5x^2 + 8x - 7$, find $f(6)$.**

Method 1 Synthetic Substitution

By the Remainder Theorem, $f(6)$ should be the remainder when you divide the polynomial by $x - 6$.

$$\begin{array}{r|rrrrr} 6 & 2 & 0 & -5 & 8 & -7 \\ & & 12 & 72 & 402 & 2460 \\ \hline & 2 & 12 & 67 & 410 & 2453 \end{array}$$

Notice that there is no x^3 term. A zero is placed in this position as a placeholder.

The remainder is 2453. Thus, by using synthetic substitution, $f(6) = 2453$.

Method 2 Direct Substitution

Replace x with 6.

$f(x) = 2x^4 - 5x^2 + 8x - 7$	Original function
$f(6) = 2(6)^4 - 5(6)^2 + 8(6) - 7$	Replace x with 6.
$= 2592 - 180 + 48 - 7$ or 2453	Simplify.

By using direct substitution, $f(6) = 2453$. Both methods give the same result.

CHECK Your Progress

1A. If $f(x) = 3x^3 - 6x^2 + x - 11$, find $f(3)$.

1B. If $g(x) = 4x^5 + 2x^3 + x^2 - 1$, find $f(-1)$.

Factors of Polynomials The synthetic division below shows that the quotient of $x^4 + x^3 - 17x^2 - 20x + 32$ and $x - 4$ is $x^3 + 5x^2 + 3x - 8$.

$$\begin{array}{r|rrrrr} 4 & 1 & 1 & -17 & -20 & 32 \\ & & 4 & 20 & 12 & -32 \\ \hline & 1 & 5 & 3 & -8 & 0 \end{array}$$

Study Tip

Depressed Polynomial

A *depressed polynomial* has a degree that is one less than the original polynomial.

When you divide a polynomial by one of its binomial factors, the quotient is called a **depressed polynomial**. From the results of the division and by using the Remainder Theorem, we can make the following statement.

$$\underbrace{x^4 + x^3 - 17x^2 - 20x + 32}_{\text{Dividend}} \underbrace{=}_{\text{equals}} \underbrace{(x^3 + 5x^2 + 3x - 8)}_{\text{quotient}} \underbrace{\cdot}_{\text{times}} \underbrace{(x - 4)}_{\text{divisor}} \underbrace{+}_{\text{plus}} \underbrace{0}_{\text{remainder.}}$$

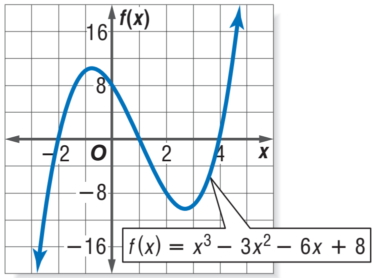
Since the remainder is 0, $f(4) = 0$. This means that $x - 4$ is a factor of $x^4 + x^3 - 17x^2 - 20x + 32$. This illustrates the **Factor Theorem,** which is a special case of the Remainder Theorem.

KEY CONCEPT *Factor Theorem*

The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

If $x - a$ is a factor of $f(x)$, then $f(a)$ has a factor of $(a - a)$, or 0. Since a factor of $f(a)$ is 0, $f(a) = 0$. Now assume that $f(a) = 0$. If $f(a) = 0$, then the Remainder Theorem states that the remainder is 0 when $f(x)$ is divided by $x - a$. This means that $x - a$ is a factor of $f(x)$. This proves the Factor Theorem.

Suppose you wanted to find the factors of $x^3 - 3x^2 - 6x + 8$. One approach is to graph the related function, $f(x) = x^3 - 3x^2 - 6x + 8$. From the graph, you can see that the graph of $f(x)$ crosses the x-axis at -2, 1, and 4. These are the zeros of the function. Using these zeros and the Zero Product Property, we can express the polynomial in factored form.



$$f(x) = [x - (-2)](x - 1)(x - 4)$$
$$= (x + 2)(x - 1)(x - 4)$$

This method of factoring a polynomial has its limitations. Most polynomial functions are not easily graphed, and once graphed, the exact zeros are often difficult to determine.

EXAMPLE Use the Factor Theorem

Personal Tutor

2 **Show that $x + 3$ is a factor of $x^3 + 6x^2 - x - 30$. Then find the remaining factors of the polynomial.**

The binomial $x + 3$ is a factor of the polynomial if -3 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

-3	1	6	-1	-30
		-3	-9	30
	1	3	-10	0

Since the remainder is 0, $x + 3$ is a factor of the polynomial. The polynomial $x^3 + 6x^2 - x - 30$ can be factored as $(x + 3)(x^2 + 3x - 10)$. The polynomial $x^2 + 3x - 10$ is the depressed polynomial. Check to see if this polynomial can be factored.

$x^2 + 3x - 10 = (x - 2)(x + 5)$ Factor the trinomial.

So, $x^3 + 6x^2 - x - 30 = (x + 3)(x - 2)(x + 5)$.

Study Tip

Factoring

The factors of a polynomial do not have to be binomials. For example, the factors of $x^3 + x^2 - x + 15$ are $x + 3$ and $x^2 - 2x + 5$.

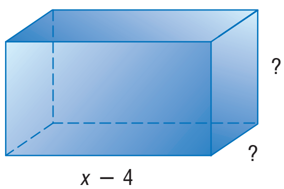
CHECK Your Progress

2. Show that $x - 2$ is a factor of $x^3 - 7x^2 + 4x + 12$. Then find the remaining factors of the polynomial.

Online Personal Tutor at algebra2.com

EXAMPLE Find All Factors

3 **GEOMETRY** **The volume of the rectangular prism is given by $V(x) = x^3 + 3x^2 - 36x + 32$. Find the missing measures.**



The volume of a rectangular prism is $\ell \times w \times h$.

You know that one measure is $x - 4$, so $x - 4$ is a factor of $V(x)$.

4	1	3	-36	32
		4	28	-32
	1	7	-8	0

The quotient is $x^2 + 7x - 8$. Use this to factor $V(x)$.

$V(x) = x^3 + 3x^2 - 36x + 32$ Volume function

$= (x - 4)(x^2 + 7x - 8)$ Factor.

$= (x - 4)(x + 8)(x - 1)$ Factor the trinomial $x^2 + 7x - 8$.

So, the missing measures of the prism are $x + 8$ and $x - 1$.

CHECK Your Progress

3. The volume of a rectangular prism is given by $V(x) = x^3 + 7x^2 - 36$. Find the expressions for the dimensions of the prism.

CHECK Your Understanding

Example 1 (p. 357)

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function.

1. $f(x) = x^3 - 2x^2 - x + 1$

2. $f(x) = 5x^4 - 6x^2 + 2$

For Exercises 3–5, use the following information.

The projected sales of e-books in millions of dollars can be modeled by the function $S(x) = -17x^3 + 200x^2 - 113x + 44$, where x is the number of years since 2000.

3. Use synthetic substitution to estimate the sales for 2008.

4. Use direct substitution to evaluate $S(8)$.

5. Which method—synthetic substitution or direct substitution—do you prefer to use to evaluate polynomials? Explain your answer.

Examples 2, 3 (pp. 358–359)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

6. $x^3 - x^2 - 5x - 3;\ x + 1$

7. $x^3 - 3x + 2;\ x - 1$

8. $6x^3 - 25x^2 + 2x + 8;\ 3x - 2$

9. $x^4 + 2x^3 - 8x - 16;\ x + 2$

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–17	1
18–29	2, 3
30–33	3

Use synthetic substitution to find $g(3)$ and $g(-4)$ for each function.

10. $g(x) = x^2 - 8x + 6$

11. $g(x) = x^3 + 2x^2 - 3x + 1$

12. $g(x) = x^3 - 5x + 2$

13. $g(x) = x^4 - 6x - 8$

14. $g(x) = 2x^3 - 8x^2 - 2x + 5$

15. $g(x) = 3x^4 + x^3 - 2x^2 + x + 12$

16. $g(x) = x^5 + 8x^3 + 2x - 15$

17. $g(x) = x^6 - 4x^4 + 3x^2 - 10$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

18. $x^3 + 2x^2 - x - 2;\ x - 1$

19. $x^3 - x^2 - 10x - 8;\ x + 1$

20. $x^3 + x^2 - 16x - 16;\ x + 4$

21. $x^3 - 6x^2 + 11x - 6;\ x - 2$

22. $2x^3 - 5x^2 - 28x + 15;\ x - 5$

23. $3x^3 + 10x^2 - x - 12;\ x + 3$

24. $2x^3 + 7x^2 - 53x - 28;\ 2x + 1$

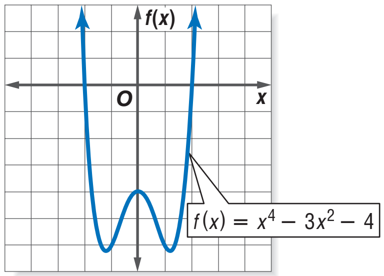
25. $2x^3 + 17x^2 + 23x - 42;\ 2x + 7$

26. $x^4 + 2x^3 + 2x^2 - 2x - 3;\ x + 1$

27. $16x^5 - 32x^4 - 81x + 162;\ x - 2$

28. Use synthetic substitution to show that $x - 8$ is a factor of $x^3 - 4x^2 - 29x - 24$. Then find any remaining factors.

29. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all the factors of the polynomial.



Cross-Curricular Project

Math Online Changes in world population can be modeled by a polynomial equation. Visit algebra2.com to continue work on your project.

BOATING **For Exercises 30 and 31, use the following information.**
A motor boat traveling against waves accelerates from a resting position. Suppose the speed of the boat in feet per second is given by the function $f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t$, where t is the time in seconds.

30. Find the speed of the boat at 1, 2, and 3 seconds.

31. It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Use synthetic substitution to find $f(6)$ and explain what this means.

ENGINEERING **For Exercises 32 and 33, use the following information.**
When a certain type of plastic is cut into sections, the length of each section determines its strength. The function $f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100$ can describe the relative strength of a section of length x feet. Sections of plastic x feet long, where $f(x) = 0$, are extremely weak. After testing the plastic, engineers discovered that sections 5 feet long were extremely weak.

32. Show that $x - 5$ is a factor of the polynomial function.

33. Are there other lengths of plastic that are extremely weak? Explain your reasoning.

Find values of k so that each remainder is 3.

34. $(x^2 - x + k) \div (x - 1)$

35. $(x^2 + kx - 17) \div (x - 2)$

36. $(x^2 + 5x + 7) \div (x + k)$

37. $(x^3 + 4x^2 + x + k) \div (x + 2)$

PERSONAL FINANCE **For Exercises 38–41, use the following information.**
Zach has purchased some home theater equipment for \$2000, which he is financing through the store. He plans to pay \$340 per month and wants to have the balance paid off after six months. The formula $B(x) = 2000x^6 - 340(x^5 + x^4 + x^3 + x^2 + x + 1)$ represents his balance after six months if x represents 1 plus the monthly interest rate (expressed as a decimal).

38. Find his balance after 6 months if the annual interest rate is 12%. (*Hint*: The monthly interest rate is the annual rate divided by 12, so $x = 1.01$.)

39. Find his balance after 6 months if the annual interest rate is 9.6%.

40. How would the formula change if Zach wanted to pay the balance in five months?

41. Suppose he finances his purchase at 10.8% and plans to pay \$410 every month. Will his balance be paid in full after five months?

EXTRA PRACTICE
See pages 904, 931.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

42. **OPEN ENDED** Give an example of a polynomial function that has a remainder of 5 when divided by $x - 4$.

43. **REASONING** Determine the dividend, divisor, quotient, and remainder represented by the synthetic division at the right.

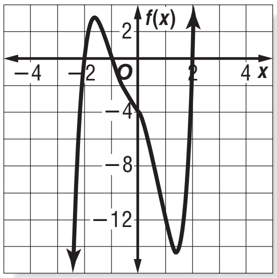
$$\begin{array}{r|rrrr} -2 & 1 & 0 & 6 & 32 \\ & & -2 & 4 & -20 \\ \hline & 1 & -2 & 10 & 12 \end{array}$$

44. **CHALLENGE** Consider the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a + b + c + d + e = 0$. Show that this polynomial is divisible by $x - 1$.

45. ***Writing in Math*** Use the information on page 356 to explain how you can use the Remainder Theorem to evaluate polynomials. Include an explanation of when it is easier to use the Remainder Theorem to evaluate a polynomial rather than substitution. Evaluate the expression for the number of international travelers to the U.S. for $x = 20$.

STANDARDIZED TEST PRACTICE

46. **ACT/SAT** Use the graph of the polynomial function at the right. Which is *not* a factor of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?



A $(x - 2)$

B $(x + 2)$

C $(x - 1)$

D $(x + 1)$

47. **REVIEW** The total area of a rectangle is $25a^4 - 16b^2$. Which factors could represent the length times width?

F $(5a^2 + 4b)(5a^2 + 4b)$

G $(5a^2 + 4b)(5a^2 - 4b)$

H $(5a - 4b)(5a - 4b)$

J $(5a + 4b)(5a - 4b)$

Spiral Review

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

48. $7xy^3 - 14x^2y^5 + 28x^3y^2$

49. $ab - 5a + 3b - 15$

50. $2x^2 + 15x + 25$

51. $c^3 - 216$

Graph each function by making a table of values. (Lesson 6-5)

52. $f(x) = x^3 - 4x^2 + x + 5$

53. $f(x) = x^4 - 6x^3 + 10x^2 - x - 3$

54. **CITY PLANNING** City planners have laid out streets on a coordinate grid before beginning construction. One street lies on the line with equation $y = 2x + 1$. Another street that intersects the first street passes through the point $(2, -3)$ and is perpendicular to the first street. What is the equation of the line on which the second street lies? (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find the exact solutions of each equation by using the Quadratic Formula.** (Lesson 5-6)

55. $x^2 + 7x + 8 = 0$

56. $3x^2 - 9x + 2 = 0$

57. $2x^2 + 3x + 2 = 0$

6-8 Roots and Zeros

Main Ideas

- Determine the number and type of roots for a polynomial equation.
- Find the zeros of a polynomial function.

GET READY for the Lesson

When doctors prescribe medication, they give patients instructions as to how much to take and how often it should be taken. The amount of medication in your body varies with time.

Suppose the equation $M(t) = 0.5t^4 + 3.5t^3 - 100t^2 + 350t$ models the number of milligrams of a certain medication in the bloodstream t hours after it has been taken. The doctor can use the roots of this equation to determine how often the patient should take the medication to maintain a certain concentration in the body.

Types of Roots You have already learned that a zero of a function $f(x)$ is any value c such that $f(c) = 0$. When the function is graphed, the real zeros of the function are the x-intercepts of the graph.

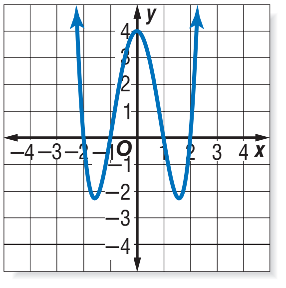
KEY CONCEPT *Zeros, Factors, and Roots*

Let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial function. Then the following statements are equivalent.

- c is a zero of the polynomial function $f(x)$.
- $x - c$ is a factor of the polynomial $f(x)$.
- c is a root or solution of the polynomial equation $f(x) = 0$.

In addition, if c is a real number, then $(c, 0)$ is an intercept of the graph of $f(x)$.

The graph of $f(x) = x^4 - 5x^2 + 4$ is shown at the right. The zeros of the function are $-2, -1, 1$, and 2. The factors of the polynomial are $x + 2$, $x + 1$, $x - 1$, and $x - 2$. The solutions of the equation $f(x) = 0$ are $-2, -1, 1$, and 2. The x-intercepts of the graph of $f(x)$ are $(-2, 0)$, $(-1, 0)$, $(1, 0)$, and $(2, 0)$.



Study Tip

Look Back
For review of **complex numbers,** see Lesson 5-4.

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra.**

KEY CONCEPT *Fundamental Theorem of Algebra*

Every polynomial equation with complex coordinates and degree greater than zero has at least one root in the set of complex numbers.

EXAMPLE 1 Determine Number and Type of Roots

Solve each equation. State the number and type of roots.

a. $x^2 - 8x + 16 = 0$

$x^2 - 8x + 16 = 0$ Original equation

$(x - 4)^2 = 0$ Factor the left side as a perfect square trinomial.

$x = 4$ Solve for x using the Square Root Property.

Since $x - 4$ is twice a factor of $x^2 - 8x + 16$, 4 is a double root. So this equation has one real repeated root, 4.

b. $x^4 - 1 = 0$

$$x^4 - 1 = 0$$

$$(x^2 + 1)(x^2 - 1) = 0$$

$$(x^2 + 1)(x + 1)(x - 1) = 0$$

$$x^2 + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x^2 = -1 \qquad x = -1 \qquad x = 1$$

$$x = \pm\sqrt{-1} \text{ or } \pm i$$

This equation has two real roots, 1 and -1, and two imaginary roots, i and $-i$.

Reading Math

Roots In addition to double roots, equations can have triple or quadruple roots. In general, these roots are referred to as *repeated roots*.

CHECK Your Progress

1A. $x^3 + 2x = 0$

1B. $x^4 - 16 = 0$

Real-World Link

René Descartes (1596–1650) was a French mathematician and philosopher. One of his best-known quotations comes from his *Discourse on Method*: "I think, therefore I am."

Source: *A History of Mathematics*

Compare the degree of each equation and the number of roots of each equation in Example 1. The following corollary of the Fundamental Theorem of Algebra is an even more powerful tool for problem solving.

KEY CONCEPT *Corollary*

A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.

Similarly, a polynomial function of nth degree has exactly n zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

KEY CONCEPT *Descartes' Rule of Signs*

If $P(x)$ is a polynomial with real coefficients, the terms of which are arranged in descending powers of the variable,

- the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

EXAMPLE 2 Find Numbers of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$.

Since $p(x)$ has degree 5, it has five zeros. However, some of them may be imaginary. Use Descartes' Rule of Signs to determine the number and type of real zeros. Count the number of changes in sign for the coefficients of $p(x)$.

$$p(x) = x^5 \quad - \quad 6x^4 \quad - \quad 3x^3 \quad + \quad 7x^2 \quad - \quad 8x \quad + \quad 1$$

yes — no — yes — yes — yes

+ to − — − to − — − to + — + to − — − to +

Since there are 4 sign changes, there are 4, 2, or 0 positive real zeros.

Find $p(-x)$ and count the number of changes in signs for its coefficients.

$$p(x) = (-x)^5 \quad - \quad 6(-x)^4 \quad - \quad 3(-x)^3 \quad + \quad 7(-x)^2 \quad - \quad 8(-x) \quad + \quad 1$$

$$= -x^5 \quad - \quad 6x^4 \quad + \quad 3x^3 \quad + \quad 7x^2 \quad + \quad 8x \quad + \quad 1$$

no — yes — no — no — no

− to − — − to + — + to + — + to + — + to +

Since there is 1 sign change, there is exactly 1 negative real zero.

Thus, the function $p(x)$ has either 4, 2, or 0 positive real zeros and exactly 1 negative real zero. Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	1	0	$4 + 1 + 0 = 5$
2	1	2	$2 + 1 + 2 = 5$
0	1	4	$0 + 1 + 4 = 5$

CHECK Your Progress

2. State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $h(x) = 2x^5 + x^4 + 3x^3 - 4x^2 - x + 9$.

Concepts in Motion
Animation
algebra2.com

Study Tip

Zero at the Origin

Recall that the number 0 has no sign. Therefore, if 0 is a zero of a function, the sum of the number of positive real zeros, negative real zeros, and imaginary zeros is reduced by how many times 0 is a zero of the function.

Find Zeros We can find all of the zeros of a function using some of the strategies you have already learned.

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EXAMPLE 3 Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^3 - 4x^2 + 6x - 4$.

Since $f(x)$ has degree 3, the function has three zeros. To determine the possible number and type of real zeros, examine the number of sign changes for $f(x)$ and $f(-x)$.

$$f(x) = x^3 - 4x^2 + 6x - 4 \qquad f(-x) = -x^3 - 4x^2 - 6x - 4$$

yes yes yes — no no no

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there are no sign changes for the coefficient of $f(-x)$, $f(x)$ has no negative real zeros. Thus, $f(x)$ has either 3 real zeros, or 1 real zero and 2 imaginary zeros.

To find these zeros, first list some possibilities and then eliminate those that are not zeros. Since none of the zeros are negative and $f(0)$ is -4, begin by evaluating $f(x)$ for positive integral values from 1 to 4. You can use a shortened form of synthetic substitution to find $f(a)$ for several values of a.

x	1	-4	6	-4
1	1	-3	3	-1
2	1	-2	2	0
3	1	-1	3	5
4	1	0	6	20

Each row in the table shows the coefficients of the depressed polynomial and the remainder.

> **Study Tip**
>
> **Finding Zeros**
> While direct substitution could be used to find each real zero of a polynomial, using synthetic substitution provides you with a depressed polynomial that can be used to find any imaginary zeros.

From the table, we can see that one zero occurs at $x = 2$. Since the depressed polynomial of this zero, $x^2 - 2x + 2$, is quadratic, use the Quadratic Formula to find the roots of the related quadratic equation, $x^2 - 2x + 2 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

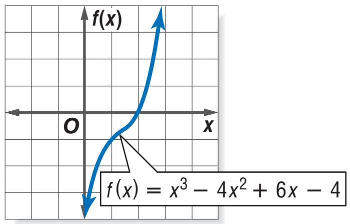
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \qquad \text{Replace } a \text{ with 1, } b \text{ with } -2\text{, and } c \text{ with 2.}$$

$$= \frac{2 \pm \sqrt{-4}}{2} \qquad \text{Simplify.}$$

$$= \frac{2 \pm 2i}{2} \qquad \sqrt{4} \times \sqrt{-1} = 2i$$

$$= 1 \pm i \qquad \text{Simplify.}$$

Thus, the function has one real zero at $x = 2$ and two imaginary zeros at $x = 1 + i$ and $x = 1 - i$. The graph of the function verifies that there is only one real zero.



CHECK Your Progress

3. Find all of the zeros of $h(x) = x^3 + 2x^2 + 9x + 18$.

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In Chapter 5, you learned that solutions of a quadratic equation that contains imaginary numbers come in pairs. This applies to the zeros of polynomial functions as well. For any polynomial function with real coefficients, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

KEY CONCEPT — *Complex Conjugates Theorem*

Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

EXAMPLE Use Zeros to Write a Polynomial Function

4 **Write a polynomial function of least degree with integral coefficients the zeros of which include 3 and $2 - i$.**

Explore If $2 - i$ is a zero, then $2 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x - 3$, $x - (2 - i)$, and $x - (2 + i)$ are factors of the polynomial function.

Plan Write the polynomial function as a product of its factors.

$f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$

Solve Multiply the factors to find the polynomial function.

$$
\begin{aligned}
f(x) &= (x - 3)[x - (2 - i)][x - (2 + i)] && \text{Write an equation.}\\
&= (x - 3)[(x - 2) + i][(x - 2) - i] && \text{Regroup terms.}\\
&= (x - 3)[(x - 2)^2 - i^2] && \text{Rewrite as the difference of two squares.}\\
&= (x - 3)[x^2 - 4x + 4 - (-1)] && \text{Square } x - 2 \text{ and replace } i^2 \text{ with } -1.\\
&= (x - 3)(x^2 - 4x + 5) && \text{Simplify.}\\
&= x^3 - 4x^2 + 5x - 3x^2 + 12x - 15 && \text{Multiply using the Distributive Property.}\\
&= x^3 - 7x^2 + 17x - 15 && \text{Combine like terms.}
\end{aligned}
$$

Check Since there are three zeros, the degree of the polynomial function must be three, so $f(x) = x^3 - 7x^2 + 17x - 15$ is a polynomial function of least degree with integral coefficients and zeros of 3, $2 - i$, and $2 + i$.

CHECK Your Progress

4. Write a polynomial function of least degree with integral coefficients the zeros of which include -1 and $1 + 2i$.

CHECK Your Understanding

Example 1 (p. 363)

Solve each equation. State the number and type of roots.

1. $x^2 + 4 = 0$

2. $x^3 + 4x^2 - 21x = 0$

Example 2 (p. 364)

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

3. $f(x) = 5x^3 + 8x^2 - 4x + 3$

4. $r(x) = x^5 - x^3 - x + 1$

Example 3 (pp. 364–365)

Find all of the zeros of each function.

5. $p(x) = x^3 + 2x^2 - 3x + 20$

6. $f(x) = x^3 - 4x^2 + 6x - 4$

7. $v(x) = x^3 - 3x^2 + 4x - 12$

8. $f(x) = x^3 - 3x^2 + 9x + 13$

Example 4 (pp. 365–366)

9. Write a polynomial function of least degree with integral coefficients the zeros of which include 2 and $4i$.

10. Write a polynomial function of least degree with integral coefficients the zeros of which include $\frac{1}{2}$, 3, and -3.

Exercises

HOMEWORK HELP

For Exercises	See Examples
11–16	1
17–22	3
23–32	2
33–38	4

Solve each equation. State the number and type of roots.

11. $3x + 8 = 0$

12. $2x^2 - 5x + 12 = 0$

13. $x^3 + 9x = 0$

14. $x^4 - 81 = 0$

15. $x^4 - 16 = 0$

16. $x^5 - 8x^3 + 16x = 0$

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

17. $f(x) = x^3 - 6x^2 + 1$

18. $g(x) = 5x^3 + 8x^2 - 4x + 3$

19. $h(x) = 4x^3 - 6x^2 + 8x - 5$

20. $q(x) = x^4 + 5x^3 + 2x^2 - 7x - 9$

21. $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$

22. $f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1$

Find all of the zeros of each function.

23. $g(x) = x^3 + 6x^2 + 21x + 26$

24. $h(x) = x^3 - 6x^2 + 10x - 8$

25. $f(x) = x^3 - 5x^2 - 7x + 51$

26. $f(x) = x^3 - 7x^2 + 25x - 175$

27. $g(x) = 2x^3 - x^2 + 28x + 51$

28. $q(x) = 2x^3 - 17x^2 + 90x - 41$

29. $h(x) = 4x^4 + 17x^2 + 4$

30. $p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40$

31. $r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13$

32. $h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156$

Write a polynomial function of least degree with integral coefficients that has the given zeros.

33. $-4, 1, 5$

34. $-2, 2, 4, 6$

35. $4i, 3, -3$

36. $2i, 3i, 1$

37. $9, 1 + 2i$

38. $6, 2 + 2i$

PROFIT **For Exercises 39–41, use the following information.**
A computer manufacturer determines that for each employee the profit for producing x computers per day is $P(x) = -0.006x^4 + 0.15x^3 - 0.05x^2 - 1.8x$.

39. How many positive real zeros, negative real zeros, and imaginary zeros exist for this function? (*Hint:* Notice that 0, which is neither positive nor negative, is a zero of this function since $d(0) = 0$.)

40. Approximate all real zeros to the nearest tenth by graphing the function using a graphing calculator.

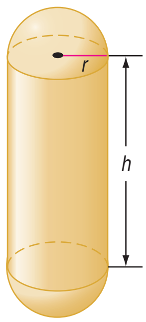
41. What is the meaning of the roots in this problem?

Real-World Link

A space shuttle is a reusable vehicle, launched like a rocket, which can put people and equipment in orbit around Earth. The first space shuttle was launched in 1981.

Source: kidsastronomy.about.com

SPACE EXPLORATION **For Exercises 42 and 43, use the following information.**

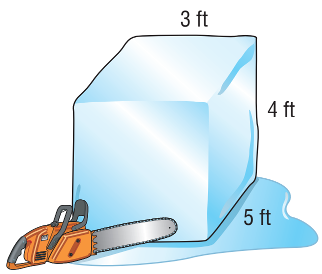


The space shuttle has an external tank for the fuel that the main engines need for the launch. This tank is shaped like a capsule, a cylinder with a hemispherical dome at either end. The cylindrical part of the tank has an approximate volume of 336π cubic meters and a height of 17 meters more than the radius of the tank. (*Hint:* $V(r) = \pi r^2 h$)

42. Write an equation that represents the volume of the cylinder.

43. What are the dimensions of the cylindrical part of the tank?

SCULPTING **For Exercises 44 and 45, use the following information.**



Antonio is preparing to make an ice sculpture. He has a block of ice that he wants to reduce in size by shaving off the same amount from the length, width, and height. He wants to reduce the volume of the ice block to 24 cubic feet.

44. Write a polynomial equation to model this situation.

45. How much should he take from each dimension?

EXTRA PRACTICE
See pages 904, 931.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

46. OPEN ENDED Sketch the graph of a polynomial function that has the indicated number and type of zeros.

a. 3 real, 2 imaginary **b.** 4 real **c.** 2 imaginary

47. CHALLENGE If a sixth-degree polynomial equation has exactly five distinct real roots, what can be said of one of its roots? Draw a graph of this situation.

48. REASONING State the least degree a polynomial equation with real coefficients can have if it has roots at $x = 5 + i$, $x = 3 - 2i$, and a double root at $x = 0$. Explain.

49. CHALLENGE Find a counterexample to disprove the following statement. *The polynomial function of least degree with integral coefficients with zeros at $x = 4$, $x = -1$, and $x = 3$, is unique.*

50. *Writing in Math* Use the information about medication on page 362 to explain how the roots of an equation can be used in pharmacology. Include an explanation of what the roots of this equation represent and an explanation of what the roots of this equation reveal about how often a patient should take this medication.

STANDARDIZED TEST PRACTICE

51. ACT/SAT How many negative real zeros does $f(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$ have?

A 3

B 2

C 1

D 0

52. REVIEW Tiles numbered from 1 to 6 are placed in a bag and are drawn out to determine which of six tasks will be assigned to six people. What is the probability that the tiles numbered 5 and 6 are drawn consecutively?

F $\frac{2}{3}$ G $\frac{2}{5}$ H $\frac{1}{2}$ J $\frac{1}{3}$

Spiral Review

Use synthetic substitution to find $f(-3)$ and $f(4)$ for each function. (Lesson 6-7)

53. $f(x) = x^3 - 5x^2 + 16x - 7$

54. $f(x) = x^4 + 11x^3 - 3x^2 + 2x - 5$

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

55. $15a^2b^2 - 5ab^2c^2$

56. $12p^2 - 64p + 45$

57. $4y^3 + 24y^2 + 36y$

58. BASKETBALL In a recent season, Monique Currie of the Duke Blue Devils scored 635 points. She made a total of 356 shots, including 3-point field goals, 2-point field goals, and 1-point free throws. She made 76 more 2-point field goals than free throws and 77 more free throws than 3-point field goals. Find the number of each type of shot she made. (Lesson 3-5)

GET READY for the Next Lesson

PREREQUISITE SKILL Find all values of $\pm\frac{a}{b}$ given each replacement set.

59. $a = \{1, 5\}; b = \{1, 2\}$

60. $a = \{1, 2\}; b = \{1, 2, 7, 14\}$

61. $a = \{1, 3\}; b = \{1, 3, 9\}$

62. $a = \{1, 2, 4\}; b = \{1, 2, 4, 8, 16\}$

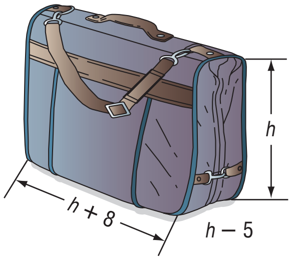
6-9 Rational Zero Theorem

Main Ideas

- Identify the possible rational zeros of a polynomial function.
- Find all the rational zeros of a polynomial function.

GET READY for the Lesson

On an airplane, carry-on baggage must fit into the overhead compartment above the passenger's seat. The length of the compartment is 8 inches longer than the height, and the width is 5 inches shorter than the height. The volume of the compartment is 2772 cubic inches. You can solve the polynomial equation $h(h + 8)(h - 5) = 2772$, where h is the height, $h + 8$ is the length, and $h - 5$ is the width, to find the dimensions of the overhead compartment.



Identify Rational Zeros Usually it is not practical to test all possible zeros of a polynomial function using only synthetic substitution. The **Rational Zero Theorem** can help you choose some possible zeros to test.

KEY CONCEPT *Rational Zero Theorem*

Words Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_0 and q is a factor of a_n.

Example Let $f(x) = 2x^3 + 3x^2 - 17x + 12$. If $\frac{3}{2}$ is a zero of $f(x)$, then 3 is a factor of 12 and 2 is a factor of 2.

Study Tip

The Rational Zero Theorem only applies to rational zeros. Not *all* of the roots of a polynomial are found using the divisibility of the coefficients.

In addition, if the coefficient of the x term with the highest degree is 1, we have the following corollary.

KEY CONCEPT *Corollary (Integral Zero Theorem)*

If the coefficients of a polynomial function are integers such that $a_n = 1$ and $a_0 \neq 0$, any rational zeros of the function must be factors of a_n.

EXAMPLE 1 **Identify Possible Zeros**

List all of the possible rational zeros of each function.

a. $f(x) = 2x^3 - 11x^2 + 12x + 9$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 9 and q is a factor of 2. The possible values of p are ±1, ±3, and ±9. The possible values for q are ±1 and ±2. So, $\frac{p}{q} = \pm1, \pm3, \pm9, \pm\frac{1}{2}, \pm\frac{3}{2}$, and $\pm\frac{9}{2}$.

(continued on the next page)

b. $f(x) = x^3 - 9x^2 - x + 105$

Since the coefficient of x^3 is 1, the possible rational zeros must be a factor of the constant term 105. So, the possible rational zeros are the integers $\pm1, \pm3, \pm5, \pm7, \pm15, \pm21, \pm35$, and ±105.

CHECK Your Progress

1A. $g(x) = 3x^3 - 4x + 10$

1B. $h(x) = x^3 + 11x^2 + 24$

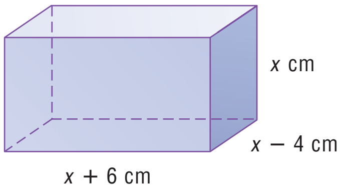
Find Rational Zeros Once you have found the possible rational zeros of a function, you can test each number using synthetic substitution to determine the zeros of the function.

EXAMPLE 2 Find Rational Zeros

GEOMETRY The volume of a rectangular solid is 675 cubic centimeters. The width is 4 centimeters less than the height, and the length is 6 centimeters more than the height. Find the dimensions of the solid.

Let x = the height, $x - 4$ = the width, and $x + 6$ = the length.

Write an equation for the volume.



$\ell wh = V$ Formula for volume

$(x - 4)(x + 6)x = 675$ Substitute.

$x^3 + 2x^2 - 24x = 675$ Multiply.

$x^3 + 2x^2 - 24x - 675 = 0$ Subtract 675.

The leading coefficient is 1, so the possible integer zeros are factors of 675, $\pm1, \pm3, \pm5, \pm9, \pm15, \pm25, \pm27, \pm45, \pm75, \pm135, \pm225$, and ±675. Since length can only be positive, we only need to check positive zeros. From Descartes' Rule of Signs, we also know there is only one positive real zero. Make a table for the synthetic division and test possible real zeros.

p	1	2	−24	−675
1	1	3	−21	−696
3	1	5	−9	−702
5	1	7	11	−620
9	1	11	75	0

One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are 9 − 4 or 5 centimeters and 9 + 6 or 15 centimeters.

CHECK Verify that the dimensions are correct. $5 \times 9 \times 15 = 675$ ✓

CHECK Your Progress

2. The volume of a rectangular solid is 1056 cubic inches. The length is 1 inch more than the width, and the height is 3 inches less than the width. Find the dimensions of the solid.

Study Tip

Descartes' Rule of Signs

Examine the signs of the coefficients in the equation, + + − −. There is one change of sign, so there is only one positive real zero.

You usually do not need to test all of the possible zeros. Once you find a zero, you can try to factor the depressed polynomial to find any other zeros.

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EXAMPLE 3 Find All Zeros

Find all of the zeros of $f(x) = 2x^4 - 13x^3 + 23x^2 - 52x + 60$.

From the corollary to the Fundamental Theorem of Algebra, we know there are exactly 4 complex roots. According to Descartes' Rule of Signs, there are 4, 2, or 0 positive real roots and 0 negative real roots. The possible rational zeros are $\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm10, \pm12, \pm15, \pm20, \pm30, \pm60, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$, and $\pm\frac{15}{2}$. Make a table and test some possible rational zeros.

$\frac{p}{q}$	2	−13	23	−52	60
1	2	−11	12	−40	20
2	2	−9	5	−42	−24
3	2	−7	2	−46	−78
5	2	−3	8	−12	0

Since $f(5) = 0$, you know that $x = 5$ is a zero. The depressed polynomial is $2x^3 - 3x^2 + 8x - 12$.

Factor $2x^3 - 3x^2 + 8x - 12$.

$2x^3 - 3x^2 + 8x - 12 = 0$	Write the depressed polynomial.
$2x^3 + 8x - 3x^2 - 12 = 0$	Regroup terms.
$2x(x^2 + 4) - 3(x^2 + 4) = 0$	Factor by grouping.
$(x^2 + 4)(2x - 3) = 0$	Distributive Property
$x^2 + 4 = 0$ or $2x - 3 = 0$	Zero Product Property
$x^2 = -4$ $\quad$ $2x = 3$	
$x = \pm 2i$ $\quad$ $x = \frac{3}{2}$	

There is another real zero at $x = \frac{3}{2}$ and two imaginary zeros at $x = 2i$ and $x = -2i$.

The zeros of this function are $5, \frac{3}{2}, 2i$ and $-2i$.

CHECK Your Progress

Find all of the zeros of each function.

3A. $h(x) = 9x^4 + 5x^2 - 4$

3B. $k(x) = 2x^4 - 5x^3 + 20x^2 - 45x + 18$

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CHECK Your Understanding

Example 1 (pp. 369–370)

List all of the possible rational zeros of each function.

1. $p(x) = x^4 - 10$

2. $d(x) = 6x^3 + 6x^2 - 15x - 2$

Example 2 (p. 370)

Find all of the rational zeros of each function.

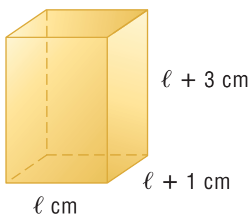
3. $p(x) = x^3 - 5x^2 - 22x + 56$

4. $f(x) = x^3 - x^2 - 34x - 56$

5. $t(x) = x^4 - 13x^2 + 36$

6. $f(x) = 2x^3 - 7x^2 - 8x + 28$

7. GEOMETRY The volume of the rectangular solid is 1430 cubic centimeters. Find the dimensions of the solid.



Example 3 (p. 371)

Find all of the zeros of each function.

8. $f(x) = 6x^3 + 5x^2 - 9x + 2$

9. $f(x) = x^4 - x^3 - x^2 - x - 2$

Exercises

For Exercises	See Examples
10–15	1
16–21	2
22–29	3

HOMEWORK HELP

List all of the possible rational zeros of each function.

10. $f(x) = x^3 + 6x + 2$

11. $h(x) = x^3 + 8x + 6$

12. $f(x) = 3x^4 + 15$

13. $n(x) = x^5 + 6x^3 - 12x + 18$

14. $p(x) = 3x^3 - 5x^2 - 11x + 3$

15. $h(x) = 9x^6 - 5x^3 + 27$

Find all of the rational zeros of each function.

16. $f(x) = x^3 + x^2 - 80x - 300$

17. $p(x) = x^3 - 3x - 2$

18. $f(x) = 2x^5 - x^4 - 2x + 1$

19. $f(x) = x^5 - 6x^3 + 8x$

20. $g(x) = x^4 - 3x^3 + x^2 - 3x$

21. $p(x) = x^4 + 10x^3 + 33x^2 + 38x + 8$

Find all of the zeros of each function.

22. $p(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$

23. $g(x) = 5x^4 - 29x^3 + 55x^2 - 28x$

24. $h(x) = 6x^3 + 11x^2 - 3x - 2$

25. $p(x) = x^3 + 3x^2 - 25x + 21$

26. $h(x) = 10x^3 - 17x^2 - 7x + 2$

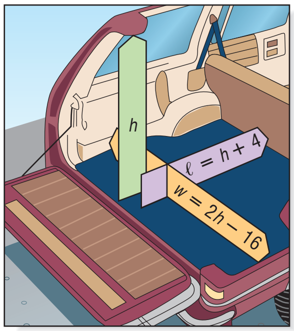
27. $g(x) = 48x^4 - 52x^3 + 13x - 3$

28. $p(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

29. $h(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$

AUTOMOBILES **For Exercises 30 and 31, use the following information.**

The length of the cargo space in a sport-utility vehicle is 4 inches greater than the height of the space. The width is sixteen inches less than twice the height. The cargo space has a total volume of 55,296 cubic inches.



30. Use a rectangular prism to model the cargo space. Write a polynomial function that represents the volume of the cargo space.

31. Will a package 34 inches long, 44 inches wide, and 34 inches tall fit in the cargo space? Explain.

FOOD **For Exercises 32–34, use the following information.**

A restaurant orders spaghetti sauce in cylindrical metal cans. The volume of each can is about 160π cubic inches, and the height of the can is 6 inches more than the radius.

32. Write a polynomial equation that represents the volume of a can. Use the formula for the volume of a cylinder, $V = \pi r^2 h$.

33. What are the possible values of r? Which values are reasonable here?

34. Find the dimensions of the can.

AMUSEMENT PARKS **For Exercises 35–37, use the following information.**

An amusement park owner wants to add a new wilderness water ride that includes a mountain that is shaped roughly like a square pyramid. Before building the new attraction, engineers must build and test a scale model.

35. If the height of the scale model is 9 inches less than its length, write a polynomial function that describes the volume of the model in terms of its length. Use the formula for the volume of a pyramid, $V = \frac{1}{3}Bh$.

36. If the volume is 6300 cubic inches, write an equation for the situation.

37. What are the dimensions of the scale model?

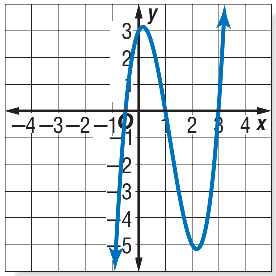
EXTRA PRACTICE
See pages 905, 931.

Math Online
Self-Check Quiz at
algebra2.com

For Exercises 38 and 39, use the following information.

38. Find all of the zeros of $f(x) = x^3 - 2x^2 + 3$ and $g(x) = 2x^3 - 7x^2 + 2x + 3$.

39. Determine which function, f or g, is shown in the graph at the right.



H.O.T. Problems

40. FIND THE ERROR Lauren and Luis are listing the possible rational zeros of $f(x) = 4x^5 + 4x^4 - 3x^3 + 2x^2 - 5x + 6$. Who is correct? Explain your reasoning.

Lauren	Luis
$\pm 1, \pm \frac{1}{2},$	$\pm 1, \pm \frac{1}{2},$
$\pm \frac{1}{3}, \pm \frac{1}{6},$	$\pm \frac{1}{4}, \pm 2,$
$\pm 2, \pm \frac{2}{3},$	$\pm 3, \pm \frac{3}{2},$
$\pm 4, \pm \frac{4}{3}$	$\pm \frac{3}{4}, \pm 6,$

41. OPEN ENDED Write a polynomial function that has possible rational zeros of $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$.

42. CHALLENGE If k and $2k$ are zeros of $f(x) = x^3 + 4x^2 + 9kx - 90$, find k and all three zeros of $f(x)$.

43. *Writing in Math* Use the information on page 369 to explain how the Rational Zero Theorem can be used to solve problems involving large numbers. Include the polynomial equation that represents the volume of the overhead baggage compartment and a list of all measures of the width of the compartment, assuming that the width is a whole number.

STANDARDIZED TEST PRACTICE

44. Which of the following is a zero of the function $f(x) = 12x^5 - 5x^3 + 2x - 9$?

A -6

B $\frac{3}{8}$

C $-\frac{2}{3}$

D 1

45. REVIEW A window is in the shape of an equilateral triangle. Each side of the triangle is 8 feet long. The window is divided in half by a support from one vertex to the midpoint of the side of the triangle opposite the vertex. Approximately how long is the support?

F 5.7 ft

G 6.9 ft

H 11.3 ft

J 13.9 ft

Spiral Review

Given a function and one of its zeros, find all of the zeros of the function. (Lesson 6-8)

46. $g(x) = x^3 + 4x^2 - 27x - 90; -3$

47. $h(x) = x^3 - 11x + 20; 2 + i$

48. $f(x) = x^3 + 5x^2 + 9x + 45; -5$

49. $g(x) = x^3 - 3x^2 - 41x + 203; -7$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. (Lesson 6-7)

50. $20x^3 - 29x^2 - 25x + 6; x - 2$

51. $3x^4 - 21x^3 + 38x^2 - 14x + 24; x - 3$

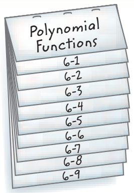
52. GEOMETRY The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. What are the lengths of all three sides? (Lesson 3-5)

CHAPTER 6 Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer — GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Properties of Exponents (Lesson 6-1)

- The properties of powers for real numbers a and b and integers m and n are as follows.

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
$a^m \cdot a^n = a^{m+n}$	$(a^m)^n = a^{mn}$
$(ab)^m = a^m b^m$	$a^{-n} = \frac{1}{a^n}, a \neq 0$

Operations with Polynomials (Lesson 6-2)

- To add or subtract: Combine like terms.
- To multiply: Use the Distributive Property.
- To divide: Use long division or synthetic division.

Polynomial Functions and Graphs (Lessons 6-4 and 6-5)

- Turning points of a function are called *relative maxima* and *relative minima.*

Solving Polynomial Equations (Lesson 6-6)

- You can factor polynomials using the GCF, grouping, or quadratic techniques.

The Remainder and Factor Theorems (Lesson 6-7)

- Factor Theorem: The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Roots, Zeros, and the Rational Zero Theorem (Lessons 6-8 and 6-9)

- Complex Conjugates Theorem: If $a + bi$ is a zero of a function, then $a - bi$ is also a zero.
- Integral Zero Theorem: If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n = 0$, any rational zeros of the function must be factors of a_n.

Key Vocabulary

degree of a polynomial (p. 320)
depressed polynomial (p. 357)
end behavior (p. 334)
leading coefficient (p. 331)
polynomial function (p. 332)
polynomial in one variable (p. 331)
quadratic form (p. 351)
relative maximum (p. 340)
relative minimum (p. 340)
scientific notation (p. 315)
simplify (p. 312)
standard notation (p. 315)
synthetic division (p. 327)
synthetic substitution (p. 356)

Vocabulary Check

Choose a term from the list above that best completes each statement or phrase.

1. A point on the graph of a polynomial function that has no other nearby points with lesser y-coordinates is a ____.
2. The _______ is the coefficient of the term in a polynomial function with the highest degree.
3. $(x^2)^2 - 17(x^2) + 16 = 0$ is written in ____.
4. A shortcut method known as ____ is used to divide polynomials by binomials.
5. A number is expressed in ________ when it is in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.
6. The __________ is the sum of the exponents of the variables of a monomial.
7. When a polynomial is divided by one of its binomial factors, the quotient is called a(n) _____________.
8. When we ________ an expression, we rewrite it without parentheses or negative exponents.
9. What a graph does as x approaches positive infinity or negative infinity is called the ________ of the graph.
10. The use of synthetic division to evaluate a function is called________.

Math Online Vocabulary Review at algebra2.com

Lesson-by-Lesson Review

6–1 Properties of Exponents (pp. 312–318)

Simplify. Assume that no variable equals 0.

11. $f^{-7} \cdot f^4$

12. $(3x^2)^3$

13. $(2y)(4xy^3)$

14. $\left(\frac{3}{5}c^2f\right)\left(\frac{4}{3}cd\right)^2$

15. MARATHON Assume that there are 10,000 runners in a marathon and each runner runs a distance of 26.2 miles. If you add together the total number of miles for all runners, how many times around the world would the marathon runners have gone? Consider the circumference of Earth to be 2.5×10^4 miles.

Example 1 **Simplify $(3x^4y^6)(-8x^3y)$.**

$(3x^4y^6)(-8x^3y)$

$= (3)(-8)x^{4+3}y^{6+1}$ Commutative Property and Product of Powers

$= -24x^7y^7$ Simplify.

Example 2 **Light travels at approximately 3.0×10^8 meters per second. How far does light travel in one week?**

Determine the number of seconds in one week.

$60 \cdot 60 \cdot 24 \cdot 7 = 604{,}800$ or 6.048×10^5 seconds

Multiply by the speed of light.

$(3.0 \times 10^8) \cdot (6.048 \times 10^5) = 1.8144 \times 10^{14}$ m

6–2 Operations with Polynomials (pp. 320–324)

Simplify.

16. $(4c - 5) - (c + 11) + (-6c + 17)$

17. $(11x^2 + 13x - 15) - (7x^2 - 9x + 19)$

18. $(d - 5)(d + 3)$

19. $(2a^2 + 6)^2$

20. CAR RENTAL The cost of renting a car is \$40 per day plus \$0.10 per mile. If a car is rented for d days and driven m miles a day, represent the cost C.

Example 3 **Find $(9k + 4)(7k - 6)$.**

$(9k + 4)(7k - 6)$

$= (9k)(7k) + (9k)(-6) + (4)(7k) + (4)(-6)$

$= 63k^2 - 54k + 28k - 24$

$= 63k^2 - 26k - 24$

6–3 Dividing Polynomials (pp. 325–330)

Simplify.

21. $(2x^4 - 6x^3 + x^2 - 3x - 3) \div (x - 3)$

22. $x^4 + 18x^3 + 10x^2 + 3x) \div (x^2 + 3x)$

23. SAILING The area of a triangular sail is $16x^4 - 60x^3 - 28x^2 + 56x - 32$ square meters. The base of the triangle is $x - 4$ meters. What is the height of the sail?

Example 4 **Use synthetic division to find $(4x^4 - x^3 - 19x^2 + 11x - 2) \div (x - 2)$.**

$$\begin{array}{r|rrrrr} 2 & 4 & -1 & -19 & 11 & -2 \\ & & 8 & 14 & -10 & 2 \\ \hline & 4 & 7 & -5 & 1 & 0 \end{array}$$

The quotient is $4x^3 + 7x^2 - 5x + 1$.

CHAPTER 6

Study Guide and Review

6–4 Polynomial Functions (pp. 331–338)

Find $p(-4)$ and $p(x + h)$ for each function.

24. $p(x) = x - 2$

25. $p(x) = -x + 4$

26. $p(x) = 6x + 3$

27. $p(x) = x^2 + 5$

28. $p(x) = x^2 - x$

29. $p(x) = 2x^3 - 1$

30. STORMS The average depth of a tsunami can be modeled by $d(s) = \left(\frac{s}{356}\right)^2$, where s is the speed in kilometers per hour and d is the average depth of the water in kilometers. Find the average depth of a tsunami when the speed is 250 kilometers per hour.

Example 5 Find $p(a + 1)$ if $p(x) = 5x - x^2 + 3x^3$.

$$\begin{aligned} p(a+1) &= 5(a+1) - (a+1)^2 + 3(a+1)^3 \\ &= 5a + 5 - (a^2 + 2a + 1) + 3(a^3 + 3a^2 + 3a + 1) \\ &= 5a + 5 - a^2 - 2a - 1 + 3a^3 + 9a^2 + 9a + 3 \\ &= 3a^3 + 8a^2 + 12a + 7 \end{aligned}$$

6–5 Analyzing Graphs of Polynomial Functions (pp. 339–347)

For Exercises 31–36, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which the real zeros are located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

31. $h(x) = x^3 - 6x - 9$

32. $f(x) = x^4 + 7x + 1$

33. $p(x) = x^5 + x^4 - 2x^3 + 1$

34. $g(x) = x^3 - x^2 + 1$

35. $r(x) = 4x^3 + x^2 - 11x + 3$

36. $f(x) = x^3 + 4x^2 + x - 2$

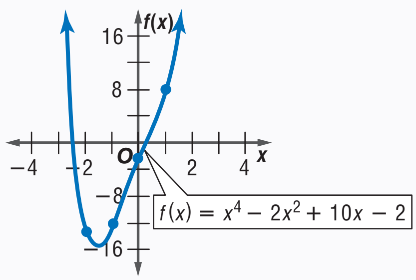
37. PROFIT A small business' monthly profits for the first half of 2006 can be modeled by (1, 550), (2, 725), (3, 680), (4, 830), (5, 920), (6, 810). How many turning points would the graph of a polynomial function through these points have? Describe them.

Example 6 Graph $f(x) = x^4 - 2x^2 + 10x - 2$ by making a table of values.

Make a table of values for several values of x.

x	−3	−2	−1	0	1	2
$f(x)$	31	−14	−13	−2	7	26

Plot the points and connect the points with a smooth curve.



Mixed Problem Solving
For mixed problem-solving practice, see page 931.

6–6 Solving Polynomial Equations (pp. 349–355)

Factor completely. If the polynomial is not factorable, write *prime*.

38. $10a^3 - 20a^2 - 2a + 4$

39. $5w^3 - 20w^2 + 3w - 12$

40. $x^4 - 7x^3 + 12x^2$ **41.** $x^2 - 7x + 5$

Solve each equation.

42. $3x^3 + 4x^2 - 15x = 0$

43. $m^4 + 3m^3 = 40m^2$

44. $x^4 - 8x^2 + 16 = 0$ **45.** $a^3 - 64 = 0$

46. HOME DECORATING The area of a dining room is 160 square feet. A rectangular rug placed in the center of the room is twice as long as it is wide. If the rug is bordered by 2 feet of hardwood floor on all sides, find the dimensions of the rug.

Example 7 **Factor $3m^2 + m - 4$.**

Find two numbers with a product of 3(−4) or −12 and a sum of 1. The two numbers must be 4 and −3 because 4(−3) = −12 and 4 + (−3) = 1.

$$\begin{aligned} 3m^2 + m - 4 &= 3m^2 + 4m - 3m - 4 \\ &= (3m^2 + 4m) - (3m + 4) \\ &= m(3m + 4) + (-1)(3m + 4) \\ &= (3m + 4)(m - 1) \end{aligned}$$

Example 8 **Solve $x^3 - 3x^2 - 54x = 0$.**

$x^3 - 3x^2 - 54x = 0$

$x(x - 9)(x + 6) = 0$

$x(x^2 - 3x - 54) = 0$

$x = 0$ or $x - 9 = 0$ or $x + 6 = 0$

$x = 0$ $\quad x = 9$ $\quad x = -6$

6–7 The Remainder and Factor Theorems (pp. 356–361)

Use synthetic substitution to find $f(3)$ and $f(-2)$ for each function.

47. $f(x) = x^2 - 5$ **48.** $f(x) = x^2 - 4x + 4$

49. $f(x) = x^3 - 3x^2 + 4x + 8$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

50. $x^3 + 5x^2 + 8x + 4;\ x + 1$

51. $x^3 + 4x^2 + 7x + 6;\ x + 2$

52. PETS The volume of water in a rectangular fish tank can be modeled by the polynomial $3x^3 - x^2 - 34x - 40$. If the depth of the tank is given by the polynomial $3x + 5$, what polynomials express the length and width of the fish tank?

Example 9 **Show that $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Then find any remaining factors of the polynomial.**

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

The remainder is 0, so $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Since $x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$, the remaining factors of $x^3 - 2x^2 - 5x + 6$ are $x - 3$ and $x - 1$.

Chapter 6 Study Guide and Review

6-8 Roots and Zeroes (pp. 362–368)

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

53. $f(x) = 2x^4 - x^3 + 5x^2 + 3x - 9$

54. $f(x) = -4x^4 - x^2 - x + 1$

55. $f(x) = 3x^4 - x^3 + 8x^2 + x - 7$

56. $f(x) = 2x^4 - 3x^3 - 2x^2 + 3$

DESIGN For Exercises 57 and 58, use the following information.

An artist has a piece he wants displayed in a gallery. The gallery told him the biggest piece they would display is 72 cubic feet. The artwork is currently 5 feet long, 8 feet wide, and 6 feet high. Joe decides to cut off the same amount from the length, width, and height.

57. Assume that a rectangular prism is a good model for the artwork. Write a polynomial equation to model this situation.

58. How much should he take from each dimension?

Example 10 **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 5x^4 + 6x^3 - 8x + 12$.**

Since $f(x)$ has two sign changes, there are 2 or 0 real positive zeros.

$f(-x) = 5x^4 - 6x^3 + 8x + 12$

Since $f(-x)$ has two sign changes, there are 0 or 2 negative real zeros.

There are 0, 2, or 4 imaginary zeros.

6-9 Rational Zero Theorem (pp. 369–373)

Find all of the rational zeros of each function.

59. $f(x) = 2x^3 - 13x^2 + 17x + 12$

60. $f(x) = x^3 - 3x^2 - 10x + 24$

61. $f(x) = x^4 - 4x^3 - 7x^2 + 34x - 24$

62. $f(x) = 2x^3 - 5x^2 - 28x + 15$

63. $f(x) = 2x^4 - 9x^3 + 2x^2 + 21x - 10$

64. **SHIPPING** The height of a shipping cylinder is 4 feet more than the radius. If the volume of the cylinder is 5π cubic feet, how tall is it? Use the formula $V = \pi \cdot r^2 \cdot h$.

Example 11 **Find all of the zeros of $f(x) = x^3 + 7x^2 - 36$.**

There are exactly three complex zeros. There are one positive real zero and two negative real zeros. The possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, ±36.

$$\begin{array}{r|rrrr} 2 & 1 & 7 & 0 & -36 \\ & & 2 & 18 & 36 \\ \hline & 1 & 9 & 18 & 0 \end{array}$$

$$x^3 + 7x^2 - 36 = (x - 2)(x^2 + 9x + 18)$$
$$= (x - 2)(x + 3)(x + 6)$$

Therefore, the zeros are 2, −3, and −6.

CHAPTER 6 Practice Test

Simplify.

1. $(5b)^4(6c)^2$
2. $(13x - 1)(x + 3)$
3. $(3x^2 - 5x + 2) - (x^2 + 12x - 7)$
4. $(8x^3 + 9x^2 + 2x - 10) + (10x - 9)$
5. $(x^4 - x^3 - 10x^2 + 4x + 24) \div (x - 2)$
6. $(2x^3 + 9x^2 - 2x + 7) \div (x + 2)$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

7. $x^3 - x^2 - 5x - 3;\ x + 1$
8. $x^3 + 8x + 24;\ x + 2$

Factor completely. If the polynomial is not factorable, write *prime*.

9. $3x^3y + x^2y^2 + x^2y$
10. $3x^2 - 2x - 2$
11. $ax^2 + 6ax + 9a$
12. $8r^3 - 64s^6$
13. $x^2 - 14x + 45$
14. $2r^2 + 3pr - 2p^2$

For Exercises 15–18, complete each of the following.

a. **Graph each function by making a table of values.**

b. **Determine consecutive integer values of x between which each real zero is located.**

c. **Estimate the x-coordinates at which the relative maxima and relative minima occur.**

15. $g(x) = x^3 + 6x^2 + 6x - 4$
16. $h(x) = x^4 + 6x^3 + 8x^2 - x$
17. $f(x) = x^3 + 3x^2 - 2x + 1$
18. $g(x) = x^4 - 2x^3 - 6x^2 + 8x + 5$

Solve each equation.

19. $a^4 = 6a^2 + 27$
20. $p^3 + 8p^2 = 18p$
21. $16x^4 - x^2 = 0$
22. $r^4 - 9r^2 + 18 = 0$
23. $p^{\frac{3}{2}} - 8 = 0$
24. $n^3 + n - 27 = n$

25. **TRAVEL** While driving in a straight line from Milwaukee to Madison, your velocity is given by $v(t) = 5t^2 - 50t + 120$, where t is driving time in hours. Estimate your speed after 1 hour of driving.

Use synthetic substitution to find $f(-2)$ and $f(3)$ for each function.

26. $f(x) = 7x^5 - 25x^4 + 17x^3 - 32x^2 + 10x - 22$
27. $f(x) = 3x^4 - 12x^3 - 21x^2 + 30x$
28. Write $36x^{\frac{2}{3}} + 18x^{\frac{1}{3}} + 5$ in quadratic form.
29. Write the polynomial equation of degree 4 with leading coefficient 1 that has roots at -2, -1, 3, and 4.

State the possible number of positive real zeros, negative real zeros, and imaginary zeros for each function.

30. $f(x) = x^3 - x^2 - 14x + 24$
31. $f(x) = 2x^3 - x^2 + 16x - 5$

Find all rational zeros of each function.

32. $g(x) = x^3 - 3x^2 - 53x - 9$
33. $h(x) = x^4 + 2x^3 - 23x^2 + 2x - 24$
34. $f(x) = 5x^3 - 29x^2 + 55x - 28$
35. $g(x) = 4x^3 + 16x^2 - x - 24$

FINANCIAL PLANNING For Exercises 36 and 37, use the following information.

Toshi will start college in six years. According to their plan, Toshi's parents will save $1000 each year for the next three years. During the fourth and fifth years, they will save $1200 each year. During the last year before he starts college, they will save $2000.

36. In the formula $A = P(1 + r)^t$, $A =$ the balance, $P =$ the amount invested, $r =$ the interest rate, and $t =$ the number of years the money has been invested. Use this formula to write a polynomial equation to describe the balance of the account when Toshi starts college.
37. Find the balance of the account if their investment yields 6% annually.

Math Online Chapter Test at algebra2.com

CHAPTER 6

Standardized Test Practice

Cumulative, Chapters 1–6

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Which expression is equivalent to $3a(2a + 1) - (2a - 2)(a + 3)$?

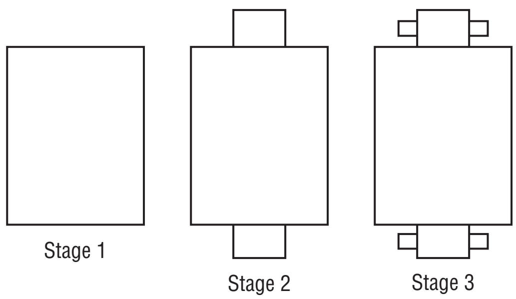
A $2a^2 + 6a + 7$

B $4a^2 - a + 6$

C $4a^2 + 6a - 6$

D $4a^2 - 3a + 7$

2. The figure below shows the first 3 stages of a fractal.



How many rectangles will the nth stage of this fractal contain?

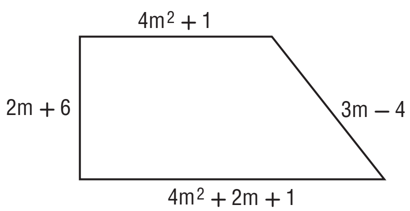
F $2n$

G 2^n

H $2n - 1$

J $2^n - 1$

3. GRIDDABLE Miguel is finding the perimeter of the quadrilateral below. What is the value of the constant term of the perimeter?



4. Which expression best represents the simplification of $(-2a^{-2}b^{-6})(-3a^{-1}b^{8})$?

A $-\frac{1}{6a^3b^2}$

B $\frac{6b^2}{a^3}$

C $\frac{a^2}{6b^{14}}$

D $\frac{6a^2}{b^{48}}$

5. Which expression is equivalent to $(6a - 2b) - \frac{1}{4}(4a + 12b)$?

F $5a + 10b$

G $10a + 10b$

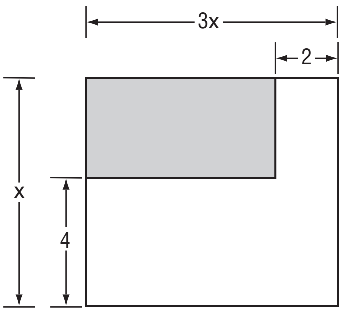
H $5a + b$

J $5a - 5b$

TEST-TAKING TIP

Question 5 If you simplify an expression and do not find your answer among the given answer choices, follow these steps. First, check your answer. Then, compare your answer with each of the given answer choices to determine whether it is equivalent to any of them.

6. What is the area of the shaded region of the rectangle expressed as a polynomial in simplest form?



A $3x^2 - 14x + 8$

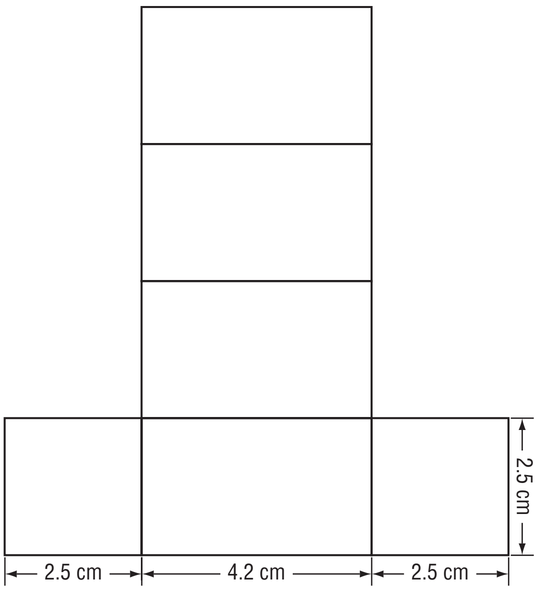
B $3x^2 + 14x + 8$

C $3x^2 - 8$

D $4x + 6$

Preparing for Standardized Tests
For test-taking strategies and more practice, see pages 941–956.

7. The figure below is the net of a rectangular prism. Use a ruler to measure the dimensions of the net to the nearest tenth of a centimeter.



Which measurement best approximates the volume of the rectangular prism represented by the net?

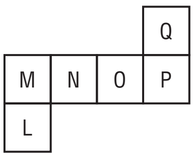
F 6.3 cm^3

G 10.5 cm^3

H 26.3 cm^3

J 44.1 cm^3

8. Which of the following is a true statement about the cube whose net is shown below?



A Faces L and M are parallel.

B Faces N and O are parallel.

C Faces M and P are perpendicular.

D Faces Q and L are perpendicular.

9. Kelly is designing a 12-inch by 12-inch scrapbook page. She cuts one picture that is 4 inches by 6 inches. She decides that she wants the next picture to be 75% as big as the first picture and the third picture to be 150% larger than the second picture. What are the approximate dimensions of the third picture?

F 0.45 in. by 0.68 in.

G 3.0 in. by 4.5 in.

H 4.5 in. by 6.75 in.

J 6.0 in. by 9.0 in.

10. **GRIDDABLE** Jalisa is a waitress. She recorded the following data about the amount that she made in tips for a certain number of hours.

Amount of Tips	Hours Worked
$12	1
$36	3
$60	5

If Jalisa continues to make the same amount of tips as shown in the table above, how much, in dollars, will she make in tips for working 9 hours?

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. Consider the polynomial function $f(x) = 3x^4 + 19x^3 + 7x^2 - 11x - 2$.

a. What is the degree of the function?

b. What is the leading coefficient of the function?

c. Evaluate $f(1)$, $f(-2)$, and $f(2a)$. Show your work.

NEED EXTRA HELP?											
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