Radical Equations and Inequalities

BIG Ideas

- Find the composition of functions.
- Determine the inverses of functions or relations.
- Graph and analyze square root functions and inequalities.
- Simplify and solve equations involving roots, radicals, and rational exponents.

Key Vocabulary

extraneous solution (p. 422)

inverse function (p. 392)

principal root (p. 402)

rationalizing the denominator (p. 409)

🌐 Real-World Link

Thrill Rides Many formulas involve square roots. For example, equations involving speeds of objects are often expressed with square roots. You can use such an equation to find the speed of a thrill ride such as the *Power Tower* free-fall ride at Cedar Point in Sandusky, Ohio.

FOLDABLES Study Organizer

Radical Equations and Inequalities Make this Foldable to help you organize your notes. Begin with four sheets of grid paper.

1. **Fold** in half along the width. On the first two sheets, cut 5 centimeters along the fold at the ends. On the second two sheets, cut in the center, stopping 5 centimeters from the ends.

First Sheets   Second Sheets
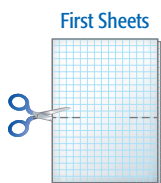
2. **Insert** the first sheets through the second sheets and align the folds. Label the pages with lesson numbers.

Joyrides
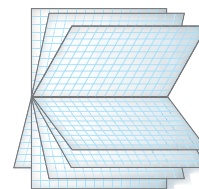
GET READY for Chapter 7

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

Option 2

Math Online Take the Online Readiness Quiz at **algebra2.com**.
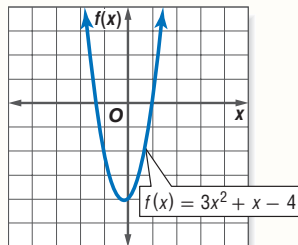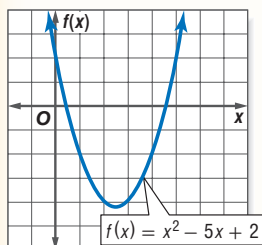
QUICK Check

Use the related graph of each equation to determine its roots. If exact roots cannot be found, state the consecutive integers between which the roots are located.
(Lesson 5-2)

1. $x^2 - 5x + 2 = 0$ **2.** $3x^2 + x - 4 = 0$

$f(x) = x^2 - 5x + 2$

$f(x) = 3x^2 + x - 4$

Simplify each expression using synthetic division. (Lesson 6-3)

3. $(3x^2 - 14x - 24) \div (x - 6)$

4. $(a^2 - 2a - 30) \div (a + 7)$

MEDICINE For Exercises 5 and 6, use the following information.

The number of students at a large high school who will catch the flu during an outbreak can be estimated by $n = \dfrac{170t^2}{t^2 + 1}$, where t is the number of weeks from the beginning of the epidemic and n is the number of ill people. (Lesson 6-3)
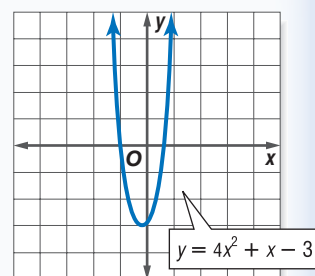
5. Perform the division indicated by $\dfrac{170t^2}{t^2 + 1}$.

6. Use the formula to estimate how many people will become ill during the first week.

QUICK Review

EXAMPLE 1

Use the related graph of $0 = 4x^2 + x - 3$ to determine its roots. If exact roots cannot be found, state the consecutive integers between which the roots are located.

$y = 4x^2 + x - 3$

The roots are the x-values where the graph crosses the x-axis.

The graph crosses the x-axis at -1 and between 0 and 1.

EXAMPLE 2

Simplify $(16x^4 - 8x^2 + 2x + 8) \div (2x + 2)$ using synthetic division.

Use division to rewrite the divisor so it has a first coefficient of 1.

$$\frac{16x^4 - 8x^2 + 2x + 8}{2x + 2} = \frac{(16x^4 - 8x^2 + 2x + 8) \div 2}{(2x + 2) \div 2}$$

Divide numerator and denominator by 2.

$$\frac{8x^4 - 4x^2 + x + 4}{x + 1} \quad \text{Simplify.}$$

Since the numerator does not have an x^3-term, use a coefficient of 0 for x^3.

$x - r = x + 1$, so $r = -1$.

$$\begin{array}{r|rrrrr} -1 & 8 & 0 & -4 & 1 & 4 \\ & \downarrow & -8 & 8 & -4 & 3 \\ \hline & 8 & -8 & 4 & -3 & \,|\,7 \end{array}$$

The result is $8x^3 - 8x^2 + 4x - 3 + \dfrac{7}{x + 1}$.

Operations on Functions

Main Ideas

- Find the sum, difference, product, and quotient of functions.
- Find the composition of functions.

New Vocabulary

composition of functions

GET READY for the Lesson

Carol Coffmon owns a store where she sells birdhouses. The revenue from birdhouse sales is given by $r(x) = 125x$. The cost of making the birdhouses is given by $c(x) = 65x + 5400$. Her profit p is the revenue minus the cost or $p = r - c$. So the profit function $p(x)$ can be defined as $p(x) = (r - c)(x)$.

Arithmetic Operations Let $f(x)$ and $g(x)$ be any two functions. You can add, subtract, multiply, and divide functions according to these rules.

KEY CONCEPT   *Operations with Functions*

Operation	Definition	Examples if $f(x) = x + 2$, $g(x) = 3x$
Sum	$(f + g)(x) = f(x) + g(x)$	$(x + 2) + 3x = 4x + 2$
Difference	$(f - g)(x) = f(x) - g(x)$	$(x + 2) - 3x = -2x + 2$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(x + 2)3x = 3x^2 + 6x$
Quotient	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, g(x) \neq 0$	$\dfrac{x + 2}{3x}, x \neq 0$

EXAMPLE   **Add and Subtract Functions**

1 Given $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$, find each function.

a. $(f + g)(x)$

$$(f + g)(x) = f(x) + g(x) \qquad \text{Addition of functions}$$
$$= (x^2 - 3x + 1) + (4x + 5) \qquad f(x) = x^2 - 3x + 1 \text{ and } g(x) = 4x + 5$$
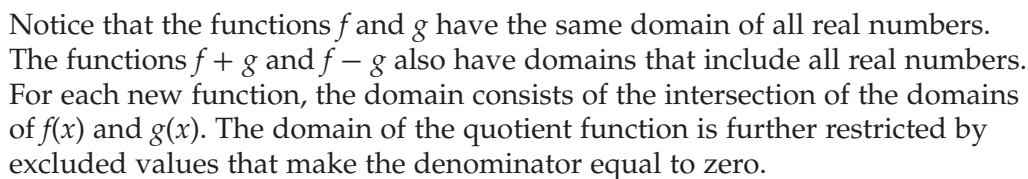$$= x^2 + x + 6 \qquad \text{Simplify.}$$

b. $(f - g)(x)$

$$(f - g)(x) = f(x) - g(x) \qquad \text{Subtraction of functions}$$
$$= (x^2 - 3x + 1) - (4x + 5) \qquad f(x) = x^2 - 3x + 1 \text{ and } g(x) = 4x + 5$$
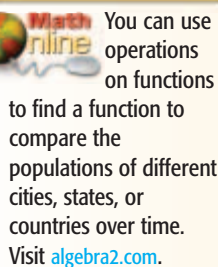$$= x^2 - 7x - 4 \qquad \text{Simplify.}$$

CHECK Your Progress

Given $f(x) = x^2 + 5x - 2$ and $g(x) = 3x - 2$, find each function.

1A. $(f + g)(x)$ **1B.** $(f - g)(x)$

Notice that the functions f and g have the same domain of all real numbers. The functions $f + g$ and $f - g$ also have domains that include all real numbers. For each new function, the domain consists of the intersection of the domains of $f(x)$ and $g(x)$. The domain of the quotient function is further restricted by excluded values that make the denominator equal to zero.

EXAMPLE **Multiply and Divide Functions**

2 **Given $f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$, find each function.**

a. $(f \cdot g)(x)$

$$
\begin{aligned}
(f \cdot g)(x) &= f(x) \cdot g(x) && \text{Product of functions} \\
&= (x^2 + 5x - 1)(3x - 2) && \text{Substitute.} \\
&= x^2(3x - 2) + 5x(3x - 2) - 1(3x - 2) && \text{Distributive Property} \\
&= 3x^3 - 2x^2 + 15x^2 - 10x - 3x + 2 && \text{Distributive Property} \\
&= 3x^3 + 13x^2 - 13x + 2 && \text{Simplify.}
\end{aligned}
$$

b. $\left(\dfrac{f}{g}\right)(x)$

$$
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Division of functions}
$$

$$
= \frac{x^2 + 5x - 1}{3x - 2}, x \neq \frac{2}{3} \quad f(x) = x^2 + 5x - 1 \text{ and } g(x) = 3x - 2
$$

Because $x = \dfrac{2}{3}$ makes $3x - 2 = 0$, $\dfrac{2}{3}$ is excluded from the domain of $\left(\dfrac{f}{g}\right)(x)$.

CHECK Your Progress

Given $f(x) = x^2 - 7x + 2$ and $g(x) = x + 4$, find each function.

2A. $(f \cdot g)(x)$ **2B.** $\left(\dfrac{f}{g}\right)(x)$

Composition of Functions Functions can also be combined using **composition of functions**. In a composition, a function is performed, and then a second function is performed on the result of the first function.
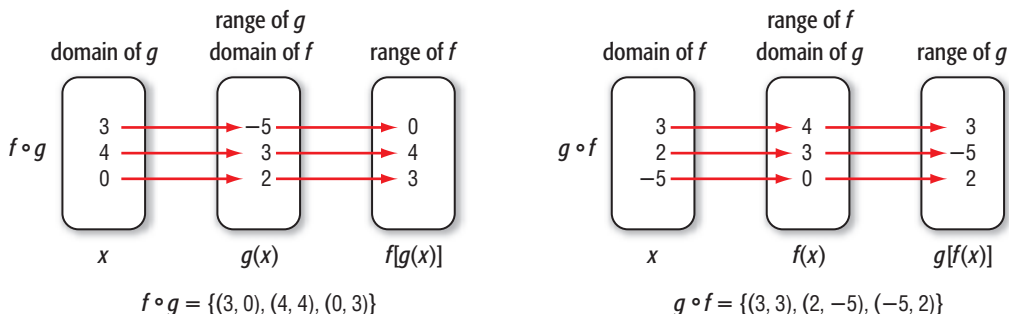
KEY CONCEPT *Composition of Functions*

Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composite function $f \circ g$ can be described by

$$
[f \circ g](x) = f[g(x)].
$$

Suppose $f = \{(3, 4), (2, 3), (-5, 0)\}$ and $g = \{(3, -5), (4, 3), (0, 2)\}$.

$f \circ g = \{(3, 0), (4, 4), (0, 3)\}$ $g \circ f = \{(3, 3), (2, -5), (-5, 2)\}$

The composition of two functions may not exist. Given two functions f and g, $[f \circ g](x)$ is defined only if the range of $g(x)$ is a subset of the domain of $f(x)$.

EXAMPLE   Evaluate Composition of Functions

③ If $f = \{(7, 8), (5, 3), (9, 8), (11, 4)\}$ and $g = \{(5, 7), (3, 5), (7, 9), (9, 11)\}$, find $f \circ g$ and $g \circ f$.

To find $f \circ g$, evaluate $g(x)$ first. Then use the range of g as the domain of f and evaluate $f(x)$.

$f[g(5)] = f(7)$ or 8 $g(5) = 7$ $f[g(7)] = f(9)$ or 8 $g(7) = 9$

$f[g(3)] = f(5)$ or 3 $g(3) = 5$ $f[g(9)] = f(11)$ or 4 $g(9) = 11$

$f \circ g = \{(5, 8), (3, 3), (7, 8), (9, 4)\}$

To find $g \circ f$, evaluate $f(x)$ first. Then use the range of f as the domain of g and evaluate $g(x)$.

$g[f(7)] = g(8)$ $g(8)$ is undefined. $g[f(9)] = g(8)$ $g(8)$ is undefined.

$g[f(5)] = g(3)$ or 5 $f(5) = 3$ $g[f(11)] = g(4)$ $g(4)$ is undefined.

Since 8 and 4 are not in the domain of g, $g \circ f$ is undefined for $x = 7$, $x = 9$, and $x = 11$. However, $g[f(5)] = 5$ so $g \circ f = \{(5, 5)\}$.

CHECK Your Progress

3. If $f = \{(3, -2), (-1, -5), (4, 7), (10, 8)\}$ and $g = \{(4, 3), (2, -1),$
   $(9, 4), (3, 10)\}$, find $f \circ g$ and $g \circ f$.

Notice that in most instances $f \circ g \neq g \circ f$. Therefore, the order in which you compose two functions is very important.

EXAMPLE   Simplify Composition of Functions

④ a. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x + 3$ and $g(x) = x^2 + x - 1$.

$[f \circ g](x) = f[g(x)]$ Composition of functions

$\qquad = f(x^2 + x - 1)$ Replace $g(x)$ with $x^2 + x - 1$.

$\qquad = (x^2 + x - 1) + 3$ Substitute $x^2 + x - 1$ for x in $f(x)$.

$\qquad = x^2 + x + 2$ Simplify.

$[g \circ f](x) = g[f(x)]$ Composition of functions

$\qquad = g(x + 3)$ Replace $f(x)$ with $x + 3$.

$\qquad = (x + 3)^2 + (x + 3) - 1$ Substitute $x + 3$ for x in $g(x)$.

$\qquad = x^2 + 6x + 9 + x + 3 - 1$ Evaluate $(x + 3)^2$.

$\qquad = x^2 + 7x + 11$ Simplify.

So, $[f \circ g](x) = x^2 + x + 2$ and $[g \circ f](x) = x^2 + 7x + 11$.

b. Evaluate $[f \circ g](x)$ **and** $[g \circ f](x)$ **for** $x = 2$.

$$[f \circ g](x) = x^2 + x + 2 \qquad \text{Function from part a}$$
$$[f \circ g](2) = (2)^2 + 2 + 2 \text{ or } 8 \qquad \text{Replace } x \text{ with 2 and simplify.}$$

$$[g \circ f](x) = x^2 + 7x + 11 \qquad \text{Function from part a}$$
$$[g \circ f](2) = (2)^2 + 7(2) + 11 \text{ or } 29 \qquad \text{Replace } x \text{ with 2 and simplify.}$$

So, $[f \circ g](2) = 8$ and $[g \circ f](2) = 29$.

CHECK Your Progress

4A. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x - 5$ and $g(x) = x^2 + 2x + 3$.

4B. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = -3$.

Real-World EXAMPLE   Use Composition of Functions

⑤ TAXES **Tyrone Davis has $180 deducted from every paycheck for retirement. He can have these deductions taken before taxes are applied, which reduces his taxable income. His federal income tax rate is 18%. If Tyrone earns $2200 every pay period, find the difference in his net income if he has the retirement deduction taken before taxes or after taxes.**

Explore   Let $x =$ Tyrone's income per paycheck, $r(x) =$ his income after the deduction for retirement, and $t(x) =$ his income after the deduction for federal income tax.

Plan   Write equations for $r(x)$ and $t(x)$.
$180 is deducted from every paycheck for retirement:
$r(x) = x - 180$.

Tyrone's tax rate is 18%: $t(x) = x - 0.18x$.

Solve   If Tyrone has his retirement deducted *before* taxes, then his net income is represented by $[t \circ r](2200)$.

$$[t \circ r](2200) = t(2200 - 180) \qquad \text{Replace } x \text{ with 2200 in } r(x) = x - 180.$$
$$= t(2020)$$
$$= 2020 - 0.18(2020) \qquad \text{Replace } x \text{ with 2020 in } t(x) = x - 0.18x.$$
$$= 1656.40$$

If Tyrone has his retirement deducted *after* taxes, then his net income is represented by $[r \circ t](2200)$.

$$[r \circ t](2200) = r[2200 - 0.18(2200)] \qquad \text{Replace } x \text{ with 2200 in } t(x) \circ x - 0.18x.$$
$$= r(1804)$$
$$= 1804 - 180 \qquad \text{Replace } x \text{ with 1804 in } r(x) = x - 180.$$
$$= 1624$$

$[t \circ r](2200) = 1656.40$ and $[r \circ t](2200) = 1624$. The difference is $1656.40 - 1624, or $32.40. So, his net pay is $32.40 more by having his retirement deducted before taxes.

Check   The answer makes sense. Since the taxes are being applied to a smaller amount, less tax will be deducted from his paycheck.

Study Tip

Combining Functions

By combining functions, you can make the evaluation of the functions more efficient.

5. All-Mart is offering both an in-store $35 rebate and a 15% discount on an MP3 player that normally costs $300. Which provides the better price: taking the discount before the rebate, or taking the rebate before the discount?

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CHECK Your Understanding

Examples 1, 2
(pp. 384–385)

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\dfrac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

1. $f(x) = 3x + 4$
$g(x) = 5 + x$

2. $f(x) = x^2 + 3$
$g(x) = x - 4$

Example 3
(p. 386)

For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist.

3. $f = \{(-1, 9), (4, 7)\}$
$g = \{(-5, 4), (7, 12), (4, -1)\}$

4. $f = \{(0, -7), (1, 2), (2, -1)\}$
$g = \{(-1, 10), (2, 0)\}$

Example 4
(pp. 386–387)

Find $[g \circ h](x)$ and $[h \circ g](x)$.

5. $g(x) = 2x$
$h(x) = 3x - 4$

6. $g(x) = x + 5$
$h(x) = x^2 + 6$

If $f(x) = 3x$, $g(x) = x + 7$, and $h(x) = x^2$, find each value.

7. $f[g(3)]$ **8.** $g[h(-2)]$ **9.** $h[h(1)]$

Example 5
(p. 387)

SHOPPING For Exercises 10–13, use the following information.
Mai-Lin is shopping for computer software. She finds a CD-ROM that costs $49.99, but is on sale at a 25% discount. She also has a $5 coupon she can use.

10. Express the price of the CD after the discount and the price of the CD after the coupon. Let x represent the price of the CD, $p(x)$ represent the price after the 25% discount, and $c(x)$ represent the price after the coupon.

11. Find $c[p(x)]$ and explain what this value represents.

12. Find $p[c(x)]$ and explain what this value represents.

13. Which method results in the lower sale price? Explain your reasoning.

Exercises

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\dfrac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

HOMEWORK HELP	
For Exercises	**See Examples**
14–21	1, 2
22–27	3
28–45	4
46, 47	5

14. $f(x) = x + 9$
$g(x) = x - 9$

15. $f(x) = 2x - 3$
$g(x) = 4x + 9$

16. $f(x) = 2x^2$
$g(x) = 8 - x$

17. $f(x) = x^2 + 6x + 9$
$g(x) = 2x + 6$

18. $f(x) = x^2 - 1$
$g(x) = \dfrac{x}{x + 1}$

19. $f(x) = x^2 - x - 6$
$g(x) = \dfrac{x - 3}{x + 2}$

WALKING For Exercises 20 and 21, use the following information.
Carlos is walking on a moving walkway. His speed is given by the function $C(x) = 3x^2 + 3x - 4$, and the speed of the walkway is $W(x) = x^2 - 4x + 7$.

20. What is his total speed as he walks along the moving walkway?

21. Carlos turned around because he left his cell phone at a restaurant. What was his speed as he walked against the moving walkway?

For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist.

22. $f = \{(1, 1), (0, -3)\}$
$g = \{(1, 0), (-3, 1), (2, 1)\}$

23. $f = \{(1, 2), (3, 4), (5, 4)\}$
$g = \{(2, 5), (4, 3)\}$

24. $f = \{(3, 8), (4, 0), (6, 3), (7, -1)\}$
$g = \{(0, 4), (8, 6), (3, 6), (-1, 8)\}$

25. $f = \{(4, 5), (6, 5), (8, 12), (10, 12)\}$
$g = \{4, 6), (2, 4), (6, 8), (8, 10)\}$

26. $f = \{(2, 5), (3, 9), (-4, 1)\}$
$g = \{(5, -4), (8, 3), (2, -2)\}$

27. $f = \{(7, 0), (-5, 3), (8, 3), (-9, 2)\}$
$g = \{(2, -5), (1, 0), (2, -9), (3, 6)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

28. $g(x) = 4x$
$h(x) = 2x - 1$

29. $g(x) = -5x$
$h(x) = -3x + 1$

30. $g(x) = x + 2$
$h(x) = x^2$

31. $g(x) = x - 4$
$h(x) = 3x^2$

32. $g(x) = 2x$
$h(x) = x^3 + x^2 + x + 1$

33. $g(x) = x + 1$
$h(x) = 2x^2 - 5x + 8$

If $f(x) = 4x$, $g(x) = 2x - 1$, and $h(x) = x^2 + 1$, find each value.

34. $f[g(-1)]$

35. $h[g(4)]$

36. $g[f(5)]$

37. $f[h(-4)]$

38. $g[g(7)]$

39. $f[f(-3)]$

40. $h\left[f\left(\frac{1}{4}\right)\right]$

41. $g\left[h\left(-\frac{1}{2}\right)\right]$

42. $[g \circ (f \circ h)](3)$

43. $[f \circ (h \circ g)](3)$

44. $[h \circ (g \circ f)](2)$

45. $[f \circ (g \circ h)](2)$

POPULATION GROWTH For Exercises 46 and 47, use the following information.
From 1990 to 2002, the number of births $b(x)$ in the United States can be modeled by the function $b(x) = -8x + 4045$, and the number of deaths $d(x)$ can be modeled by the function $d(x) = 24x + 2160$, where x is the number of years since 1990 and $b(x)$ and $d(x)$ are in thousands.

46. The net increase in population P is the number of births per year minus the number of deaths per year, or $P = b - d$. Write an expression that can be used to model the population increase in the U.S. from 1990 to 2002 in function notation.

47. Assume that births and deaths continue at the same rates. Estimate the net increase in population in 2015.

SHOPPING For Exercises 48–50, use the following information.
Liluye wants to buy a pair of inline skates that are on sale for 30% off the original price of $149. The sales tax is 5.75%.

48. Express the price of the inline skates after the discount and the price of the inline skates after the sales tax using function notation. Let x represent the price of the inline skates, $p(x)$ represent the price after the 30% discount, and $s(x)$ represent the price after the sales tax.

49. Which composition of functions represents the price of the inline skates, $p[s(x)]$ or $s[p(x)]$? Explain your reasoning.

50. How much will Liluye pay for the inline skates?

51. FINANCE Regina pays $50 each month on a credit card that charges 1.6% interest monthly. She has a balance of $700. The balance at the beginning of the nth month is given by $f(n) = f(n - 1) + 0.016\,f(n - 1) - 50$. Find the balance at the beginning of the first five months. No additional charges are made on the card. (*Hint:* $f(1) = 700$)

52. OPEN ENDED Write a set of ordered pairs for functions f and g, given that $f \circ g = \{(4, 3), (-1, 9), (-2, 7)\}$.

53. FIND THE ERROR Danette and Marquan are trying to find $[g \circ f](3)$ for $f(x) = x^2 + 4x + 5$ and $g(x) = x - 7$. Who is correct? Explain your reasoning.

Danette
$[g \cdot f](3) = g((3)2 + 4(3) + 5)$
$= g(26)$
$= 26 - 7$
$= 19$

Marquan
$[g \cdot f](3) = f(3 - 7)$
$= f(-4)$
$= (-4)^2 + 4(-4) + 5$
$= 5$

54. CHALLENGE If $f(0) = 4$ and $f(x + 1) = 3f(x) - 2$, find $f(4)$.

55. *Writing in Math* Refer to the information on page 384 to explain how combining functions can be important to business. Describe how to write a new function that represents the profit, using the revenue and cost functions. What are the benefits of combining two functions into one function?

STANDARDIZED TEST PRACTICE

56. ACT/SAT What is the value of $f(g(6))$ if $f(x) = 2x + 4$ and $g(x) = x^2 + 5$?

A 38

B 43

C 86

D 261

57. REVIEW If $g(x) = x^2 + 9x + 21$ and $h(x) = 2(x + 5)^2$, which is an equivalent form of $h(x) - g(x)$?

F $-x^2 - 11x - 29$

G $x^2 + 11x + 29$

H $x + 4$

J $x^2 + 7x + 11$

Spiral Review

List all of the possible rational zeros of each function. (Lesson 6-9)

58. $r(x) = x^2 - 6x + 8$

59. $f(x) = 4x^3 - 2x^2 + 6$

60. $g(x) = 9x^2 - 1$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function. (Lesson 6-8)

61. $f(x) = 7x^4 + 3x^3 - 2x^2 - x + 1$

62. $g(x) = 2x^4 - x^3 - 3x + 7$

63. CHEMISTRY The mass of a proton is about 1.67×10^{-27} kilogram. The mass of an electron is about 9.11×10^{-31} kilogram. About how many times as massive as an electron is a proton? (Lesson 6-1)

GET READY for the Next Lesson

PREREQUISITE SKILL Solve each equation or formula for the specified variable. (Lesson 1-3)

64. $2x - 3y = 6$, for x

65. $4x^2 - 5xy + 2 = 3$, for y

66. $3x + 7xy = -2$, for x

67. $I = prt$, for t

68. $C = \frac{5}{9}(F - 32)$, for F

69. $F = G\frac{Mm}{r^2}$, for m

Inverse Functions and Relations

Main Ideas

- Find the inverse of a function or relation.
- Determine whether two functions or relations are inverses.

New Vocabulary

inverse relation

inverse function

identity function

one-to-one

GET READY for the Lesson

Most scientific formulas involve measurements given in SI (International System) units. The SI units for speed are meters per second. However, the United States uses customary measurements such as miles per hour.

To convert x miles per hour to an approximate equivalent in meters per second, you can evaluate the following.

$$f(x) = \frac{x \text{ miles}}{1 \text{ hour}} \cdot \frac{1600 \text{ meters}}{1 \text{ mile}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}} \text{ or } f(x) = \frac{4}{9}x$$

To convert x meters per second to an approximate equivalent in miles per hour, you can evaluate the following.

$$g(x) = \frac{x \text{ meters}}{1 \text{ second}} \cdot \frac{3600 \text{ seconds}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{1600 \text{ meters}} \text{ or } g(x) = \frac{9}{4}x$$

Notice that $f(x)$ multiplies a number by 4 and divides it by 9. The function $g(x)$ does the inverse operation of $f(x)$. It divides a number by 4 and multiplies it by 9. These functions are inverses.

Find Inverses Recall that a relation is a set of ordered pairs. The **inverse relation** is the set of ordered pairs obtained by reversing the coordinates of each ordered pair. The domain of a relation becomes the range of the inverse, and the range of a relation becomes the domain of the inverse.

KEY CONCEPT   Inverse Relations

Words   Two relations are inverse relations if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).

Examples $Q = \{(1, 2), (3, 4), (5, 6)\}$ $S = \{(2, 1), (4, 3), (6, 5)\}$
         Q and S are inverse relations.

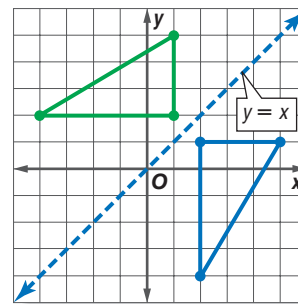### EXAMPLE   Find an Inverse Relation

1 **GEOMETRY** The ordered pairs of the relation $\{(2, 1), (5, 1), (2, -4)\}$ are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.

To find the inverse of this relation, reverse the coordinates of the ordered pairs.

The inverse of the relation is $\{(1, 2), (1, 5), (-4, 2)\}$.

Plotting the points shows that the ordered pairs also describe the vertices of a right triangle. Notice that the graphs of the relation and the inverse relation are reflections over the graph of $y = x$.

CHECK Your Progress

1. The ordered pairs of the relation $\{(-8, -3), (-8, -6), (-3, -6)\}$ are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.

The ordered pairs of **inverse functions** are also related. We can write the inverse of function $f(x)$ as $f^{-1}(x)$.

KEY CONCEPT
Property of Inverse Functions

Suppose f and f^{-1} are inverse functions. Then, $f(a) = b$ if and only if $f^{-1}(b) = a$.

Let's look at the inverse functions $f(x) = x + 2$ and $f^{-1}(x) = x - 2$.

Evaluate $f(5)$.
$f(x) = x + 2$
$f(5) = 5 + 2$ or 7

Now, evaluate $f^{-1}(7)$.
$f^{-1}(x) = x - 2$
$f^{-1}(7) = 7 - 2$ or 5

Since $f(x)$ and $f^{-1}(x)$ are inverses, $f(5) = 7$ and $f^{-1}(7) = 5$. The inverse function can be found by exchanging the domain and range of the function.

EXAMPLE   Find and Graph an Inverse Function

2 a. **Find the inverse of $f(x) = \dfrac{x + 6}{2}$.**

Step 1 Replace $f(x)$ with y in the original equation.

$$f(x) = \frac{x + 6}{2} \qquad y = \frac{x + 6}{2}$$

Step 2 Interchange x and y.

$$x = \frac{y + 6}{2}$$

Step 3 Solve for y.

$$x = \frac{y + 6}{2} \qquad \text{Inverse}$$

$$2x = y + 6 \qquad \text{Multiply each side by 2.}$$

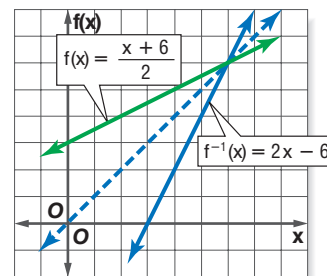$$2x - 6 = y \qquad \text{Subtract 6 from each side.}$$

Step 4 Replace y with $f^{-1}(x)$.

$$y = 2x - 6 \qquad f^{-1}(x) = 2x - 6$$

The inverse of $f(x) = \dfrac{x + 6}{2}$ is $f^{-1}(x) = 2x - 6$.

b. Graph the function and its inverse.

Graph both functions on the coordinate plane. The graph of $f^{-1}(x) = 2x - 6$ is the reflection of the graph of $f(x) = \dfrac{x + 6}{2}$ over the line $y = x$.

CHECK Your Progress

2A. Find the inverse of $f(x) = \dfrac{x - 3}{5}$.

2B. Graph the function and its inverse.

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Inverses of Relations and Functions You can determine whether two functions are inverses by finding both of their compositions. If both equal the **identity function** $I(x) = x$, then the functions are inverse functions.

KEY CONCEPT   *Inverse Functions*

Words   Two functions f and g are inverse functions if and only if both of their compositions are the identity function.

Symbols $[f \circ g](x) = x$ and $[g \circ f](x) = x$

EXAMPLE   **Verify that Two Functions are Inverses**

3 Determine whether $f(x) = 5x + 10$ and $g(x) = \frac{1}{5}x - 2$ are inverse functions.

Check to see if the compositions of $f(x)$ and $g(x)$ are identity functions.

$$[f \circ g](x) = f[g(x)] \qquad\qquad [g \circ f](x) = g[f(x)]$$
$$= f\left(\frac{1}{5}x - 2\right) \qquad\qquad\quad = g(5x + 10)$$
$$= 5\left(\frac{1}{5}x - 2\right) + 10 \qquad\quad = \frac{1}{5}(5x + 10) - 2$$
$$= x - 10 + 10 \qquad\qquad\quad = x + 2 - 2$$
$$= x \qquad\qquad\qquad\qquad = x$$

The functions are inverses since both $[f \circ g](x)$ and $[g \circ f](x)$ equal x.

CHECK Your Progress

3. Determine whether $f(x) = 3x - 3$ and $g(x) = \frac{1}{3}x + 4$ are inverse functions.

You can also determine whether two functions are inverse functions by graphing. The graphs of a function and its inverse are mirror images with respect to the graph of the identity function $I(x) = x$.

Inverses of Functions

- Use a full sheet of grid paper. Draw and label the x- and y-axes.
- Graph $y = 2x - 3$. Then graph $y = x$ as a dashed line.
- Place a geomirror so that the drawing edge is on the line $y = x$. Carefully plot the points that are part of the reflection of the original line. Draw a line through the points.

ANALYZE

1. What is the equation of the drawn line?

2. What is the relationship between the line $y = 2x - 3$ and the line that you drew? Justify your answer.

3. Try this activity with the function $y = |x|$. Is the inverse also a function? Explain.

When the inverse of a function is a function, then the original function is said to be **one-to-one**. Recall that the *vertical line test* can be used to determine if a graph represents a function. Similarly, the *horizontal line test* can be used to determine if the inverse of a function is a function.
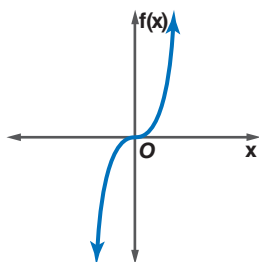
No horizontal line can be drawn so that it passes through more than point. The inverse of this function is a function.
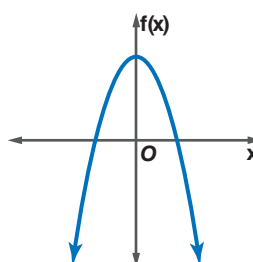
A horizontal line can be drawn so that it passes through more than one point. The inverse of this function is not a function.

CHECK Your Understanding

Example 1
(pp. 391–392)

Find the inverse of each relation.

1. {(2, 4), (−3, 1), (2, 8)}

2. {(1, 3), (1, −1), (1, −3), (1, 1)}

Example 2
(pp. 392–393)

Find the inverse of each function. Then graph the function and its inverse.

3. $f(x) = -x$

4. $g(x) = 3x + 1$

5. $y = \frac{1}{2}x + 5$

PHYSICS For Exercises 6 and 7, use the following information.

The acceleration due to gravity is 9.8 meters per second squared (m/s^2). To convert to feet per second squared, you can use the following operations.

$$\frac{9.8 \ m}{s^2} \times \frac{100 \ cm}{1 \ m} \times \frac{1 \ in.}{2.54 \ cm} \times \frac{1 \ ft}{12 \ in.}$$

6. Find the value of the acceleration due to gravity in feet per second squared.

7. An object is accelerating at 50 feet per second squared. How fast is it accelerating in meters per second squared?

Determine whether each pair of functions are inverse functions.

8. $f(x) = x + 7$
$\quad g(x) = x - 7$

9. $g(x) = 3x - 2$
$\quad f(x) = \dfrac{x - 2}{3}$

Exercises

Find the inverse of each relation.

10. $\{(2, 6), (4, 5), (-3, -1)\}$

11. $\{(3, 8), (4, -2), (5, -3)\}$

12. $\{(7, -4), (3, 5), (-1, 4), (7, 5)\}$

13. $\{(-1, -2), (-3, -2), (-1, -4), (0, 6)\}$

14. $\{(6, 11), (-2, 7), (0, 3), (-5, 3)\}$

15. $\{(2, 8), (-6, 5), (8, 2), (5, -6)\}$

Find the inverse of each function. Then graph the function and its inverse.

16. $y = -3$

17. $g(x) = -2x$

18. $f(x) = x - 5$

19. $g(x) = x + 4$

20. $f(x) = 3x + 3$

21. $y = -2x - 1$

22. $y = \dfrac{1}{3}x$

23. $f(x) = \dfrac{5}{8}x$

24. $f(x) = \dfrac{1}{3}x + 4$

25. $f(x) = \dfrac{4}{5}x - 7$

26. $g(x) = \dfrac{2x + 3}{6}$

27. $f(x) = \dfrac{7x - 4}{8}$

GEOMETRY The formula for the area of a circle is $A = \pi r^2$.

28. Find the inverse of the function.

29. Use the inverse to find the radius of the circle whose area is 36 square centimeters.

Determine whether each pair of functions are inverse functions.

30. $f(x) = x - 5$
$\quad g(x) = x + 5$

31. $f(x) = 3x + 4$
$\quad g(x) = 3x - 4$

32. $f(x) = 6x + 2$
$\quad g(x) = x - \dfrac{1}{3}$

33. $g(x) = 2x + 8$
$\quad f(x) = \dfrac{1}{2}x - 4$

34. $h(x) = 5x - 7$
$\quad g(x) = \dfrac{1}{5}(x + 7)$

35. $g(x) = 2x + 1$
$\quad f(x) = \dfrac{x - 1}{2}$

NUMBER GAMES For Exercises 36–38, use the following information.
Damaso asked Emilia to choose a number between 1 and 35. He told her to subtract 12 from that number, multiply by 2, add 10, and divide by 4.

36. Write an equation that models this problem.

37. Find the inverse.

38. Emilia's final number was 9. What was her original number?

TEMPERATURE For Exercises 39 and 40, use the following information.
A formula for converting degrees Celsius to Fahrenheit is $F(x) = \dfrac{9}{5}x + 32$.

39. Find the inverse $F^{-1}(x)$. Show that $F(x)$ and $F^{-1}(x)$ are inverses.

40. Explain what purpose $F^{-1}(x)$ serves.

41. REASONING Determine the values of n for which $f(x) = x^n$ has an inverse that is a function. Assume that n is a whole number.

42. OPEN ENDED Sketch a graph of a function f that satisfies the following conditions: f does not have an inverse function, $f(x) > x$ for all x, and $f(1) > 0$.

43. CHALLENGE Give an example of a function that is its own inverse.

44. *Writing in Math* Refer to the information on page 391 to explain how inverse functions can be used in measurement conversions. Point out why it might be helpful to know the customary units if you are given metric units. Demonstrate how to convert the speed of light $c = 3.0 \times 10^8$ meters per second to miles per hour.

45. ACT/SAT Which of the following is the inverse of the function $f(x) = \dfrac{3x - 5}{2}$?

A $g(x) = \dfrac{2x + 5}{3}$ **C** $g(x) = 2x + 5$

B $g(x) = \dfrac{3x + 5}{2}$ **D** $g(x) = \dfrac{2x - 5}{3}$

46. REVIEW Which expression represents $f(g(x))$ if $f(x) = x^2 + 3$ and $g(x) = -x + 1$?

F $x^2 - x + 2$ **H** $-x^3 + x^2 - 3x + 3$

G $-x^2 - 2$ **J** $x^2 - 2x + 4$

Spiral Review

If $f(x) = 2x + 4$, $g(x) = x - 1$, and $h(x) = x^2$, find each value. (Lesson 7-1)

47. $f[g(2)]$ **48.** $g[h(-1)]$ **49.** $h[f(-3)]$

List all of the possible rational zeros of each function. (Lesson 6-9)

50. $f(x) = x^3 + 6x^2 - 13x - 42$ **51.** $h(x) = -4x^3 - 86x^2 + 57x + 20$

Perform the indicated operations. (Lesson 4-2)

52. $\begin{bmatrix} 3 & -4 \\ 2 & 8 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 7 & 7 \\ 3 & -6 \end{bmatrix}$

53. $\begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}$

54. Find the maximum and minimum values of the function $f(x, y) = 2x + 3y$ for the polygonal region with vertices at $(2, 4)$, $(-1, 3)$, $(-3, -3)$, and $(2, -5)$. (Lesson 3-4)

55. State whether the system of equations shown at the right is *consistent and independent*, *consistent and dependent*, or *inconsistent*. (Lesson 3-1)

56. BUSINESS The amount that a mail-order company charges for shipping and handling is given by the function $c(x) = 3 + 0.15x$, where x is the weight in pounds. Find the charge for an 8-pound order. (Lesson 2-2)
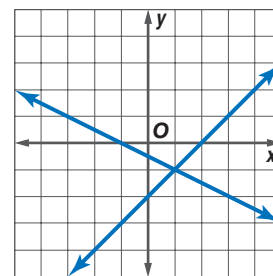
Solve each equation or inequality. Check your solutions. (Lessons 1-3, 1-4, and 1-5)

57. $2x + 7 = -3$ **58.** $-5x + 6 = -4$ **59.** $|x - 1| = 3$

60. $|3x + 2| = 5$ **61.** $2x - 4 > 8$ **62.** $-x - 3 \le 4$

GET READY for the Next Lesson

PREREQUISITE SKILL Graph each inequality. (Lesson 2-7)

63. $y > \dfrac{2}{3}x - 3$ **64.** $y \le -4x + 5$ **65.** $y < -x - 1$

Square Root Functions and Inequalities

Main Ideas

- Graph and analyze square root functions.
- Graph square root inequalities.

New Vocabulary

square root function

square root inequality

GET READY for the Lesson

The Sunshine Skyway Bridge across Tampa Bay, Florida, is supported by 21 steel cables, each 9 inches in diameter. The amount of weight that a steel cable can support is given by $w = 8d^2$, where d is the diameter of the cable in inches and w is the weight in tons.

If you need to know what diameter a steel cable should have to support a given weight, you can use the equation
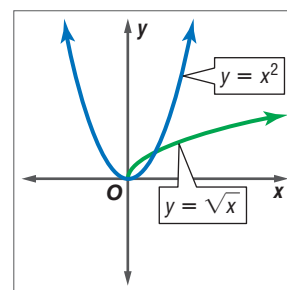
$$d = \sqrt{\frac{w}{8}}.$$

Square Root Functions If a function contains a square root of a variable, it is called a **square root function**. The parent function of the family of square root functions is $y = \sqrt{x}$. The inverse of a quadratic function is a square root function only if the range is restricted to nonnegative numbers.
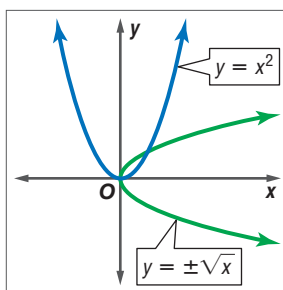
$y = \pm\sqrt{x}$ is not a function. $y = \sqrt{x}$ is a function.

In order for a square root to be a real number, the radicand cannot be negative. When graphing a square root function, determine when the radicand would be negative and exclude those values from the domain.

Graph a Square Root Function

1 **Graph $y = \sqrt{3x + 4}$. State the domain, range, and x- and y-intercepts.**

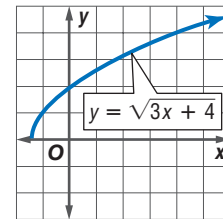Since the radicand cannot be negative, identify the domain.

$3x + 4 \geq 0$ Write the expression inside the radicand as ≥ 0.

$x \geq -\dfrac{4}{3}$ Solve for x.

The x-intercept is $-\dfrac{4}{3}$.

x	y
$-\dfrac{4}{3}$	0
-1	1
0	2
2	3.2
4	4

Make a table of values and graph the function. From the graph, you can see that the domain is $x \geq -\dfrac{4}{3}$, and the range is $y \geq 0$. The y-intercept is 2.

CHECK Your Progress

1. Graph $y = \sqrt{-2x + 3}$. State the domain, range, and x- and y-intercepts.


Real-World EXAMPLE

2 **SUBMARINES** A lookout on a submarine is h feet above the surface of the water. The greatest distance d in miles that the lookout can see on a clear day is given by the square root of the quantity h multiplied by $\dfrac{3}{2}$.

a. Graph the function. State the domain and range.

The function is $d = \sqrt{\dfrac{3h}{2}}$. Make a table of values and graph the function.

The domain is $h \geq 0$, and the range is $d \geq 0$.

h	d
0	0
2	$\sqrt{3}$ or 1.73
4	$\sqrt{6}$ or 2.45
6	$\sqrt{9}$ or 3.00
8	$\sqrt{12}$ or 3.46
10	$\sqrt{15}$ or 3.87
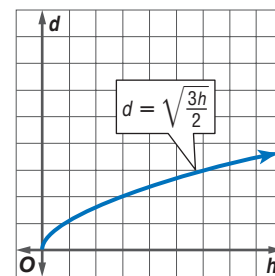
b. A ship is 3 miles from a submarine. How high would the submarine have to raise its periscope in order to see the ship?

$d = \sqrt{\dfrac{3h}{2}}$ Original equation

$3 = \sqrt{\dfrac{3h}{2}}$ Replace d with 3.

$9 = \dfrac{3h}{2}$ Square each side.

$18 = 3h$ Multiply each side by 2.

$6 = h$ Divide each side by 3.

The periscope would have to be 6 feet above the water. Check the reasonableness of this result by comparing it to the graph.

The speed v of a ball can be determined by the equation $v = \sqrt{\dfrac{2k}{m}}$, where k is the kinetic energy and m is the mass of the ball. Assume that the mass of the ball is 5 kg.

2A. Graph the function. State the domain and range.

2B. The ball is traveling 6 meters per second. What is the ball's kinetic energy in Joules?

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Like quadratic functions, graphs of square root functions can be transformed.

GRAPHING CALCULATOR LAB

Square Root Functions

You can use a TI-83/84 Plus graphing calculator to graph square root functions. Use ⎡2nd⎤ [√] to enter the functions in the Y=list.

THINK AND DISCUSS

1. Graph $y = \sqrt{x}$, $y = \sqrt{x} + 1$, and $y = \sqrt{x} - 2$ in the viewing window $[-2, 8]$ by $[-4, 6]$. State the domain and range of each function and describe the similarities and differences among the graphs.

2. Graph $y = \sqrt{x}$, $y = \sqrt{2x}$, and $y = \sqrt{8x}$ in the viewing window $[0, 10]$ by $[0, 10]$. State the domain and range of each function and describe the similarities and differences among the graphs.

3. Make a conjecture about an equation that translates the graph of $y = \sqrt{x}$ to the left three units. Test your conjecture with the graphing calculator.

Square Root Inequalities A **square root inequality** is an inequality involving square roots.

EXAMPLE Graph a Square Root Inequality

3 Graph $y < \sqrt{2x - 6}$.

Graph $y = \sqrt{2x - 6}$. Since the boundary should not be included, the graph should be dashed.

The domain includes values for $x \geq 3$, so the graph includes $x = 3$ and values for which $x > 3$. Select a point to see if it is in the shaded region.

Test (4, 1): $\quad 1 \overset{?}{<} \sqrt{2(4) - 6}$

$\qquad\qquad\quad 1 < \sqrt{2}$ true

Shade the region that includes the point (4, 1).

$y = \sqrt{2x - 6}$
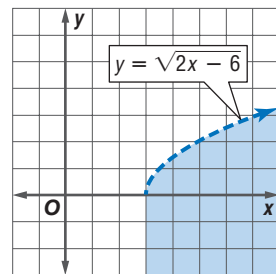
✓ CHECK Your Progress

3. Graph $y \geq \sqrt{x + 1}$.

Example 1
(p. 398)

Graph each function. State the domain and range of the function.

1. $y = \sqrt{x} + 2$ **2.** $y = \sqrt{4x}$ **3.** $y = \sqrt{x - 1} + 3$

Example 2
(p. 398)

FIREFIGHTING For Exercises 4 and 5, use the following information.
When fighting a fire, the velocity v of water being pumped into the air is the square root of twice the product of the maximum height h and g, the acceleration due to gravity (32 ft/s^2).

4. Determine an equation that will give the maximum height of the water as a function of its velocity.

5. The Coolville Fire Department must purchase a pump that will propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 ft/s meet the fire department's need? Explain.

Example 3
(p. 399)

Graph each inequality.

6. $y \leq \sqrt{x - 4} + 1$ **7.** $y > \sqrt{2x + 4}$ **8.** $y \geq \sqrt{x + 2} - 1$

Exercises

Graph each function. State the domain and range of each function.

9. $y = \sqrt{3x}$ **10.** $y = -\sqrt{5x}$ **11.** $y = -4\sqrt{x}$

12. $y = \frac{1}{2}\sqrt{x}$ **13.** $y = \sqrt{x + 2}$ **14.** $y = \sqrt{x - 7}$

15. $y = -\sqrt{2x + 1}$ **16.** $y = \sqrt{5x - 3}$ **17.** $y = \sqrt{x + 6} - 3$

18. $y = 5 - \sqrt{x + 4}$ **19.** $y = \sqrt{3x - 6} + 4$ **20.** $y = 2\sqrt{3 - 4x} + 3$

21. ROLLER COASTERS The velocity of a roller coaster as it moves down a hill is $v = \sqrt{v_0^2 + 64h}$, where v_0 is the initial velocity and h is the vertical drop in feet. An engineer wants a new coaster to have a velocity greater than 90 feet per second when it reaches the bottom of the hill. If the initial velocity of the coaster at the top of the hill is 10 feet per second, how high should the engineer make the hill? Is your answer reasonable?

AEROSPACE For Exercises 22 and 23, use the following information.
The force due to gravity decreases with the square of the distance from the center of Earth. As an object moves farther from Earth, its weight decreases. The radius of Earth is approximately 3960 miles. The formula relating weight and distance is $r = \sqrt{\dfrac{3960^2 W_E}{W_S}} - 3960$, where W_E represents the weight of a body on Earth, W_S represents its weight a certain distance from the center of Earth, and r represents the distance above Earth's surface.

22. An astronaut weighs 140 pounds on Earth and 120 pounds in space. How far is he above Earth's surface?

23. An astronaut weighs 125 pounds on Earth. What is her weight in space if she is 99 miles above the surface of Earth?

Graph each inequality.

24. $y \leq -6\sqrt{x}$ **25.** $y < \sqrt{x + 5}$ **26.** $y > \sqrt{2x + 8}$

27. $y \geq \sqrt{5x - 8}$ **28.** $y \geq \sqrt{x - 3} + 4$ **29.** $y < \sqrt{6x - 2} + 1$

30. OPEN ENDED Write a square root function with a domain of $\{x \mid x \geq 2\}$.

31. CHALLENGE Recall how values of a, h, and k can affect the graph of a quadratic function of the form $y = a(x - h)^2 + k$. Describe how values of a, h, and k can affect the graph of a square root function of the form $y = a\sqrt{x - h} + k$.

32. REASONING Describe the difference between the graphs of $y = \sqrt{x} - 4$ and $y = \sqrt{x - 4}$.

33. *Writing in Math* Refer to the information on page 397 to explain how square root functions can be used in bridge design. Assess the weights for which a diameter less than 1 is reasonable. Evaluate the amount of weight that the Sunshine Skyway Bridge can support.

STANDARDIZED TEST PRACTICE

34. ACT/SAT Given the graph of the square root function at the right, which must be true?
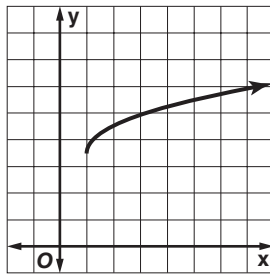
I. The domain is all real numbers.

II. The function is $y = \sqrt{x} + 3.5$.

III. The range is about $\{y \mid y \geq 3.5\}$.

A I only

B I, II, and III

C II and III

D III only

35. REVIEW For a game, Patricia must roll a die and draw a card from a deck of 26 cards, with each one having a letter of the alphabet on it. What is the probability that Patricia will roll an odd number and draw a letter in her name?

F $\frac{2}{3}$

G $\frac{3}{26}$

H $\frac{1}{13}$

J $\frac{1}{26}$

Spiral Review

Determine whether each pair of functions are inverse functions. (Lesson 7-2)

36. $f(x) = 3x$
$g(x) = \frac{1}{3}x$

37. $f(x) = 4x - 5$
$g(x) = \frac{1}{4}x - \frac{5}{16}$

38. $f(x) = \dfrac{3x + 2}{7}$
$g(x) = \dfrac{7x - 2}{3}$

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\dfrac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. (Lesson 7-1)

39. $f(x) = x + 5$
$g(x) = x - 3$

40. $f(x) = 10x - 20$
$g(x) = x - 2$

41. $f(x) = 4x^2 - 9$
$g(x) = \dfrac{1}{2x + 3}$

42. BIOLOGY Humans blink their eyes about once every 5 seconds. How many times do humans blink their eyes in two hours? (Lesson 1-1)

GET READY for the Next Lesson

PREREQUISITE SKILL Determine whether each number is *rational* or *irrational*. (Lesson 1-2)

43. 4.63

44. π

**45.** $\dfrac{16}{3}$

46. 8.333…

47. 7.323223222…

*n*th Roots

Main Ideas

- Simplify radicals.
- Use a calculator to approximate radicals.

New Vocabulary

*n*th root

principal root

Study Tip

Look Back

Review **square roots** in Lesson 5-4.

GET READY for the Lesson

The radius r of a sphere with volume V can be found using the formula $r = \sqrt[3]{\dfrac{3V}{4\pi}}$. This is an example of an equation that contains an *n*th root. In this case, $n = 3$.
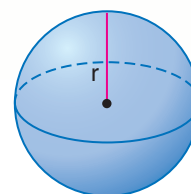
Simplify Radicals Finding the square root of a number and squaring a number are inverse operations. To find the square root of a number n, you must find a number whose square is n.

Similarly, the inverse of raising a number to the *n*th power is finding the **nth root** of a number. The table below shows the relationship between raising a number to a power and taking that root of a number.

| Powers	Factors	Roots
$a^3 = 125$	$5 \cdot 5 \cdot 5 = 125$	5 is a cube root of 125.
$a^4 = 81$	$3 \cdot 3 \cdot 3 \cdot 3 = 81$	3 is a fourth root of 81.
$a^5 = 32$	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$	2 is a fifth root of 32.
$a^n = b$	$\underbrace{a \cdot a \cdot a \cdot a \cdot \ldots \cdot a = b}_{n \text{ factors of } a}$	a is an *n*th root of b.

This pattern suggests the following formal definition of an *n*th root.

KEY CONCEPT   *Definition of nth Root*

Word For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an *n*th root of b.

Example Since $2^5 = 32$, 2 is a fifth root of 32.

The symbol $\sqrt[n]{}$ indicates an *n*th root.

index → $\sqrt[n]{50}$ ← radical sign, ← radicand
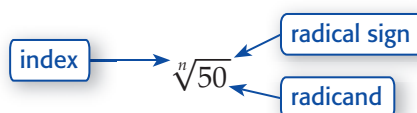
Some numbers have more than one real *n*th root. For example, 36 has two square roots, 6 and -6. When there is more than one real root, the nonnegative root is called the **principal root**. When no index is given, as in $\sqrt{36}$, the radical sign indicates the principal square root. The symbol $\sqrt[n]{b}$ stands for the principal *n*th root of b. If n is odd and b is negative, there will be no nonnegative root. In this case, the principal root is negative.

$$\sqrt{16} = 4 \qquad \sqrt{16} \text{ indicates the principal square root of 16.}$$

$$-\sqrt{16} = -4 \qquad -\sqrt{16} \text{ indicates the opposite of the principal square root of 16.}$$

$$\pm\sqrt{16} = \pm4 \qquad \pm\sqrt{16} \text{ indicates both square roots of 16.} \quad \pm \text{ means positive or negative.}$$

$$\sqrt[3]{-125} = -5 \qquad \sqrt[3]{-125} \text{ indicates the principal cube root of } -125.$$

$$-\sqrt[4]{81} = -3 \qquad -\sqrt[4]{81} \text{ indicates the opposite of the principal fourth root of 81.}$$

CONCEPT SUMMARY			Real nth roots of b, $\sqrt[n]{b}$, or $-\sqrt[n]{b}$
n	$\sqrt[n]{b}$ **if** $b > 0$	$\sqrt[n]{b}$ **if** $b < 0$	$b = 0$
even	one positive root, one negative root $\pm\sqrt[4]{625} = \pm5$	no real roots $\sqrt{-4}$ not a real number	one real root, 0 $\sqrt[n]{0} = 0$
odd	one positive root, no negative roots $\sqrt[3]{8} = 2$	no positive roots, one negative root $\sqrt[5]{-32} = -2$	

EXAMPLE **Find Roots**

① Simplify.

Study Tip

Fractional Exponents

For any real number b and any positive integer n, $\sqrt[n]{b} = b^{\frac{1}{n}}$.

a. $\pm\sqrt{25x^4}$

$$\pm25x^4 = \pm\sqrt{(5x^2)^2}$$

$$= \pm5x^2$$

The square roots of $25x^4$ are $\pm5x^2$.

b. $-\sqrt{(y^2 + 2)^8}$

$$-\sqrt{(y^2 + 2)^8} = -\sqrt{[(y^2 + 2)^4]^2}$$

$$= -(y^2 + 2)^4$$

The opposite of the principal square root of $(y^2 + 2)^8$ is $-(y^2 + 2)^4$.

c. $\sqrt[5]{32x^{15}y^{20}}$

$$\sqrt[5]{32x^{15}y^{20}} = \sqrt[5]{(2x^3y^4)^5}$$

$$= 2x^3y^4$$

The principal fifth root of $32x^{15}y^{20}$ is $2x^3y^4$.

d. $\sqrt{-9}$

$$\sqrt{-9} = \sqrt[2]{-9} \quad \leftarrow \text{n is even.} \quad \leftarrow \text{b is negative.}$$

Thus, $\sqrt{-9}$ is not a real number.

CHECK Your Progress

1A. $\pm\sqrt{81y^6}$

1B. $-\sqrt{(x - 3)^{12}}$

1C. $\sqrt[6]{729x^{30}y^{18}}$

1D. $\sqrt{-25}$

When you find the nth root of an even power and the result is an odd power, you must take the absolute value of the result to ensure that the answer is nonnegative.

$$\sqrt{(-5)^2} = |-5| \text{ or } 5 \qquad \sqrt{(-2)^6} = |(-2)^3| \text{ or } 8$$

If the result is an even power or you find the nth root of an odd power, there is no need to take the absolute value. *Why?*

EXAMPLE   Simplify Using Absolute Value

2 **Simplify.**

a. $\sqrt[8]{x^8}$

Note that x is an eighth root of x^8. The index is even, so the principal root is nonnegative. Since x could be negative, you must take the absolute value of x to identify the principal root.

$$\sqrt[8]{x^8} = |x|$$

b. $\sqrt[4]{81(a+1)^{12}}$

$$\sqrt[4]{81(a+1)^{12}} = \sqrt[4]{[3(a+1)^3]^4}$$

Since the index 4 is even and the exponent 3 is odd, you must use an absolute value.

$$\sqrt[4]{81(a+1)^{12}} = 3|(a+1)^3|$$

CHECK Your Progress

2A. $\sqrt{100x^{10}}$

2B. $\sqrt{64(y+1)^{14}}$

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Approximate Radicals with a Calculator Recall that real numbers that cannot be expressed as terminating or repeating decimals are *irrational numbers*. Approximations for irrational numbers are often used in real-world problems.

EXAMPLE

3 **PHYSICS** The distance a planet is from the Sun is a function of the length of its year. The formula is $d = \sqrt[3]{6t^2}$, where d is the distance of the planet from the Sun in millions of miles and t is the number of Earth-days in the planet's year. If the length of a year on Mars is 687 Earth-days, how far from the Sun is Mars?

$$d = \sqrt[3]{6t^2} \qquad\qquad \text{Original formula}$$

$$= \sqrt[3]{6(687)^2} \text{ or about } 141.48 \quad t = 687$$

Mars is approximately 141.48 million miles from the Sun.

CHECK   According to NASA, Mars is approximately 142 million miles from the Sun. So, 141.48 million miles is reasonable. ✔

CHECK Your Progress

3. Approximately how far away from the Sun is Earth?

Examples 1, 2
(pp. 403–404)

Simplify.

1. $\sqrt[3]{64}$

2. $\sqrt{(-2)^2}$

3. $\sqrt[5]{-243}$

4. $\sqrt[4]{-4096}$

5. $\sqrt[3]{x^3}$

6. $\sqrt[4]{y^4}$

7. $\sqrt{36a^2b^4}$

8. $\sqrt{(4x+3y)^2}$

Example 3
(p. 404)

Use a calculator to approximate each value to three decimal places.

9. $\sqrt{77}$

10. $-\sqrt[3]{19}$

11. $\sqrt[4]{48}$

12. **SHIPPING** Golden State Manufacturing wants to increase the size of the boxes it uses to ship its products. The new volume N is equal to the old volume V times the scale factor F cubed, or $N = V \cdot F^3$. What is the scale factor if the old volume was 8 cubic feet and the new volume is 216 cubic feet?

Exercises

Simplify.

13. $\sqrt{225}$

14. $\pm\sqrt{169}$

15. $\sqrt{-(-7)^2}$

16. $\sqrt{(-18)^2}$

17. $\sqrt[3]{-27}$

18. $\sqrt[7]{-128}$

19. $\sqrt{\dfrac{1}{16}}$

20. $\sqrt[3]{\dfrac{1}{125}}$

21. $\sqrt{0.25}$

22. $\sqrt[3]{-0.064}$

23. $\sqrt[4]{z^8}$

24. $-\sqrt[6]{x^6}$

25. $\sqrt{49m^6}$

26. $\sqrt{64a^8}$

27. $\sqrt[3]{27r^3}$

28. $\sqrt[3]{-c^6}$

29. $\sqrt{(5g)^4}$

30. $\sqrt[3]{(2z)^6}$

31. $\sqrt{25x^4y^6}$

32. $\sqrt{36x^4z^4}$

33. $\sqrt{169x^8y^4}$

34. $\sqrt{9p^{12}q^6}$

35. $\sqrt[3]{8a^3b^3}$

36. $\sqrt[3]{-27c^9d^{12}}$

Use a calculator to approximate each value to three decimal places.

37. $\sqrt{129}$

38. $-\sqrt{147}$

39. $\sqrt{0.87}$

40. $\sqrt{4.27}$

41. $\sqrt[3]{59}$

42. $\sqrt[3]{-480}$

43. $\sqrt[4]{602}$

44. $\sqrt[5]{891}$

45. $\sqrt[6]{4123}$

46. $\sqrt[7]{46,815}$

47. $\sqrt[6]{(723)^3}$

48. $\sqrt[4]{(3500)^2}$

49. **AEROSPACE** The radius r of the orbit of a satellite is given by $r = \sqrt[3]{\dfrac{GMt^2}{4\pi^2}}$, where G is the universal gravitational constant, M is the mass of the central object, and t is the time it takes the satellite to complete one orbit. Find the radius of the orbit if G is 6.67×10^{-11} N $\cdot$ m^2/kg^2, M is 5.98×10^{24} kg, and t is 2.6×10^6 seconds.

50. **SHOPPING** A certain store found that the number of customers that will attend a limited time sale can be modeled by $N = 125\sqrt[3]{100Pt}$, where N is the number of customers expected, P is the percent of the sale discount, and t is the number of hours the sale will last. Find the number of customers the store should expect for a sale that is 50% off and will last four hours.

H.O.T. Problems

51. **OPEN ENDED** Write a number whose principal square root and cube root are both integers.

52. **REASONING** Determine whether the statement $\sqrt[4]{(-x)^4} = x$ is *sometimes*, *always*, or *never* true.

53. CHALLENGE Under what conditions is $\sqrt{x^2 + y^2} = x + y$ true?

54. REASONING Explain why it is not always necessary to take the absolute value of a result to indicate the principal root.

55. *Writing in Math* Refer to the information on page 402 to explain how nth roots apply to geometry. Analyze what happens to the value of r as the value of V increases.

Spiral Review

Graph each function. State the domain and range. (Lesson 7-3)

58. $y = \sqrt{x - 2}$ **59.** $y = \sqrt{x} - 1$ **60.** $y = 2\sqrt{x} + 1$

61. Determine whether the functions $f(x) = x - 2$ and $g(x) = 2x$ are inverse functions. (Lesson 7-2)

Simplify. (Lesson 5-4)

62. $(3 + 2i) - (1 - 7i)$ **63.** $(8 - i)(4 - 3i)$ **64.** $\dfrac{2 + 3i}{1 + 2i}$

Solve each system of equations. (Lesson 3-2)

65. $2x - y = 7$
    $x + 3y = 0$

66. $4x + y = 7$
    $3x + \frac{4}{5}y = 5.5$

67. $\frac{1}{4}x + \frac{2}{3}y = 3$
    $2x + y = -2$

68. BUSINESS A dry cleaner ordered 7 drums of two different types of cleaning fluid. One type costs $30 per drum, and the other type costs $20 per drum. The total cost was $160. How much of each type of fluid did the company order? Write a system of equations and solve by graphing. (Lesson 3-1)

Graph each function. (Lesson 2-6)

69. $f(x) = 5$ **70.** $f(x) = |x - 3|$ **71.** $f(x) = |2x| + 3$

GET READY for the Next Lesson

PREREQUISITE SKILL Find each product. (Lesson 6-2)

72. $(x + 3)(x + 8)$ **73.** $(y - 2)(y + 5)$ **74.** $(a + 2)(a - 9)$

75. $(a + b)(a + 2b)$ **76.** $(x - 3y)(x + 3y)$ **77.** $(2w + z)(3w - 5z)$

Given $f(x) = 2x^2 - 5x + 3$ and $g(x) = 6x + 4$, find each function. (Lesson 7-1)

1. $(f + g)(x)$ **2.** $(f - g)(x)$

3. $(f \cdot g)(x)$ **4.** $\left(\dfrac{f}{g}\right)(x)$

5. $[f \circ g](x)$ **6.** $[g \circ f](x)$

DINING For Exercises 7 and 8, use the following information. (Lesson 7-1)

The Rockwell family goes out to dinner at Jack's Fancy Steak House. They have a coupon for 10% off their meal, but this restaurant adds an 18% gratuity.

7. Express the price of the meal after the discount and the price of the meal after the gratuity gets added using function notation. Let x represent the price of the meal, $p(x)$ represent the price after the 10% discount, and $g(x)$ represent the price after the gratuity is added to the bill.

8. Which composition of functions represents the price of the meal, $p[g(x)]$ or $g[p(x)]$? Explain your reasoning.

Determine whether each pair of functions are inverse functions. (Lesson 7-2)

9. $f(x) = x + 73$
$g(x) = x - 73$

10. $g(x) = 7x - 11$
$h(x) = \dfrac{1}{7}x + 11$

REMODELING For Exercises 11 and 12, use the following information. (Lesson 7-2)

Kimi is replacing the carpet in her 12-foot by 15-foot living room. The new carpet costs $13.99 per square yard. The formula $f(x) = 9x$ converts square yards to square feet.

11. Find the inverse $f^{-1}(x)$. What is the significance of $f^{-1}(x)$ for Kimi?

12. What will the new carpet cost Kimi?

Graph each inequality. (Lesson 7-3)

13. $y < \sqrt{x + 3}$ **14.** $y \geq -5\sqrt{x}$

Graph each function. State the domain and range of each function. (Lesson 7-3)

15. $y = 3 - \sqrt{x}$ **16.** $y = \sqrt{5x}$

17. $y = \sqrt{2x - 7} + 4$ **18.** $y = -2\sqrt{6x - 1}$

19. MULTIPLE CHOICE What is the domain of $f(x) = \sqrt{5x - 3}$?

A $\left\{x \middle| x > \dfrac{3}{5}\right\}$

B $\left\{x \middle| x > -\dfrac{3}{5}\right\}$

C $\left\{x \middle| x \geq \dfrac{3}{5}\right\}$

D $\left\{x \middle| x \geq -\dfrac{3}{5}\right\}$

Simplify. (Lesson 7-4)

20. $\sqrt{36x^2 y^6}$ **21.** $\sqrt[3]{-64a^6 b^9}$

22. $\sqrt{4n^2 + 12n + 9}$ **23.** $\sqrt{\dfrac{x^4}{y^3}}$

24. MULTIPLE CHOICE The relationship between the length and mass of Pacific Halibut can be approximated by the equation $L = 0.46\sqrt[3]{M}$, where L is the length in meters and M is the mass in kilograms. Use this equation to predict the length of a 25-kilogram Pacific Halibut. (Lesson 7-4)

F 1.03 m

G 1.35 m

H 1.97 m

J 2.30 m

25. BASEBALL Refer to the drawing below. How far does the catcher have to throw a ball from home plate to second base? (Lesson 7-4)
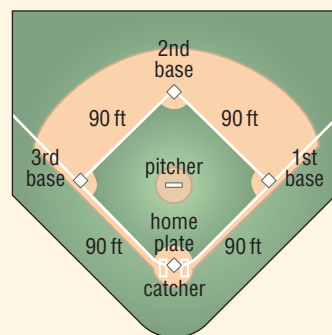
Operations with Radical Expressions

Main Ideas

• Simplify radical expressions.

• Add, subtract, multiply, and divide radical expressions.

New Vocabulary

rationalizing the denominator

like radical expressions

conjugates

▶ GET READY for the Lesson

Golden rectangles have been used by artists and architects to create beautiful designs. For example, if you draw a rectangle around the Mona Lisa's face, the resulting quadrilateral is the golden rectangle. The ratio of the lengths of the sides of a golden rectangle is $\dfrac{2}{\sqrt{5}-1}$.

In this lesson, you will learn how to simplify radical expressions like $\dfrac{2}{\sqrt{5}-1}$.
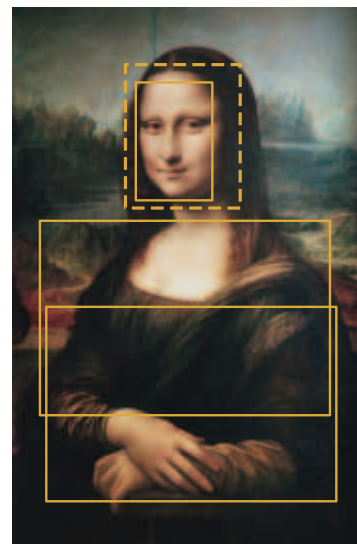
Simplify Radicals The properties you have used to simplify radical expressions involving square roots also hold true for expressions involving nth roots.

KEY CONCEPT   *Properties of Radicals*

For any real numbers a and b and any integer $n > 1$, the following properties hold true.

Property	Words	Examples
Product Property	1. If n is even and a and b are both nonnegative, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, and 2. If n is odd, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.	$\sqrt{2} \cdot \sqrt{8} = \sqrt{16}$, or 4, and $\sqrt[3]{3} \cdot \sqrt[3]{9} = \sqrt[3]{27}$, or 3
Quotient Property	$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, if all roots are defined and $b \neq 0$.	$\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\dfrac{54}{2}} = \sqrt[3]{27}$, or 3

Follow these steps to simplify a square root.

Step 1 Factor the radicand into as many squares as possible.

Step 2 Use the Product Property to isolate the perfect squares.

Step 3 Simplify each radical.

You can use the properties of radicals to write expressions in simplified form.

> **CONCEPT SUMMARY** *Simplifying Radical Expressions*
>
> A radical expression is in simplified form when the following conditions are met.
> - The index n is as small as possible.
> - The radicand contains no factors (other than 1) that are nth powers of an integer or polynomial.
> - The radicand contains no fractions.
> - No radicals appear in a denominator.

To eliminate radicals from a denominator or fractions from a radicand, you can use a process called **rationalizing the denominator**. To rationalize a denominator, multiply the numerator and denominator by a quantity so that the radicand has an exact root.

EXAMPLE **Simplify Expressions**

1 Simplify.

a. $\sqrt{16p^8q^7}$

$$\sqrt{16p^8q^7} = \sqrt{4^2 \cdot (p^4)^2 \cdot (q^3)^2 \cdot q} \qquad \text{Factor into squares where possible.}$$

$$= \sqrt{4^2} \cdot \sqrt{(p^4)^2} \cdot \sqrt{(q^3)^2} \cdot \sqrt{q} \qquad \text{Product Property of Radicals}$$

$$= 4p^4|q^3|\sqrt{q} \qquad \text{Simplify.}$$

However, for $\sqrt{16p^8q^7}$ to be defined, $16p^8q^7$ must be nonnegative. If that is true, q must be nonnegative, since it is raised to an odd power. Thus, the absolute value is unnecessary, and $\sqrt{16p^8q^7} = 4p^4q^3\sqrt{q}$.

Study Tip

Rationalizing the Denominator

You may want to think of rationalizing the denominator as making the denominator a rational number.

b. $\sqrt{\dfrac{x^4}{y^5}}$

$$\sqrt{\dfrac{x^4}{y^5}} = \dfrac{\sqrt{x^4}}{\sqrt{y^5}} \qquad \text{Quotient Property}$$

$$= \dfrac{\sqrt{(x^2)^2}}{\sqrt{(y^2)^2 \cdot y}} \qquad \text{Factor into squares.}$$

$$= \dfrac{\sqrt{(x^2)^2}}{\sqrt{(y^2)^2} \cdot \sqrt{y}} \qquad \text{Product Property}$$

$$= \dfrac{x^2}{y^2\sqrt{y}} \qquad \sqrt{(x^2)^2} = x^2$$

$$= \dfrac{x^2}{y^2\sqrt{y}} \cdot \dfrac{\sqrt{y}}{\sqrt{y}} \qquad \text{Rationalize the denominator.}$$

$$= \dfrac{x^2\sqrt{y}}{y^3} \qquad \sqrt{y} \cdot \sqrt{y} = y$$

c. $\sqrt[5]{\dfrac{5}{4a}}$

$$\sqrt[5]{\dfrac{5}{4a}} = \dfrac{\sqrt[5]{5}}{\sqrt[5]{4a}} \qquad \text{Quotient Property}$$

$$= \dfrac{\sqrt[5]{5}}{\sqrt[5]{4a}} \cdot \dfrac{\sqrt[5]{8a^4}}{\sqrt[5]{8a^4}} \qquad \text{Rationalize the denominator.}$$

$$= \dfrac{\sqrt[5]{5 \cdot 8a^4}}{\sqrt[5]{4a \cdot 8a^4}} \qquad \text{Product Property}$$

$$= \dfrac{\sqrt[5]{40a^4}}{\sqrt[5]{32a^5}} \qquad \text{Multiply.}$$

$$= \dfrac{\sqrt[5]{40a^4}}{2a} \qquad \sqrt[5]{32a^5} = 2a$$

Simplify.

1A. $\sqrt{36r^5s^{10}}$ **1B.** $\sqrt{\dfrac{m^9}{n^7}}$ **1C.** $\sqrt[4]{\dfrac{3x}{2}}$

Operations with Radicals You can use the Product and Quotient Properties to multiply and divide some radicals, respectively.

EXAMPLE **Multiply Radicals**

2 Simplify $6\sqrt[3]{9n^2} \cdot 3\sqrt[3]{24n}$.

$$6\sqrt[3]{9n^2} \cdot 3\sqrt[3]{24n} = 6 \cdot 3 \cdot \sqrt[3]{9n^2 \cdot 24n} \qquad \text{Product Property of Radicals}$$

$$= 18 \cdot \sqrt[3]{2^3 \cdot 3^3 \cdot n^3} \qquad \text{Factor into cubes where possible.}$$

$$= 18 \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{3^3} \cdot \sqrt[3]{n^3} \qquad \text{Product Property of Radicals}$$

$$= 18 \cdot 2 \cdot 3 \cdot n \text{ or } 108n \qquad \text{Multiply.}$$

Simplify.
2A. $5\sqrt[4]{24x^3} \cdot 4\sqrt[4]{54x}$ **2B.** $7\sqrt[3]{75a^4} \cdot 3\sqrt[3]{45a^2}$

Can you add radicals in the same way that you multiply them? In other words, if $\sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a}$, does $\sqrt{a} + \sqrt{a} = \sqrt{a + a}$?
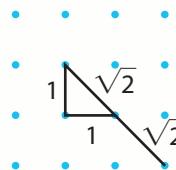
ALGEBRA LAB

Adding Radicals

You can use dot paper to show the sum of two like radicals, such as $\sqrt{2} + \sqrt{2}$.

Step 1 First, find a segment of length $\sqrt{2}$ units by using the Pythagorean Theorem with the dot paper.

$$a^2 + b^2 = c^2$$
$$1^2 + 1^2 = c^2$$
$$2 = c^2$$

Step 2 Extend the segment to twice its length to represent $\sqrt{2} + \sqrt{2}$.

Study Tip

Simplifying Radicals

In general, $\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$. In fact, $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ only when $a = 0$, $b = 0$, or both $a = 0$ and $b = 0$.

ANALYZE THE RESULTS

1. Is $\sqrt{2} + \sqrt{2} = \sqrt{2 + 2}$ or 2? Justify your answer using the geometric models above.

2. Use this method to model other irrational numbers. Do these models support your conjecture?

You add radicals in the same manner as adding monomials. That is, you can add only the like terms or like radicals. Two radical expressions are called **like radical expressions** if both the indices and the radicands are alike.

Like: $2\sqrt[4]{3a}$ and $5\sqrt[4]{3a}$ Radicands are $3a$; indices are 4.

Unlike: $\sqrt{3}$ and $\sqrt[3]{3}$ Different indices

$\sqrt[4]{5x}$ and $\sqrt[4]{5}$ Different radicands

Concepts
in MOtion

Animation
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EXAMPLE   Add and Subtract Radicals

3 Simplify $2\sqrt{12} - 3\sqrt{27} + 2\sqrt{48}$.

$2\sqrt{12} - 3\sqrt{27} + 2\sqrt{48}$

$= 2\sqrt{2^2 \cdot 3} - 3\sqrt{3^2 \cdot 3} + 2\sqrt{2^2 \cdot 2^2 \cdot 3}$ Factor using squares.

$= 2\sqrt{2^2} \cdot \sqrt{3} - 3\sqrt{3^2} \cdot \sqrt{3} + 2\sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{3}$ Product Property

$= 2 \cdot 2 \cdot \sqrt{3} - 3 \cdot 3 \cdot \sqrt{3} + 2 \cdot 2 \cdot 2 \cdot \sqrt{3}$ $\sqrt{2^2} = 2, \sqrt{3^2} = 3$

$= 4\sqrt{3} - 9\sqrt{3} + 8\sqrt{3}$ Multiply.

$= 3\sqrt{3}$ Combine like radicals.

CHECK Your Progress Simplify.

3A. $3\sqrt{8} + 5\sqrt{32} - 4\sqrt{18}$ **3B.** $5\sqrt{12} - 2\sqrt{27} + 6\sqrt{108}$

Just as you can add and subtract radicals like monomials, you can multiply radicals using the FOIL method as you do when multiplying binomials.

EXAMPLE   Multiply Radicals

4 Simplify.

a. $\left(3\sqrt{5} - 2\sqrt{3}\right)\left(2 + \sqrt{3}\right)$

$\qquad\qquad\qquad\qquad\qquad$ F $\qquad$ O $\qquad$ I $\qquad$ L

$\left(3\sqrt{5} - 2\sqrt{3}\right)\left(2 + \sqrt{3}\right) = 3\sqrt{5} \cdot 2 + 3\sqrt{5} \cdot \sqrt{3} - 2\sqrt{3} \cdot 2 - 2\sqrt{3} \cdot \sqrt{3}$

$\qquad\qquad\qquad\qquad\qquad\quad = 6\sqrt{5} + 3\sqrt{5 \cdot 3} - 4\sqrt{3} - 2\sqrt{3^2}$ Product Property

$\qquad\qquad\qquad\qquad\qquad\quad = 6\sqrt{5} + 3\sqrt{15} - 4\sqrt{3} - 6$ $2\sqrt{3^2} = 2 \cdot 3$ or 6

b. $\left(5\sqrt{3} - 6\right)\left(5\sqrt{3} + 6\right)$

$\left(5\sqrt{3} - 6\right)\left(5\sqrt{3} + 6\right) = 5\sqrt{3} \cdot 5\sqrt{3} + 5\sqrt{3} \cdot 6 - 6 \cdot 5\sqrt{3} - 6 \cdot 6$ FOIL

$\qquad\qquad\qquad\qquad\qquad\quad = 25\sqrt{3^2} + 30\sqrt{3} - 30\sqrt{3} - 36$ Multiply.

$\qquad\qquad\qquad\qquad\qquad\quad = 75 - 36$ $25\sqrt{3^2} = 25 \cdot 3$ or 75

$\qquad\qquad\qquad\qquad\qquad\quad = 39$ Subtract.

Study Tip

Conjugates
The product of conjugates of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ is always a rational number.

CHECK Your Progress

4A. $\left(4\sqrt{2} + 2\sqrt{6}\right)\left(\sqrt{5} - 3\right)$ **4B.** $\left(3\sqrt{5} + 4\right)\left(3\sqrt{5} - 4\right)$

Binomials like those in Example 4b, of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers, are called **conjugates** of each other. You can use conjugates to rationalize denominators.

Lesson 7-5 Operations with Radical Expressions   **411**

5 Simplify $\dfrac{1 - \sqrt{3}}{5 + \sqrt{3}}$.

$\dfrac{1 - \sqrt{3}}{5 + \sqrt{3}} = \dfrac{\left(1 - \sqrt{3}\right)\left(5 - \sqrt{3}\right)}{\left(5 + \sqrt{3}\right)\left(5 - \sqrt{3}\right)}$ 　Multiply by $\dfrac{5 - \sqrt{3}}{5 - \sqrt{3}}$ because $5 - \sqrt{3}$ is the conjugate of $5 + \sqrt{3}$.

$= \dfrac{1 \cdot 5 - 1 \cdot \sqrt{3} - \sqrt{3} \cdot 5 + \left(\sqrt{3}\right)^2}{5^2 - \left(\sqrt{3}\right)^2}$ 　FOIL

　Difference of squares

$= \dfrac{5 - \sqrt{3} - 5\sqrt{3} + 3}{25 - 3}$ 　Multiply.

$= \dfrac{8 - 6\sqrt{3}}{22}$ 　Combine like terms.

$= \dfrac{4 - 3\sqrt{3}}{11}$ 　Divide numerator and denominator by 2.

CHECK Your Progress **Simplify.**

5A. $\dfrac{4 + \sqrt{2}}{5 - \sqrt{2}}$ 　　　　　　**5B.** $\dfrac{3 - 2\sqrt{5}}{6 + \sqrt{5}}$

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CHECK Your Understanding

Example 1
(pp. 409–410)

Simplify.

1. $5\sqrt{63}$ 　　　　**2.** $\sqrt[4]{16x^5y^4}$ 　　　　**3.** $\sqrt{75x^3y^6}$

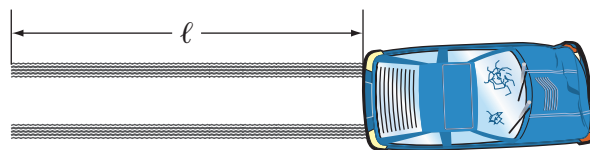**4.** $\sqrt{\dfrac{7}{8y}}$ 　　　　**5.** $\sqrt{\dfrac{a^7}{b^9}}$ 　　　　**6.** $\sqrt[3]{\dfrac{2}{3x}}$

LAW ENFORCEMENT **For Exercises 7 and 8, use the following information.**

Under certain conditions, a police accident investigator can use the formula $s = \dfrac{10\sqrt{\ell}}{\sqrt{5}}$ to estimate the speed s of a car in miles per hour based on the length ℓ in feet of the skid marks it left.

7. Write the formula without a radical in the denominator.

8. How fast was a car traveling that left skid marks 120 feet long?

Examples 2–5
(pp. 410–412)

Simplify.

9. $\left(-2\sqrt{15}\right)\left(4\sqrt{21}\right)$ 　　**10.** $\sqrt{2ab^2} \cdot \sqrt{6a^3b^2}$ 　　**11.** $\dfrac{\sqrt[3]{625}}{\sqrt[3]{25}}$

12. $\sqrt{3} - 2\sqrt[4]{3} + 4\sqrt{3} + 5\sqrt[4]{3}$ 　　　**13.** $3\sqrt[3]{128} + 5\sqrt[3]{16}$

14. $\left(3 - \sqrt{5}\right)\left(1 + \sqrt{3}\right)$ 　　　　**15.** $\left(2 + \sqrt{2}\right)\left(2 - \sqrt{2}\right)$

16. $\dfrac{1 + \sqrt{5}}{3 - \sqrt{5}}$ 　　　　　　**17.** $\dfrac{4 - \sqrt{7}}{3 + \sqrt{7}}$

Simplify.

For Exercises	See Examples
18–23	1
34–35	2
36–41	3
42–45	4
46–51	5

18. $\sqrt{243}$

19. $\sqrt{72}$

20. $\sqrt[3]{54}$

21. $\sqrt[4]{96}$

22. $\sqrt{50x^4}$

23. $\sqrt[3]{16y^3}$

24. $\sqrt{18x^2y^3}$

25. $\sqrt{40a^3b^4}$

26. $3\sqrt[3]{56y^6z^3}$

27. $2\sqrt[3]{24m^4n^5}$

28. $\sqrt[4]{\dfrac{1}{81}c^5d^4}$

29. $\sqrt[5]{\dfrac{1}{32}w^6z^7}$

30. $\sqrt[3]{\dfrac{3}{4}}$

31. $\sqrt[4]{\dfrac{2}{3}}$

32. $\sqrt{\dfrac{a^4}{b^3}}$
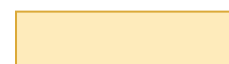
33. $\sqrt{\dfrac{4r^8}{t^9}}$

34. $\left(3\sqrt{12}\right)\left(2\sqrt{21}\right)$

35. $\left(-3\sqrt{24}\right)\left(5\sqrt{20}\right)$

36. **GEOMETRY** Find the perimeter and area of the rectangle.

$3 + 6\sqrt{2}$ yd

$\sqrt{8}$ yd

37. **GEOMETRY** Find the perimeter of a regular pentagon whose sides measure $\left(2\sqrt{3} + 3\sqrt{12}\right)$ feet.

Simplify.

38. $\sqrt{12} + \sqrt{48} - \sqrt{27}$

39. $\sqrt{98} - \sqrt{72} + \sqrt{32}$

40. $\sqrt{3} + \sqrt{72} - \sqrt{128} + \sqrt{108}$

41. $5\sqrt{20} + \sqrt{24} - \sqrt{180} + 7\sqrt{54}$

42. $\left(5 + \sqrt{6}\right)\left(5 - \sqrt{2}\right)$

43. $\left(3 + \sqrt{7}\right)\left(2 + \sqrt{6}\right)$

44. $\left(\sqrt{11} - \sqrt{2}\right)^2$

45. $\left(\sqrt{3} - \sqrt{5}\right)^2$

46. $\dfrac{7}{4 - \sqrt{3}}$

47. $\dfrac{\sqrt{6}}{5 + \sqrt{3}}$

48. $\dfrac{-2 - \sqrt{3}}{1 + \sqrt{3}}$

49. $\dfrac{2 + \sqrt{2}}{5 - \sqrt{2}}$

50. $\dfrac{x + 1}{\sqrt{x^2 - 1}}$

51. $\dfrac{x - 1}{\sqrt{x} - 1}$

52. What is $\sqrt{39}$ divided by $\sqrt{26}$?

53. Divide $\sqrt{14}$ by $\sqrt{35}$.

AMUSEMENT PARKS For Exercises 54 and 55, use the following information.
The velocity v in feet per second of a roller coaster at the bottom of a hill is related to the vertical drop h in feet and the velocity v_0 in feet per second of the coaster at the top of the hill by the formula $v_0 = \sqrt{v^2 - 64h}$.

54. Explain why $v_0 = v - 8\sqrt{h}$ is not equivalent to the given formula.

55. What velocity must a coaster have at the top of a 225-foot hill to achieve a velocity of 120 feet per second at the bottom?

SPORTS For Exercises 56 and 57, use the following information.
A ball that is hit or thrown horizontally with a velocity of v meters per second will travel a distance of d meters before hitting the ground, where $d = v\sqrt{\dfrac{h}{4.9}}$ and h is the height in meters from which the ball is hit or thrown.

56. Use the properties of radicals to rewrite the formula.

57. How far will a ball that is hit with a velocity of 45 meters per second at a height of 0.8 meter above the ground travel before hitting the ground?

H.O.T. Problems

58. **REASONING** Determine whether the statement $\dfrac{1}{\sqrt[n]{a}} = \sqrt[n]{a}$ is *sometimes*, *always*, or *never* true. Explain.

59. **OPEN ENDED** Write a sum of three radicals that contains two like terms. Explain how you would combine the terms. Defend your answer.

Six Flags

60. FIND THE ERROR Ethan and Alexis are simplifying $\dfrac{4 + \sqrt{5}}{2 - \sqrt{5}}$. Who is correct? Explain your reasoning.

Ethan

$$\dfrac{4 + \sqrt{5}}{2 - \sqrt{5}} = \dfrac{4 + \sqrt{5}}{2 - \sqrt{5}} \cdot \dfrac{2 + \sqrt{5}}{2 + \sqrt{5}}$$

$$= \dfrac{13 + 6\sqrt{5}}{-1}$$

Alexis

$$\dfrac{4 + \sqrt{5}}{2 - \sqrt{5}} = \dfrac{4 + \sqrt{5}}{2 - \sqrt{5}} \cdot \dfrac{4 - \sqrt{5}}{4 - \sqrt{5}}$$

$$= \dfrac{11}{13 - 6\sqrt{5}}$$

61. *Writing in Math* Refer to the information given on page 408 to explain how radical expressions relate to the Mona Lisa. Use the properties in this lesson to explain how you could rewrite the radical expression.

STANDARDIZED TEST PRACTICE

62. ACT/SAT The expression $\sqrt{180a^2b^8}$ is equivalent to which of the following?

A $5\sqrt{6}|a|b^4$

B $6\sqrt{5}|a|b^4$

C $3\sqrt{10}|a|b^4$

D $36\sqrt{5}|a|b^4$

63. REVIEW When the number of a year is divisible by 4, then a leap year occurs. However, when the year is divisible by 100, then a leap year does not occur unless the year is divisible by 400. Which is *not* an example of a leap year?

F 1884   H 1904

G 1900   J 1940

Spiral Review

Simplify. (Lesson 7-4)

64. $\sqrt{144z^8}$

65. $\sqrt[3]{216a^3b^9}$

66. $\sqrt{(y + 2)^2}$

67. Graph $y \le \sqrt{x + 1}$. (Lesson 7-3)

68. ELECTRONICS There are three basic things to be considered in an electrical circuit: the flow of the electrical current I, the resistance to the flow Z, called impedance, and electromotive force E, called voltage. These quantities are related in the formula $E = I \cdot Z$. The current of a circuit is to be $35 - 40j$ amperes. Electrical engineers use the letter j to represent the imaginary unit. Find the impedance of the circuit if the voltage is to be $430 - 330j$ volts. (Lesson 5-4)

Find the inverse of each matrix, if it exists. (Lesson 4-7)

69. $\begin{bmatrix} 8 & 6 \\ 7 & 5 \end{bmatrix}$

70. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

71. $\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}$

GET READY for the Next Lesson

PREREQUISITE SKILL Evaluate each expression.

72. $2\left(\dfrac{1}{8}\right)$

73. $3\left(\dfrac{1}{6}\right)$

74. $\dfrac{1}{2} + \dfrac{1}{3}$

75. $\dfrac{1}{3} + \dfrac{3}{4}$

76. $\dfrac{1}{8} + \dfrac{5}{12}$

77. $\dfrac{5}{6} - \dfrac{1}{5}$

78. $\dfrac{5}{8} - \dfrac{1}{4}$

79. $\dfrac{1}{4} - \dfrac{2}{3}$
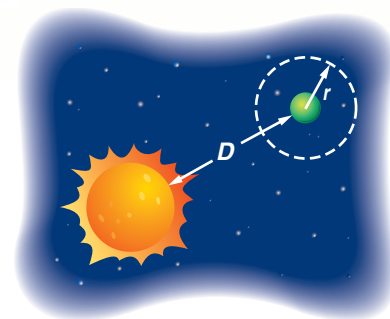
Rational Exponents

Main Ideas

- Write expressions with rational exponents in radical form and vice versa.
- Simplify expressions in exponential or radical form.

▶ **GET READY** for the Lesson

Astronomers refer to the space around a planet where the planet's gravity is stronger than the Sun's as the *sphere of influence* of the planet. The radius r of the sphere of influence is given by the formula

$r = D\left(\dfrac{M_p}{M_S}\right)^{\frac{2}{5}}$, where M_p is the mass

of the planet, M_S is the mass of the Sun, and D is the distance between the planet and the Sun.

Rational Exponents and Radicals You know that squaring a number and taking the square root of a number are inverse operations. But how would you evaluate an expression that contains a fractional exponent such as the one above? You can investigate such an expression by assuming that fractional exponents behave as integral exponents.

$$\left(b^{\frac{1}{2}}\right)^2 = b^{\frac{1}{2}} \cdot b^{\frac{1}{2}} \quad \text{Write the square as multiplication.}$$

$$= b^{\frac{1}{2} + \frac{1}{2}} \quad \text{Add the exponents.}$$

$$= b^1 \text{ or } b \quad \text{Simplify.}$$

Thus, $b^{\frac{1}{2}}$ is a number whose square equals b. So $b^{\frac{1}{2}} = \sqrt{b}$.

KEY CONCEPT $b^{\frac{1}{n}}$

Words   For any real number b and for any positive integer n, $b^{\frac{1}{n}} = \sqrt[n]{b}$, except when $b < 0$ and n is even.

Example $8^{\frac{1}{3}} = \sqrt[3]{8}$ or 2

EXAMPLE   Radical and Exponential Forms

1 **a.** Write $a^{\frac{1}{4}}$ in radical form.   **b.** Write $\sqrt[3]{y}$ in exponential form.

$a^{\frac{1}{4}} = \sqrt[4]{a}$ Definition of $b^{\frac{1}{n}}$ $\sqrt[3]{y} = y^{\frac{1}{3}}$ Definition of $b^{\frac{1}{n}}$

✓ **CHECK Your Progress**

1A. $x^{\frac{1}{5}}$ **1B.** $\sqrt[8]{c}$

EXAMPLE **Evaluate Expressions with Rational Exponents**

2 Evaluate each expression.

a. $16^{-\frac{1}{4}}$

Method 1

$$16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}} \qquad b^{-n} = \frac{1}{b^n}$$

$$= \frac{1}{\sqrt[4]{16}} \qquad 16^{\frac{1}{4}} = \sqrt[4]{16}$$

$$= \frac{1}{\sqrt[4]{2^4}} \qquad 16 = 2^4$$

$$= \frac{1}{2} \qquad \text{Simplify.}$$

Method 2

$$16^{-\frac{1}{4}} = (2^4)^{-\frac{1}{4}} \qquad 16 = 2^4$$

$$= 2^{4\left(-\frac{1}{4}\right)} \qquad \text{Power of a Power}$$

$$= 2^{-1} \qquad \text{Multiply exponents.}$$

$$= \frac{1}{2} \qquad 2^{-1} = \frac{1}{2^1}$$

b. $243^{\frac{3}{5}}$

Method 1

$$243^{\frac{3}{5}} = 243^{3\left(\frac{1}{5}\right)} \qquad \text{Factor.}$$

$$= (243^3)^{\frac{1}{5}} \qquad \text{Power of a Power}$$

$$= \sqrt[5]{243^3} \qquad b^{\frac{1}{5}} = \sqrt[5]{b}$$

$$= \sqrt[5]{(3^5)^3} \qquad 243 = 3^5$$

$$= \sqrt[5]{3^5 \cdot 3^5 \cdot 3^5} \qquad \text{Expand the cube.}$$

$$= 3 \cdot 3 \cdot 3 \text{ or } 27 \qquad \text{Find the fifth root.}$$

Method 2

$$243^{\frac{3}{5}} = (3^5)^{\frac{3}{5}} \qquad 243 = 3^5$$

$$= 3^{5\left(\frac{3}{5}\right)} \qquad \text{Power of a Power}$$

$$= 3^3 \qquad \text{Multiply exponents.}$$

$$= 27 \qquad 3^3 = 3 \cdot 3 \cdot 3$$

CHECK Your Progress

2A. $27^{-\frac{1}{3}}$ **2B.** $64^{\frac{2}{3}}$

In Example 2b, Method 1 uses a combination of the definition of $b^{\frac{1}{n}}$ and the properties of powers. This example suggests the following general definition of rational exponents.

KEY CONCEPT   *Rational Exponents*

Words   For any nonzero real number b, and any integers m and n, with
$n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$, except when $b < 0$ and n is even.

Example $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2$, or 4

In general, we define $b^{\frac{m}{n}}$ as $\left(b^{\frac{1}{n}}\right)^m$ or $(b^m)^{\frac{1}{n}}$. Now apply the definition of $b^{\frac{1}{n}}$ to $\left(b^{\frac{1}{n}}\right)^m$ and $(b^m)^{\frac{1}{n}}$.

$$\left(b^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{b}\right)^m \qquad\qquad (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$$

3 **WEIGHTLIFTING** The formula $M = 512 - 146{,}230B^{-\frac{8}{5}}$ can be used to estimate the maximum total mass that a weightlifter of mass B kilograms can lift using the snatch and the clean and jerk.

a. According to the formula, what is the maximum amount that 2004 Olympic champion Hossein Reza Zadeh of Iran can lift if he weighs 163 kilograms?

$$M = 512 - 146{,}230B^{-\frac{8}{5}}$$ Original formula

$$= 512 - 146{,}230(163)^{-\frac{8}{5}} \text{ or about } 470 \quad B = 163$$

The formula predicts that he can lift at most 470 kilograms.

Real-World Link

With origins in both the ancient Egyptian and Greek societies, weightlifting was among the sports on the program of the first Modern Olympic Games, in 1896, in Athens, Greece.

Source: International Weightlifting Association

b. **Hossein Reza Zadeh's winning total in the 2004 Olympics was 472.50 kg. Compare this to the value predicted by the formula.**

The formula prediction is close to the actual weight, but slightly lower.

CHECK Your Progress

3. The radius r of a sphere with volume V is given by $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$. Find the radius of a ball whose volume is 77 cm^3.

Simplify Expressions All of the properties of powers you learned in Lesson 6-1 apply to rational exponents. When simplifying expressions containing rational exponents, leave the exponent in rational form rather than writing the expression as a radical. Write the expression with all positive exponents. Also, any exponents in the denominator of a fraction must be positive *integers*. So, it may be necessary to rationalize a denominator.

EXAMPLE **Simplify Expressions with Rational Exponents**

4 Simplify each expression.

a. $x^{\frac{1}{5}} \cdot x^{\frac{7}{5}}$

$$x^{\frac{1}{5}} \cdot x^{\frac{7}{5}} = x^{\frac{1}{5} + \frac{7}{5}} \quad \text{Multiply powers.}$$

$$= x^{\frac{8}{5}} \quad \text{Add exponents.}$$

b. $y^{-\frac{3}{4}}$

$$y^{-\frac{3}{4}} = \frac{1}{y^{\frac{3}{4}}} \quad b^{-n} = \frac{1}{b^n}$$

$$= \frac{1}{y^{\frac{3}{4}}} \cdot \frac{y^{\frac{1}{4}}}{y^{\frac{1}{4}}} \quad \text{Why use } \frac{y^{\frac{1}{4}}}{y^{\frac{1}{4}}}?$$

$$= \frac{y^{\frac{1}{4}}}{y^{\frac{4}{4}}} \quad y^{\frac{3}{4}} \cdot y^{\frac{1}{4}} = y^{\frac{3}{4} + \frac{1}{4}}$$

$$= \frac{y^{\frac{1}{4}}}{y} \quad y^{\frac{4}{4}} = y^1 \text{ or } y$$

CHECK Your Progress

4A. $a^{\frac{1}{4}} \cdot a^{\frac{9}{4}}$

4B. $r^{-\frac{4}{5}}$

Andrea Comas/Reuters/CORBIS

EXAMPLE Simplify Radical Expressions

5 Simplify each expression.

a. $\dfrac{\sqrt[8]{81}}{\sqrt[6]{3}}$

$\dfrac{\sqrt[8]{81}}{\sqrt[6]{3}} = \dfrac{81^{\frac{1}{8}}}{3^{\frac{1}{6}}}$ Rational exponents

$= \dfrac{(3^4)^{\frac{1}{8}}}{3^{\frac{1}{6}}}$ $81 = 3^4$

$= \dfrac{3^{\frac{1}{2}}}{3^{\frac{1}{6}}}$ Power of a Power

$= 3^{\frac{1}{2} - \frac{1}{6}}$ Quotient of Powers

$= 3^{\frac{1}{3}}$ Simplify.

$= \sqrt[3]{3}$ Rewrite in radical form.

b. $\sqrt[4]{9z^2}$

$\sqrt[4]{9z^2} = (9z^2)^{\frac{1}{4}}$ Rational exponents

$= (3^2 \cdot z^2)^{\frac{1}{4}}$ $9 = 3^2$

$= 3^{2\left(\frac{1}{4}\right)} \cdot z^{2\left(\frac{1}{4}\right)}$ Power of a Power

$= 3^{\frac{1}{2}} \cdot z^{\frac{1}{2}}$ Multiply.

$= \sqrt{3} \cdot \sqrt{z}$ $3^{\frac{1}{2}} = \sqrt{3}$, $z^{\frac{1}{2}} = \sqrt{z}$

$= \sqrt{3z}$ Simplify.

c. $\dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1}$

$\dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1} = \dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1} \cdot \dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} - 1}$ $m^{\frac{1}{2}} - 1$ is the conjugate of $m^{\frac{1}{2}} + 1$.

$= \dfrac{m - 2m^{\frac{1}{2}} + 1}{m - 1}$ Multiply.

CHECK Your Progress

5A. $\dfrac{\sqrt[4]{32}}{\sqrt[3]{2}}$

5B. $\sqrt[3]{16x^4}$

5C. $\dfrac{y^{\frac{1}{2}} + 2}{y^{\frac{1}{2}} - 2}$

A STANDARDIZED TEST EXAMPLE

6 If x is a positive number, then $\dfrac{x^{\frac{4}{3}} \cdot x^{\frac{2}{3}}}{x^{\frac{1}{3}}} = ?$

A $x^{\frac{2}{3}}$ **B** $x^{\frac{3}{2}}$ **C** $\sqrt[3]{x^5}$ **D** $\sqrt[5]{x^3}$

$\dfrac{x^{\frac{4}{3}} \cdot x^{\frac{2}{3}}}{x^{\frac{1}{3}}} = \dfrac{x^{\frac{6}{3}}}{x^{\frac{1}{3}}}$ Add the exponents in the numerator.

$= \dfrac{x^2}{x^{\frac{1}{3}}}$ Simplify.

$= x^2 \cdot x^{-\frac{1}{3}}$ Quotient of Powers

$= x^{\frac{5}{3}}$ or $\sqrt[3]{x^5}$ The answer is C.

6. If y is positive, then $\dfrac{y \cdot y^2}{y^{\frac{1}{2}}} = ?$

F $y^{\frac{3}{2}}$ **G** $y^{\frac{5}{2}}$ **H** $\sqrt[2]{y^3}$ **J** $\sqrt[5]{y^2}$

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CONCEPT SUMMARY *Expressions with Rational Exponents*

An expression with rational exponents is simplified when all of the following conditions are met.

- It has no negative exponents.
- It has no fractional exponents in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

CHECK Your Understanding

Example 1
(p. 415)

Write each expression in radical form.

1. $7^{\frac{1}{3}}$ **2.** $x^{\frac{2}{3}}$

Write each radical using rational exponents.

3. $\sqrt[4]{26}$ **4.** $\sqrt[3]{6x^5 y^7}$

Example 2
(p. 416)

Evaluate each expression.

5. $125^{\frac{1}{3}}$ **6.** $81^{-\frac{1}{4}}$ **7.** $27^{\frac{2}{3}}$ **8.** $\dfrac{54}{9^{\frac{3}{2}}}$

Example 3
(p. 417)

9. ECONOMICS When inflation causes the price of an item to increase, the new cost C and the original cost c are related by the formula $C = c(1 + r)^n$, where r is the rate of inflation per year as a decimal and n is the number of years. What would be the price of a $4.99 item after six months of 5% inflation?

Examples 4, 5
(pp. 417–418)

Simplify each expression.

10. $a^{\frac{2}{3}} \cdot a^{\frac{1}{4}}$ **11.** $\dfrac{x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$ **12.** $\dfrac{a^2}{b^{\frac{1}{3}}} \cdot \dfrac{b}{a^{\frac{1}{2}}}$

13. $\sqrt[6]{27x^3}$ **14.** $\dfrac{\sqrt[4]{27}}{\sqrt[4]{3}}$ **15.** $\dfrac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}} - 1}$

Example 6
(pp. 418–419)

16. STANDARDIZED TEST PRACTICE If a is positive, then $\dfrac{a^5 \cdot a^{\frac{2}{3}}}{a^{\frac{4}{3}}} = ?$

A a **B** a^2 **C** $\sqrt[3]{a^{13}}$ **D** $\sqrt[13]{a^3}$

Exercises

Write each expression in radical form.

17. $6^{\frac{1}{5}}$ **18.** $4^{\frac{1}{3}}$ **19.** $c^{\frac{2}{5}}$ **20.** $(x^2)^{\frac{4}{3}}$

Write each radical using rational exponents.

21. $\sqrt{23}$ **22.** $\sqrt[3]{62}$ **23.** $\sqrt[4]{16z^2}$ **24.** $\sqrt[3]{5x^2y}$

Evaluate each expression.

25. $16^{\frac{1}{4}}$ **26.** $216^{\frac{1}{3}}$ **27.** $25^{-\frac{1}{2}}$ **28.** $(-27)^{-\frac{2}{3}}$

29. $81^{-\frac{1}{2}} \cdot 81^{\frac{3}{2}}$ **30.** $8^{\frac{3}{2}} \cdot 8^{\frac{5}{2}}$ **31.** $\dfrac{16^{\frac{1}{2}}}{9^{\frac{1}{2}}}$ **32.** $\dfrac{8^{\frac{1}{3}}}{64^{\frac{1}{3}}}$

BASKETBALL For Exercises 33 and 34, use the following information.
A women's regulation-sized basketball is slightly smaller than a men's basketball. The radius r of the ball that holds V volume of air is $r = \left(\dfrac{3V}{4\pi}\right)^{\frac{1}{3}}$.

33. Find the radius of a women's basketball if it will hold 413 cubic inches of air.

34. Find the radius of a men's basketball if it will hold 455 cubic inches of air.

Simplify each expression.

35. $y^{\frac{5}{3}} \cdot y^{\frac{7}{3}}$ **36.** $x^{\frac{3}{4}} \cdot x^{\frac{9}{4}}$ **37.** $\left(b^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **38.** $\left(a^{-\frac{2}{3}}\right)^{-\frac{1}{6}}$

39. $w^{-\frac{4}{5}}$ **40.** $\dfrac{r^{\frac{2}{3}}}{r^{\frac{1}{6}}}$ **41.** $\dfrac{a^{-\frac{1}{2}}}{6a^{\frac{1}{3}} \cdot a^{-\frac{1}{4}}}$ **42.** $\dfrac{y^{\frac{3}{2}}}{y^{\frac{1}{2}} + 2}$

43. $\sqrt[4]{25}$ **44.** $\sqrt[6]{27}$ **45.** $\sqrt{17} \cdot \sqrt[3]{17^2}$ **46.** $\sqrt[8]{25x^4y^4}$

47. $\dfrac{xy}{\sqrt{z}}$ **48.** $\sqrt[3]{\sqrt{8}}$ **49.** $\dfrac{8^{\frac{1}{6}} - 9^{\frac{1}{4}}}{\sqrt{3} + \sqrt{2}}$ **50.** $\dfrac{x^{\frac{5}{3}} - x^{\frac{1}{3}}z^{\frac{4}{3}}}{x^{\frac{2}{3}} + z^{\frac{2}{3}}}$

51. GEOMETRY A triangle has a base of $3r^{\frac{1}{2}}s^{\frac{1}{4}}$ units and a height of $4r^{\frac{1}{4}}s^{\frac{1}{2}}$ units. Find the area of the triangle.

52. GEOMETRY Find the area of a circle whose radius is $3x^{\frac{2}{3}}y^{\frac{1}{5}}z^2$ centimeters.

53. Find the simplified form of $32^{\frac{1}{2}} + 3^{\frac{1}{2}} - 8^{\frac{1}{2}}$.

54. What is the simplified form of $81^{\frac{1}{3}} - 24^{\frac{1}{3}} + 3^{\frac{1}{3}}$?

55. BIOLOGY Suppose a culture has 100 bacteria to begin with and the number of bacteria doubles every 2 hours. Then the number N of bacteria after t hours is given by $N = 100 \cdot 2^{\frac{t}{2}}$. How many bacteria will be present after $3\frac{1}{2}$ hours?

H.O.T. Problems

56. OPEN ENDED Determine a value of b for which $b^{\frac{1}{6}}$ is an integer.

57. REASONING Explain why $(-16)^{\frac{1}{2}}$ is not a real number.

58. CHALLENGE Explain how to solve $9^x = 3^{x + \frac{1}{2}}$ for x.

59. REASONING Determine whether $\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$ is *always*, *sometimes*, or *never* true. Explain.

60. *Writing in Math* Refer to the information on page 415 to explain how rational exponents can be applied to astronomy. Explain how to write the formula $r = D\left(\dfrac{M_p}{M_S}\right)^{\frac{2}{5}}$ in radical form and simplify it.

Spiral Review

Simplify. (Lessons 7-4 and 7-5)

63. $\sqrt{4x^3y^2}$

64. $\left(2\sqrt{6}\right)\left(3\sqrt{12}\right)$

65. $\sqrt{32} + \sqrt{18} - \sqrt{50}$

66. $\sqrt[4]{(-8)^4}$

67. $\sqrt[4]{(x-5)^2}$

68. $\sqrt{\dfrac{9}{36}x^4}$

TEMPERATURE For Exercises 69 and 70, use the following information.
There are three temperature scales: Fahrenheit (°F), Celsius (°C), and Kelvin (K). The function $K(C) = C + 273$ can be used to convert Celsius temperatures to Kelvin. The function $C(F) = \dfrac{5}{9}(F - 32)$ can be used to convert Fahrenheit temperatures to Celsius. (Lesson 7-1)

69. Write a composition of functions that could be used to convert Fahrenheit temperatures to Kelvin.

70. Find the temperature in Kelvin for the boiling point of water and the freezing point of water if water boils at 212°F and freezes at 32°F.

71. PHYSICS A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. (Lesson 5-6)

GET READY for the Next Lesson

PREREQUISITE SKILL Find each power. (Lesson 7-5)

72. $\left(\sqrt{x-2}\right)^2$

73. $\left(\sqrt[3]{2x-3}\right)^3$

74. $\left(\sqrt{x}+1\right)^2$

75. $\left(2\sqrt{x}-3\right)^2$

Solving Radical Equations and Inequalities

Main Ideas

- Solve equations containing radicals.
- Solve inequalities containing radicals.

New Vocabulary

radical equation

extraneous solution

radical inequality

▶GET READY for the Lesson

Computer chips are made from the element silicon, which is found in sand. Suppose a company that manufactures computer chips uses the formula $C = 10n^{\frac{2}{3}} + 1500$ to estimate the cost C in dollars of producing n chips. This can be rewritten as a radical equation.

Solve Radical Equations Equations with radicals that have variables in the radicands are called **radical equations**. To solve this type of equation, raise each side of the equation to a power equal to the index of the radical to eliminate the radical.

It is very important that you check your solution. Sometimes you will obtain a number that does not satisfy the original equation. Such a number is called an **extraneous solution.**

EXAMPLE   Solve Radical Equations

1 Solve each equation.

a. $\sqrt{x + 1} + 2 = 4$

$\sqrt{x + 1} + 2 = 4$ Original equation

$\sqrt{x + 1} = 2$ Subtract 2 from each side to isolate the radical.

$(\sqrt{x + 1})^2 = 2^2$ Square each side to eliminate the radical.

$x + 1 = 4$ Find the squares.

$x = 3$ Subtract 1 from each side.

CHECK $\sqrt{x + 1} + 2 = 4$ Original equation

$\sqrt{3 + 1} + 2 \stackrel{?}{=} 4$ Replace x with 3.

$4 = 4$ ✔   Simplify.

The solution checks. The solution is 3.

b. $\sqrt{x - 15} = 3 - \sqrt{x}$

$\sqrt{x - 15} = 3 - \sqrt{x}$ Original equation

$(\sqrt{x - 15})^2 = (3 - \sqrt{x})^2$ Square each side.

$x - 15 = 9 - 6\sqrt{x} + x$ Find the squares.

$-24 = -6\sqrt{x}$ Isolate the radical.

$4 = \sqrt{x}$ Divide each side by -6.

$4^2 = (\sqrt{x})^2$ Square each side again.

$16 = x$ Evaluate the squares.

CHECK
$$\sqrt{x - 15} = 3 - \sqrt{x}$$
$$\sqrt{16 - 15} \overset{?}{=} 3 - \sqrt{16}$$
$$\sqrt{1} \overset{?}{=} 3 - 4$$
$$1 \neq -1$$

The solution does not check, so the equation has no real solution. The graphs of $y = \sqrt{x - 15}$ and $y = 3 - \sqrt{x}$ are shown. The graphs do not intersect, which confirms that there is no solution.
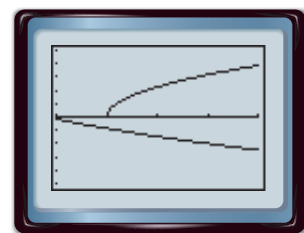
[10, 30] scl: 5 by [−5, 5] scl: 1

CHECK Your Progress

1A. $5 = \sqrt{x - 2} - 1$

1B. $\sqrt{x + 15} = 5 + \sqrt{x}$

To undo a square root, you square the expression. To undo an nth root, you must raise the expression to the nth power.

EXAMPLE **Solve a Cube Root Equation**

2 Solve $3(5n - 1)^{\frac{1}{3}} - 2 = 0$.

In order to remove the $\frac{1}{3}$ power, or cube root, you must first isolate it and then raise each side of the equation to the third power.

$$3(5n - 1)^{\frac{1}{3}} - 2 = 0 \qquad \text{Original equation}$$
$$3(5n - 1)^{\frac{1}{3}} = 2 \qquad \text{Add 2 to each side.}$$
$$(5n - 1)^{\frac{1}{3}} = \frac{2}{3} \qquad \text{Divide each side by 3.}$$
$$\left[(5n - 1)^{\frac{1}{3}}\right]^3 = \left(\frac{2}{3}\right)^3 \qquad \text{Cube each side.}$$
$$5n - 1 = \frac{8}{27} \qquad \text{Evaluate the cubes.}$$
$$5n = \frac{35}{27} \qquad \text{Add 1 to each side.}$$
$$n = \frac{7}{27} \qquad \text{Divide each side by 5.}$$

CHECK
$$3(5n - 1)^{\frac{1}{3}} - 2 = 0 \qquad \text{Original equation}$$
$$3\left(5 \cdot \frac{7}{27} - 1\right)^{\frac{1}{3}} - 2 \overset{?}{=} 0 \qquad \text{Replace } n \text{ with } \frac{7}{27}.$$
$$3\left(\frac{8}{27}\right)^{\frac{1}{3}} - 2 \overset{?}{=} 0 \qquad \text{Simplify.}$$
$$3\left(\frac{2}{3}\right) - 2 \overset{?}{=} 0 \qquad \text{The cube root of } \frac{8}{27} \text{ is } \frac{2}{3}.$$
$$0 = 0 \checkmark \qquad \text{Subtract.}$$

CHECK Your Progress **Solve each equation.**

2A. $(3n + 2)^{\frac{1}{3}} + 1 = 0$

2B. $(2y + 6)^{\frac{1}{4}} - 2 = 0$

Solve Radical Inequalities A **radical inequality** is an inequality that has a variable in a radicand.

EXAMPLE Solve a Radical Inequality

3 Solve $2 + \sqrt{4x - 4} \leq 6$.

Since the radicand of a square root must be greater than or equal to zero, first solve $4x - 4 \geq 0$ to identify the values of x for which the left side of the given inequality is defined.

$$4x - 4 \geq 0$$
$$4x \geq 4$$
$$x \geq 1$$

Now solve $2 + \sqrt{4x - 4} \leq 6$.

$2 + \sqrt{4x - 4} \leq 6$	Original inequality
$\sqrt{4x - 4} \leq 4$	Isolate the radical.
$4x - 4 \leq 16$	Eliminate the radical.
$4x \leq 20$	Add 4 to each side.
$x \leq 5$	Divide each side by 4.

It appears that $1 \leq x \leq 5$. You can test some x-values to confirm the solution. Let $f(x) = 2 + \sqrt{4x - 4}$. Use three test values: one less than 1, one between 1 and 5, and one greater than 5. Organize the test values in a table.

$x = 0$	$x = 2$	$x = 7$
$f(0) = 2 + \sqrt{4(0) - 4}$ $= 2 + \sqrt{-4}$ Since $\sqrt{-4}$ is not a real number, the inequality is not satisfied.	$f(2) = 2 + \sqrt{4(2) - 4}$ $= 4$ Since $4 \leq 6$, the inequality is satisfied.	$f(7) = 2 + \sqrt{4(7) - 4}$ ≈ 6.90 Since $6.90 \not\leq 6$, the inequality is not satisfied.

The solution checks. Only values in the interval $1 \leq x \leq 5$ satisfy the inequality. You can summarize the solution with a number line.

CHECK Your Progress

Solve each inequality.

3A. $\sqrt{2x + 2} + 1 \geq 5$ **3B.** $\sqrt{4x - 4} - 2 < 4$

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CONCEPT SUMMARY — Solving Radical Inequalities

To solve radical inequalities, complete the following steps.

Step 1 If the index of the root is even, identify the values of the variable for which the radicand is nonnegative.
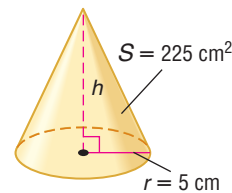
Step 2 Solve the inequality algebraically.

Step 3 Test values to check your solution.

Example 1
(pp. 422–423)

Solve each equation.

1. $\sqrt{4x+1}=3$

2. $4-(7-y)^{\frac{1}{2}}=0$

3. $1+\sqrt{x+2}=0$

4. GEOMETRY The surface area S of a cone can be found by using $S=\pi r\sqrt{r^2+h^2}$, where r is the radius of the base and h is the height of the cone. Find the height of the cone.

$S=225\text{ cm}^2$

h

$r=5\text{ cm}$

Example 2
(p. 423)

Solve each equation.

5. $\frac{1}{6}(12a)^{\frac{1}{3}}=1$

6. $\sqrt[3]{x-4}=3$

7. $(3y)^{\frac{1}{3}}+2=5$

Example 3
(p. 424)

Solve each inequality.

8. $\sqrt{2x+3}-4\le 5$

9. $\sqrt{b+12}-\sqrt{b}>2$

10. $\sqrt{y-7}+5\ge 10$

Exercises

Solve each equation.

11. $\sqrt{x}=4$

12. $\sqrt{y}-7=0$

13. $a^{\frac{1}{2}}+9=0$

14. $2+4z^{\frac{1}{2}}=0$

15. $7+\sqrt{4x+8}=9$

16. $5+\sqrt{4y-5}=12$

17. $\sqrt{x-5}=\sqrt{2x-4}$

18. $\sqrt{2t-7}=\sqrt{t+2}$

19. $\sqrt{x-6}-\sqrt{x}=3$

20. $\sqrt{y+21}-1=\sqrt{y+12}$

21. $\sqrt{b+1}=\sqrt{b+6}-1$

22. $\sqrt{4z+1}=3+\sqrt{4z-2}$

23. $\sqrt[3]{c-1}=2$

24. $\sqrt[3]{5m+2}=3$

25. $(6n-5)^{\frac{1}{3}}+3=-2$

26. $(5x+7)^{\frac{1}{5}}+3=5$

27. $(3x-2)^{\frac{1}{5}}+6=5$

28. $(7x-1)^{\frac{1}{3}}+4=2$

29. The formula $s=2\pi\sqrt{\dfrac{\ell}{32}}$ represents the swing of a pendulum, where s is the time in seconds to swing back and forth, and ℓ is the length of the pendulum in feet. Find the length of a pendulum that makes one swing in 1.5 seconds.

Real-World Link

A ponderal index p is a measure of a person's body based on height h in meters and mass m in kilograms. One such formula is $p=\dfrac{\sqrt[3]{m}}{h}$.

Source: *A Dictionary of Food and Nutrition*

30. HEALTH Refer to the information at the left.
A 70-kilogram person who is 1.8 meters tall has a ponderal index of about 2.29. How much weight could such a person gain and still have an index of at most 2.5?

Solve each inequality.

31. $1 + \sqrt{7x - 3} > 3$

32. $\sqrt{3x + 6} + 2 \le 5$

33. $-2 + \sqrt{9 - 5x} \ge 6$

34. $6 - \sqrt{2y + 1} < 3$

35. $\sqrt{2} - \sqrt{x + 6} \le -\sqrt{x}$

36. $\sqrt{a + 9} - \sqrt{a} > \sqrt{3}$

37. $\sqrt{b - 5} - \sqrt{b + 7} \le 4$

38. $\sqrt{c + 5} + \sqrt{c + 10} > 2$

39. PHYSICS When an object is dropped from the top of a 50-foot tall building, the object will be h feet above the ground after t seconds, where $\frac{1}{4}\sqrt{50 - h} = t$. How far above the ground will the object be after 1 second?

40. FISH The relationship between the length and mass of certain fish can be approximated by the equation $L = 0.46 \sqrt[3]{M}$, where L is the length in meters and M is the mass in kilograms. Solve this equation for M.

H.O.T. Problems

41. REASONING Determine whether the equation $\dfrac{\sqrt{(x^2)^2}}{-x} = x$ is *sometimes*, *always*, or *never* true when x is a real number. Explain your reasoning.

42. Which One Doesn't Belong? Which equation does *not* have a solution?

| $\sqrt{x-1} + 3 = 4$ | $\sqrt{x+1} + 3 = 4$ | $\sqrt{x-2} + 7 = 10$ | $\sqrt{x+2} - 7 = -10$ |

43. OPEN ENDED Write an equation containing two radicals for which 1 is a solution.

44. CHALLENGE Explain how you know that $\sqrt{x + 2} + \sqrt{2x - 3} = -1$ has no real solution without actually solving it.

45. *Writing in Math* Refer to the information on page 422 to describe how the cost and the number of units manufactured are related. Rewrite the manufacturing equation $C = 10n^{\frac{2}{3}} + 1500$ as a radical equation, and write a step-by-step explanation of how to determine the maximum number of chips the company could make for $10,000.

STANDARDIZED TEST PRACTICE

46. ACT/SAT If $\sqrt{x + 5} + 1 = 4$, what is the value of x?

A 4 **B** 10 **C** 11 **D** 20

47. REVIEW Which set of points describes a function?

F $\{(3, 0), (-2, 5), (2, -1), (2, 9)\}$

G $\{(-3, 5), (-2, 3), (-1, 5), (0, 7)\}$

H $\{(2, 5), (2, 4), (2, 3), (2, 2)\}$

J $\{(3, 1), (-3, 2), (3, 3), (-3, 4)\}$

48. REVIEW What is an equivalent form of $\dfrac{4}{5 + i}$?

A $\dfrac{10 - 2i}{13}$

B $\dfrac{5 - i}{6}$

C $\dfrac{6 - i}{6}$

D $\dfrac{6 - i}{13}$

Spiral Review

Write each radical using rational exponents. (Lesson 7-6)

49. $\sqrt[7]{5^3}$

50. $\sqrt{x + 7}$

51. $\left(\sqrt[3]{x^2 + 1}\right)^2$

Simplify. (Lesson 7-5)

52. $\sqrt{72x^6y^3}$

53. $\dfrac{1}{\sqrt[3]{10}}$

54. $\left(5 - \sqrt{3}\right)^2$

55. SALES Sales associates at Electronics Unlimited earn $8 an hour plus a 4% commission on the merchandise they sell. Write a function to describe their income, and find how much merchandise they must sell in order to earn $500 in a 40-hour week. (Lesson 7-2)

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\dfrac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. (Lesson 7-1)

56. $f(x) = x + 5$
$g(x) = x - 3$

57. $f(x) = 10x - 20$
$g(x) = x - 2$

58. $f(x) = 4x^2 - 9$
$g(x) = \dfrac{1}{2x + 3}$

59. ENTERTAINMENT A magician asked a member of his audience to choose any number. He said, "Multiply your number by 3. Add the sum of your number and 8 to that result. Now divide by the sum of your number and 2." The magician announced the final answer without asking the original number. What was the final answer? How did he know what it was?
(Lesson 6-4)

Simplify. (Lesson 6-2)

60. $(x + 2)(2x - 8)$

61. $(3p + 5)(2p - 4)$

62. $(a^2 + a + 1)(a - 1)$

CONSTRUCTION For Exercises 63 and 64, use the graph at right that shows the amount of money awarded for construction in Texas.
(Lesson 2-5)

63. Let the independent variable be years since 1999. Write a prediction equation from the data for 1999, 2000, 2001, and 2002.

64. Use your prediction equation to predict the amount for 2010.

Money Awarded for Construction in Texas

4.94, 5.23, 6.07, 7.3

Dollars (billions) — 1999, 2000, 2001, 2002 — Years

Source: *Texas Almanac*

Cross-Curricular Project

Algebra and Social Studies

Population Explosion It is time to complete your project. Use the information and data you have gathered about the population to prepare a Web page. Be sure to include graphs, tables, and equations in the presentation.

Math Online **Cross-Curricular Project at** algebra2.com

Graphing Calculator Lab
Solving Radical Equations and Inequalities

You can use a TI-83/84 Plus graphing calculator to solve radical equations and inequalities. One way to do this is by rewriting the equation or inequality so that one side is 0 and then using the zero feature on the calculator.

ACTIVITY 1 Solve $\sqrt{x} + \sqrt{x + 2} = 3$.

Step 1 Rewrite the equation.

- Subtract 3 from each side of the equation to obtain $\sqrt{x} + \sqrt{x + 2} - 3 = 0$.

- Enter the function $y = \sqrt{x} + \sqrt{x + 2} - 3$ in the Y= list.

   KEYSTROKES: *Review entering a function on page 399.*

Step 2 Use a table.

- You can use the TABLE function to locate intervals where the solution(s) lie. First, enter the starting value and the interval for the table.

   KEYSTROKES: 2nd [TBLSET] 0 ENTER
   1 ENTER

Step 3 Estimate the solution.

- Complete the table and estimate the solution(s).

   KEYSTROKES: 2nd [TABLE]
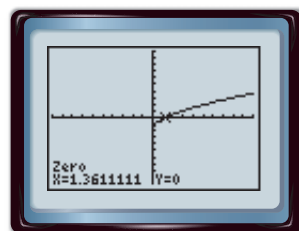

Since the function changes sign from negative to positive between $x = 1$ and $x = 2$, there is a solution between 1 and 2.

Step 4 Use the zero feature.

- Graph, then select zero from CALC menu.

   KEYSTROKES: GRAPH 2nd [CALC] 2

[−10, 10] scl: 1 by [−10, 10] scl: 1

Place the cursor to the left of the zero and press ENTER for the Left Bound. Then place the cursor to the right of the zero and press ENTER for the Right Bound. Press ENTER to solve. The solution is about 1.36. This agrees with the estimate made by using the TABLE.

Other Calculator Keystrokes at algebra2.com

ACTIVITY 2 Solve $2\sqrt{x} > \sqrt{x + 2} + 1$.

Step 1 Graph each side of the inequality and use the trace feature.

- In the Y= list, enter $y_1 = 2\sqrt{x}$ and $y_2 = \sqrt{x + 2} + 1$. Then press GRAPH.
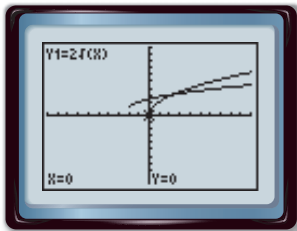
[−10, 10] scl: 1 by [−10, 10] scl: 1

- Press TRACE. You can use ▲ or ▼ to switch the cursor between the two curves.

The calculator screen above shows that, for points to the left of where the curves cross, Y1 < Y2 or $2\sqrt{x} < \sqrt{x + 2} + 1$. To solve the original inequality, you must find points for which Y1 > Y2. These are the points to the right of where the curves cross.

Step 2 Use the intersect feature.

- You can use the INTERSECT feature on the CALC menu to approximate the x-coordinate of the point at which the curves cross.

  KEYSTROKES: 2nd [CALC] 5

- Press ENTER for each of First curve?, Second curve?, and Guess?.
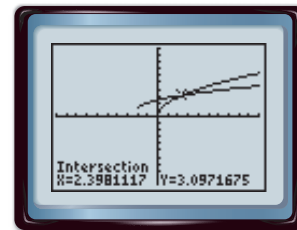
[−10, 10] scl: 1 by [−10, 10] scl: 1

The calculator screen shows that the x-coordinate of the point at which the curves cross is about 2.40. Therefore, the solution of the inequality is about $x > 2.40$. *Use the symbol > in the solution because the symbol in the original inequality is >.*

Step 3 Use the table feature to check your solution.

Start the table at 2 and show x-values in increments of 0.1. Scroll through the table.

KEYSTROKES: 2nd [TBLSET] 2 ENTER .1 ENTER
2nd [TABLE]

Notice that when x is less than or equal to 2.4, Y1 < Y2. This verifies the solution $\{x \mid x > 2.40\}$.

EXERCISES

Solve each equation or inequality.

1. $\sqrt{x + 4} = 3$ **2.** $\sqrt{3x - 5} = 1$ **3.** $\sqrt{x + 5} = \sqrt{3x + 4}$

4. $\sqrt{x + 3} + \sqrt{x - 2} = 4$ **5.** $\sqrt{3x - 7} = \sqrt{2x - 2} - 1$ **6.** $\sqrt{x + 8} - 1 = \sqrt{x + 2}$

7. $\sqrt{x - 3} \geq 2$ **8.** $\sqrt{x + 3} > 2\sqrt{x}$ **9.** $\sqrt{x} + \sqrt{x - 1} < 4$

10. Explain how you could apply the technique in the first example to solving an inequality.
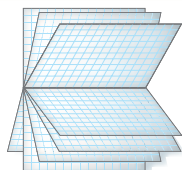
STUDY TO GO

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FOLDABLES™
Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.

Key Concepts

Operations on Functions (Lesson 7-1)

Operation	Definition
Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, g(x) \neq 0$
Composition	$[f \circ g](x) = f[g(x)]$

Inverse and Square Root Functions
(Lesson 7-2 and 7-3)

- Reverse the coordinates of ordered pairs to find the inverse of a relation.
- Two functions are inverses if and only if both of their compositions are the identity function.

Roots of Real Numbers (Lesson 7-4)

	Real nth roots of b, $\sqrt[n]{b}$, or $-\sqrt[n]{b}$		
n	$\sqrt[n]{b}$ if $b > 0$	$\sqrt[n]{b}$ if $b < 0$	$\sqrt[n]{b}$ if $b = 0$
even	one positive root one negative root	no real roots	one real root, 0
odd	one positive root no negative roots	no positive roots one negative root	

Radicals (Lessons 7-5 through 7-7)

For any real numbers a and b and any integer $n > 1$,

- Product Property: $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- Quotient Property: $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- For any nonzero real number b, and any integers m and n, with $n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$.
- To solve a radical equation, isolate the radical. Then raise each side of the equation to a power equal to the index of the radical.

Key Vocabulary

composition of functions (p. 385)
conjugates (p. 411)
extraneous solution (p. 422)
identity function (p. 393)
inverse function (p. 392)
inverse relation (p. 391)
like radical expressions (p. 411)
nth root (p. 402)
one-to-one (p. 394)
principal root (p. 402)
radical equation (p. 422)
radical inequality (p. 424)
rationalizing the denominator (p. 409)
square root function (p. 397)
square root inequality (p. 399)

Vocabulary Check

Choose a word or term from the list above that best completes each statement or phrase.

1. A(n)_____ is an equation with radicals that have variables in the radicands.

2. A solution of a transformed equation that is not a solution of the original equation is a(n) _____.

3. _____ have the same index and the same radicand.

4. When a number has more than one real root, the _____ is the nonnegative root.

5. $f(x) = 6x - 2$ and $g(x) = \dfrac{x + 2}{6}$ are _____ since $[f \circ g](x) = x$ and $[g \circ f](x) = x$.

6. A(n) _____ is when a function is performed, and then a second function is performed on the result of the first function.

7. A(n) _____ function is a function whose inverse is a function.

8. The process of eliminating radicals from a denominator or fractions from a radicand is called _____.

9. Two relations are _____ if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).

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Lesson-by-Lesson Review

Operations on Functions (pp. 384–390)

Find $[g \circ h](x)$ and $[h \circ g](x)$.

10. $h(x) = 2x - 1$
$g(x) = 3x + 4$

11. $h(x) = x^2 + 2$
$g(x) = x - 3$

12. $h(x) = x^2 + 1$
$g(x) = -2x + 1$

13. $h(x) = -5x$
$g(x) = 3x - 5$

14. $h(x) = x^3$
$g(x) = x - 2$

15. $h(x) = x + 4$
$g(x) = |x|$

16. **TIME** The formula $h = \dfrac{m}{60}$ converts minutes m to hours h, and $d = \dfrac{h}{24}$ converts hours to days d. Write a composition of functions that converts minutes to days.

Example 1 If $f(x) = x^2 - 2$ and $g(x) = 8x - 1$, find $g[f(x)]$ and $f[g(x)]$.

$$\begin{aligned} g[f(x)] &= 8(x^2 - 2) - 1 &&\text{Replace } f(x) \text{ with } x^2 - 2. \\ &= 8x^2 - 16 - 1 &&\text{Multiply.} \\ &= 8x^2 - 17 &&\text{Simplify.} \end{aligned}$$

$$\begin{aligned} f[g(x)] &= (8x - 1)^2 - 2 &&\text{Replace } g(x) \text{ with } 8x - 1. \\ &= 64x^2 - 16x + 1 - 2 &&\text{Expand.} \\ &= 64x^2 - 16x - 1 &&\text{Simplify.} \end{aligned}$$

Inverse Functions and Relations (pp. 391–396)

Find the inverse of each function. Then graph the function and its inverse.

17. $f(x) = 3x - 4$

18. $f(x) = -2x - 3$

19. $g(x) = \frac{1}{3}x + 2$

20. $f(x) = \dfrac{-3x + 1}{2}$

21. $y = x^2$

22. $y = (2x + 3)^2$

23. **SALES** Jim earns $10 an hour plus a 10% commission on sales. Write a function to describe Jim's income. If Jim wants to earn $1000 in a 40-hour week, what should his sales be?

24. **BANKING** During the last month, Jonathan has made two deposits of $45, made a deposit of double his original balance, and withdrawn $35 five times. His balance is now $189. Write an equation that models this problem. How much money did Jonathan have in his account at the beginning of the month?

Example 2 Find the inverse of $f(x) = -3x + 1$.

Rewrite $f(x)$ as $y = -3x + 1$. Then interchange the variables and solve for y.

$$\begin{aligned} x &= -3y + 1 &&\text{Interchange the variables.} \\ 3y &= -x + 1 &&\text{Solve for } y. \\ y &= \frac{-x + 1}{3} &&\text{Divide each side by 3.} \\ f^{-1}(x) &= \frac{-x + 1}{3} &&\text{Rewrite in function notation.} \end{aligned}$$

Square Root Functions and Inequalities (pp. 397–401)

Graph each function.
25. $y = \frac{1}{3}\sqrt{x + 2}$
26. $y = \sqrt{5x - 3}$
27. $y = 4 + 2\sqrt{x - 3}$

**Graph each inequality.**
28. $y \geq \sqrt{x} - 2$
29. $y < \sqrt{4x - 5}$

30. **OCEAN** The speed a tsunami, or tidal wave, can travel is modeled by the equation $s = 356\sqrt{d}$, where s is the speed in kilometers per hour and d is the average depth of the water in kilometers. A tsunami is found to be traveling at 145 kilometers per hour. What is the average depth of the water? Round to the nearest hundredth.

Example 3 Graph $y = 2 + \sqrt{x - 1}$.
Make a table of values and graph the function.

x	1	2	3	4	5
y	2	3	$2 + \sqrt{2}$ or 3.4	$2 + \sqrt{3}$ or 3.7	7
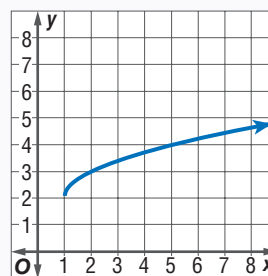
***n*th Roots** (pp. 402–406)

Simplify.
31. $\pm\sqrt{256}$
32. $\sqrt[3]{-216}$
33. $\sqrt{(-8)^2}$
34. $\sqrt[5]{c^5 d^{15}}$
35. $\sqrt{(x^4 - 3)^2}$
36. $\sqrt[3]{(512 + x^2)^3}$
37. $\sqrt[4]{16m^8}$
38. $\sqrt{a^2 - 10a + 25}$

39. **PHYSICS** The velocity v of an object can be defined as $v = \sqrt{\frac{2K}{m}}$, where m is the mass of an object and K is the kinetic energy. Find the velocity of an object with a mass of 15 grams and a kinetic energy of 750.

Example 4 Simplify $\sqrt{81x^6}$.
$\sqrt{81x^6} = \sqrt{(9x^3)^2}$ $81x^6 = (9x^3)^2$
$= 9|x^3|$ Use absolute value since x could be negative.

Example 5 Simplify $\sqrt[7]{2187x^{14}y^{35}}$.
$\sqrt[7]{2187x^{14}y^{35}}$
$= \sqrt[7]{(3x^2y^5)^7}$ $2187x^{14}y^{35} = (3x^2y^5)^7$
$= 3x^2y^5$ Evaluate.

7–5 **Operations with Radical Expressions** (pp. 408–414)

Simplify.

40. $\sqrt[6]{128}$

41. $5\sqrt{12} - 3\sqrt{75}$

42. $6\sqrt[5]{11} - 8\sqrt[5]{11}$

43. $\left(\sqrt{8} + \sqrt{12}\right)^2$

44. $\sqrt{8} \cdot \sqrt{15} \cdot \sqrt{21}$

45. $\dfrac{\sqrt{243}}{\sqrt{3}}$

46. $\dfrac{1}{3 + \sqrt{5}}$

47. $\dfrac{\sqrt{10}}{4 + \sqrt{2}}$

48. GEOMETRY The measures of the legs of a right triangle can be represented by the expressions $4x^2y^2$ and $8x^2y^2$. Use the Pythagorean Theorem to find a simplified expression for the measure of the hypotenuse.

Example 6 Simplify $6\sqrt[5]{32m^3} \cdot 5\sqrt[5]{1024m^2}$.

$6\sqrt[5]{32m^3} \cdot 5\sqrt[5]{1024m^2}$

$= 6 \cdot 5 \sqrt[5]{(32m^3 \cdot 1024m^2)}$ Product Property of Radicals

$= 30\sqrt[5]{2^5 \cdot 4^5 \cdot m^5}$ Factor into exponents of 5 if possible.

$= 30\sqrt[5]{2^5} \cdot \sqrt[5]{4^5} \cdot \sqrt[5]{m^5}$ Product Property of Radicals

$= 30 \cdot 2 \cdot 4 \cdot m$ or $240m$ Write the fifth roots.

7–6 **Rational Exponents** (pp. 415–421)

Evaluate.

49. $27^{-\frac{2}{3}}$

50. $9^{\frac{1}{3}} \cdot 9^{\frac{5}{3}}$

51. $\left(\dfrac{8}{27}\right)^{-\frac{2}{3}}$

Simplify.

52. $\dfrac{1}{y^{\frac{2}{5}}}$

53. $\dfrac{xy}{\sqrt[3]{z}}$

54. $\dfrac{3x + 4x^2}{x^{-\frac{2}{3}}}$

55. ELECTRICITY The amount of current in amperes I that an appliance uses can be calculated using the formula $I = \left(\dfrac{P}{R}\right)^{\frac{1}{2}}$, where P is the power in watts and R is the resistance in ohms. How much current does an appliance use if $P = 120$ watts and $R = 3$ ohms? Round your answer to the nearest tenth.

Example 7 Write $32^{\frac{4}{5}} \cdot 32^{\frac{2}{5}}$ in radical form.

$32^{\frac{4}{5}} \cdot 32^{\frac{2}{5}} = 32^{\frac{4}{5} + \frac{2}{5}}$ Product of powers

$= 32^{\frac{6}{5}}$ Add.

$= \left(2^5\right)^{\frac{6}{5}}$ $32 = 2^5$

$= 2^6$ or 64 Power of a power

Example 8 Simplify $\dfrac{3x}{\sqrt[3]{z}}$.

$\dfrac{3x}{\sqrt[3]{z}} = \dfrac{3x}{z^{\frac{1}{3}}}$ Rational exponents

$= \dfrac{3x}{z^{\frac{1}{3}}} \cdot \dfrac{z^{\frac{2}{3}}}{z^{\frac{2}{3}}}$ Rationalize the denominator.

$= \dfrac{3xz^{\frac{2}{3}}}{z}$ or $\dfrac{3x\sqrt[3]{z^2}}{z}$ Rewrite in radical form.

7-7 **Solving Radical Equations and Inequalities** (pp. 422–427)

Solve each equation or inequality.

56. $\sqrt{x} = 6$

57. $y^{\frac{1}{3}} - 7 = 0$

58. $(x - 2)^{\frac{3}{2}} = -8$

59. $\sqrt{x + 5} - 3 = 0$

60. $\sqrt{3t - 5} - 3 = 4$

61. $\sqrt{2x - 1} = 3$

62. $\sqrt[4]{2x - 1} = 2$

63. $\sqrt{y + 5} = \sqrt{2y - 3}$

64. $\sqrt{y + 1} + \sqrt{y - 4} = 5$

65. $1 + \sqrt{5x - 2} > 4$

66. $\sqrt{-2x + 14} - 6 \geq -4$

67. $10 - \sqrt{2x + 7} \leq 3$

68. $6 + \sqrt{3y + 4} < 6$

69. $\sqrt{d + 3} + \sqrt{d + 7} > 4$

70. $\sqrt{2x + 5} - \sqrt{9 + x} > 0$

71. GRAVITY Hugo drops his keys from the top of a Ferris wheel. The formula $t = \frac{1}{4}\sqrt{65 - h}$ describes the time t in seconds when the keys are h feet above the boardwalk. If Hugo was 65 meters high when he dropped the keys, how many meters above the boardwalk will the keys be after 2 seconds?

Example 9 Solve $\sqrt{3x - 8} + 1 = 3$.

$\sqrt{3x - 8} + 1 = 3$ Original equation

$\sqrt{3x - 8} = 2$ Subtract 1 from each side.

$\left(\sqrt{3x - 8}\right)^2 = 2^2$ Square each side.

$3x - 8 = 4$ Evaluate the squares.

$3x = 12$ Add 8 to each side.

$x = 4$ Divide each side by 3.
   Check this solution.

Example 10 Solve $\sqrt{4x - 3} - 2 > 3$.

$\sqrt{4x - 3} - 2 > 3$ Original inequality

$\sqrt{4x - 3} > 5$ Add 2 to each side.

$\left(\sqrt{4x - 3}\right)^2 > 5^2$ Square each side.

$4x - 3 > 25$ Evaluate the squares.

$4x > 28$ Add 3 to each side.

$x > 7$ Divide each side by 4.

Determine whether each pair of functions are inverse functions.

1. $f(x) = 4x - 9$, $g(x) = \frac{x-9}{4}$

2. $f(x) = \frac{1}{x+2}$, $g(x) = \frac{1}{x} - 2$

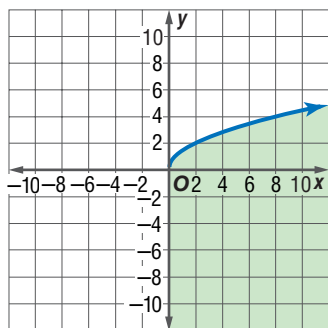**If $f(x) = 2x - 4$ and $g(x) = x^2 + 3$, find each value.**

3. $(f + g)(x)$

4. $(f - g)(x)$

5. $(f \cdot g)(x)$

6. $\left(\frac{f}{g}\right)(x)$

7. **MULTIPLE CHOICE** Which inequality represents the graph below?

   A $y \geq \sqrt{2x}$

   B $y \leq \sqrt{2x}$

   C $y < 2\sqrt{x}$

   D none of these

Solve each equation.

8. $\sqrt{b+15} = \sqrt{3b+1}$

9. $\sqrt{2x} = \sqrt{x-4}$

10. $\sqrt[4]{y+2} + 9 = 14$

11. $\sqrt[3]{2w-1} + 11 = 18$

12. $\sqrt{4x+28} = \sqrt{6x+38}$

13. $1 + \sqrt{x+5} = \sqrt{x+12}$

Simplify.

14. $\sqrt{175}$

15. $\left(5 + \sqrt{3}\right)\left(7 - 2\sqrt{3}\right)$

16. $\left(6 - 4\sqrt{2}\right)\left(-5 + \sqrt{2}\right)$

17. $3\sqrt{6} + 5\sqrt{54}$

18. $\dfrac{9}{5 - \sqrt{3}}$

19. $\dfrac{16}{-2 + \sqrt{5}}$

20. $\left(9^{\frac{1}{2}} \cdot 9^{\frac{2}{3}}\right)^{\frac{1}{6}}$

21. $11^{\frac{1}{2}} \cdot 11^{\frac{7}{3}} \cdot 11^{\frac{1}{6}}$

22. $\sqrt[6]{256s^{11}t^{18}}$

23. $\dfrac{b^{\frac{1}{2}}}{b^{\frac{3}{2}} - b^{\frac{1}{2}}}$
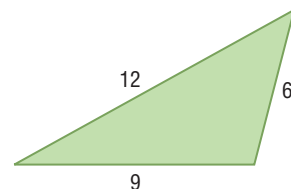
Solve each inequality.

24. $\sqrt{3x+1} \geq 5$

25. $3 + \sqrt{5x-1} < 11$

26. $1 - \sqrt{2y+1} < -6$

27. **SKYDIVING** The approximate time t in seconds that it takes an object to fall a distance of d feet is given by $t = \sqrt{\dfrac{d}{16}}$. Suppose a parachutist falls 11 seconds before the parachute opens. How far does the parachutist fall during this time period?

28. **GEOMETRY** The area of a triangle with sides of length a, b, and c is given by

$\sqrt{s(s-a)(s-b)(s-c)}$,

where $s = \frac{1}{2}(a + b + c)$. If the lengths of the sides of a triangle are 6, 9, and 12 feet, what is the area of the triangle expressed in radical form?

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Marilyn bought a pair of jeans and a sweater at her favorite clothing store. She spent $120 not including tax. If the price of the sweater s was $12 less than twice the cost of the jeans j which system of linear equations could be used to determine the price of each item?

   A $j + s = 120a$
      $s = 2j - 12$
   B $j + s = 120$
      $j = 2s - 12$
   C $j + 120 = s$
      $s = 2j - 12$
   D $j + s = 12$
      $s = 2j - 120$

2. **GRIDDABLE** If $f(x) = 3x$ and $g(x) = x^2 - 1$, what is the value of $f(g(-3))$?

3. What is the effect on the graph of the equation $y = 3x^2$ when the equation is changed to $y = 2x^2$?
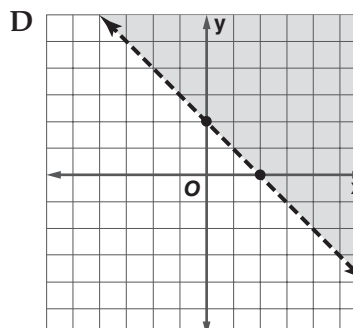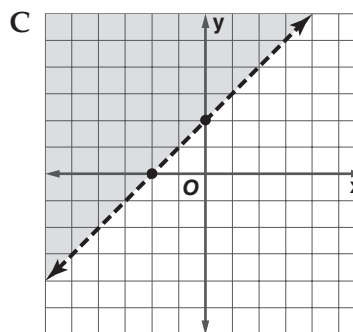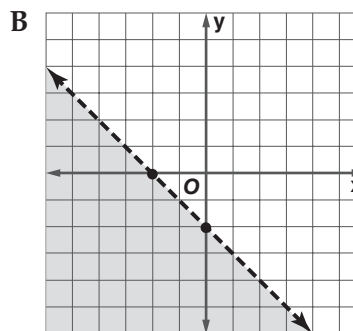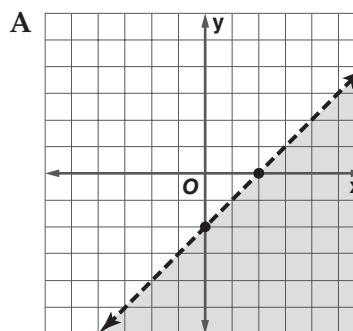
   F The graph of $y = 2x^2$ is a reflection of the graph of $y = 3x^2$ across the y-axis.
   G The graph is rotated 90 degrees about the origin.
   H The graph is narrower.
   **J** The graph is wider.

TEST-TAKING TIP

QUESTION 3 If the question involves a graph but does not include the graph, draw one. A diagram can help you see relationships among the given values that will help you answer the question.

4. Which graph best represents all the pairs of numbers (x, y) such that $x - y > 2$?

A

B

C

D

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5. The table below shows the cost of a pizza depending on the diameter of the pizza.

Diameter (inches)	Cost ($)
6	5.00
10	8.10
15	11.70
20	15.00

Which conclusion can be made based on the information in the table?

F The cost of a 12-inch would be less than $9.00.

G The cost of a 24-inch would be less than $18.00.

H The cost of an 18-inch would be more than $13.70.

J The cost of an 8-inch would be less than $6.00.

6. A rectangle is graphed on the coordinate grid.
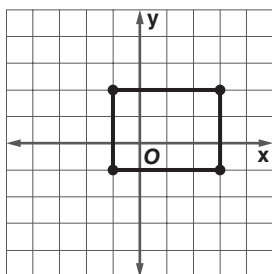
Which two points lie on the same line of symmetry of the square?

A $(-1, -1)$ and $(-1, 2)$

B $(0, 2)$ and $(0, -1)$

C $(1, 2)$ and $(1, -1)$

D $(3, 2)$ and $(-1, -1)$

Pre-AP

Record your answers on a sheet of paper. Show your work.

7. The period of a pendulum is the time it takes for the pendulum to make one complete swing back and forth. The formula $T = 2\pi\sqrt{\dfrac{L}{32}}$ gives the period T in seconds for a pendulum L feet long.

6 in.
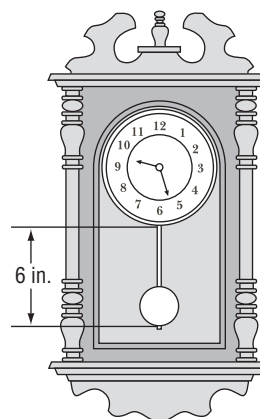
a. What is the period of the pendulum in the wall clock shown? Round to the nearest hundredth of a second.

b. Solve the formula for the length of the pendulum L in terms of the time T. Show each step of the process.

c. If you are building a grandfather clock and you want the pendulum to have a period of 2 seconds, how long should you make the pendulum? Round to the nearest tenth of a foot.

NEED EXTRA HELP?								
If You Missed Question...		1	2	3	4	5	6	7
Go to Lesson or Page...		2-4	7-1	5-7	2-7	1-3	5-1	5-6