UNIT 3

Advanced Functions and Relations

Focus

Use a variety of representations, tools, and technology to model mathematical situations to solve meaningful problems.

CHAPTER 8

Rational Expressions and Equations

BIG Idea Formulate equations and inequalities based on rational functions, use a variety of methods to solve them, and analyze the solutions in terms of the situation.

BIG Idea Connect algebraic and geometric representations of functions.

CHAPTER 9

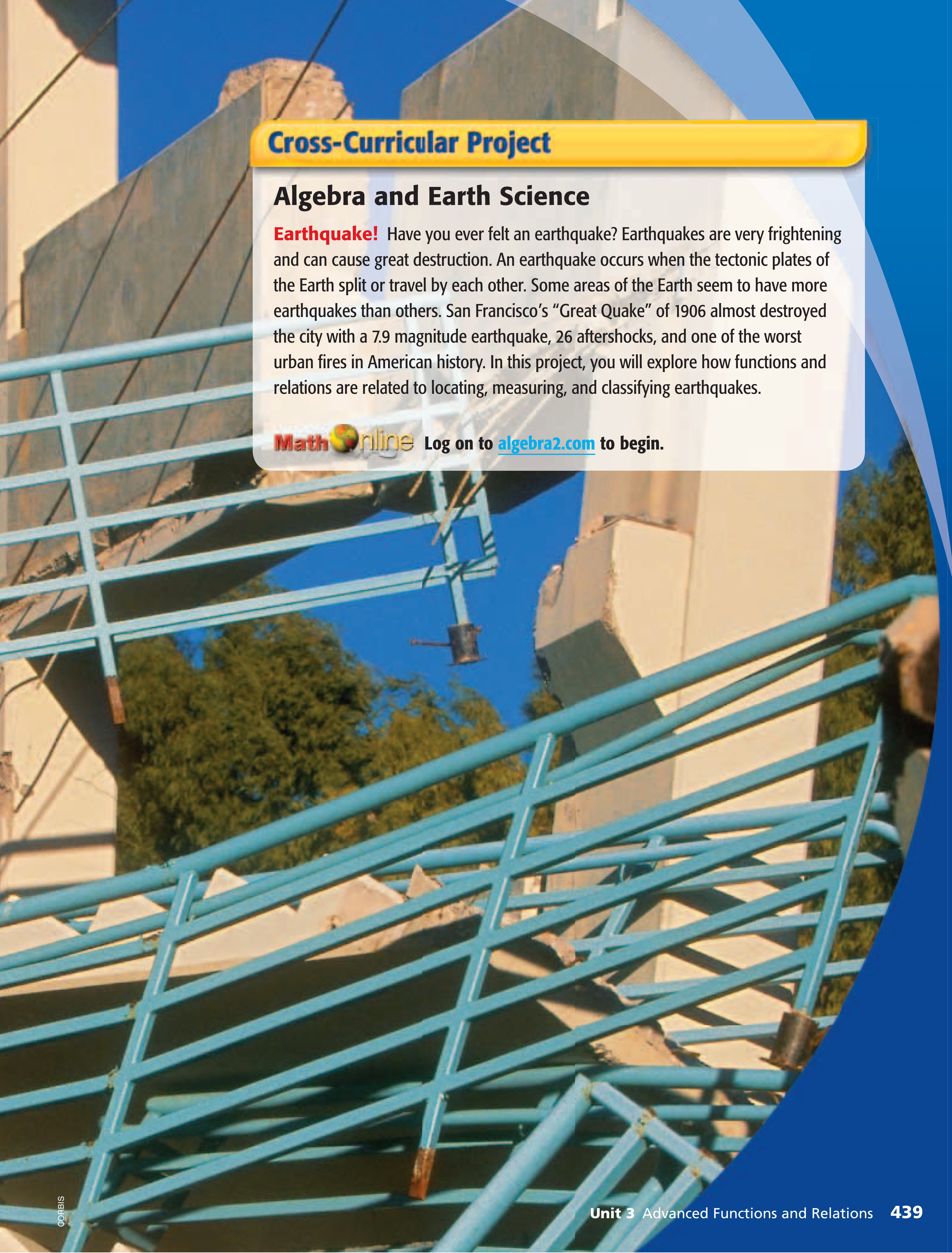
Exponential and Logarithmic Relations

BIG Idea Formulates equations and inequalities based on exponential and logarithmic functions, use a variety of methods to solve them, and analyze the solutions in terms of the situation.

CHAPTER 10

Conic Sections

BIG Idea Explore the relationship between the geometric and algebraic descriptions of conic sections.



CORBIS

Cross-Curricular Project

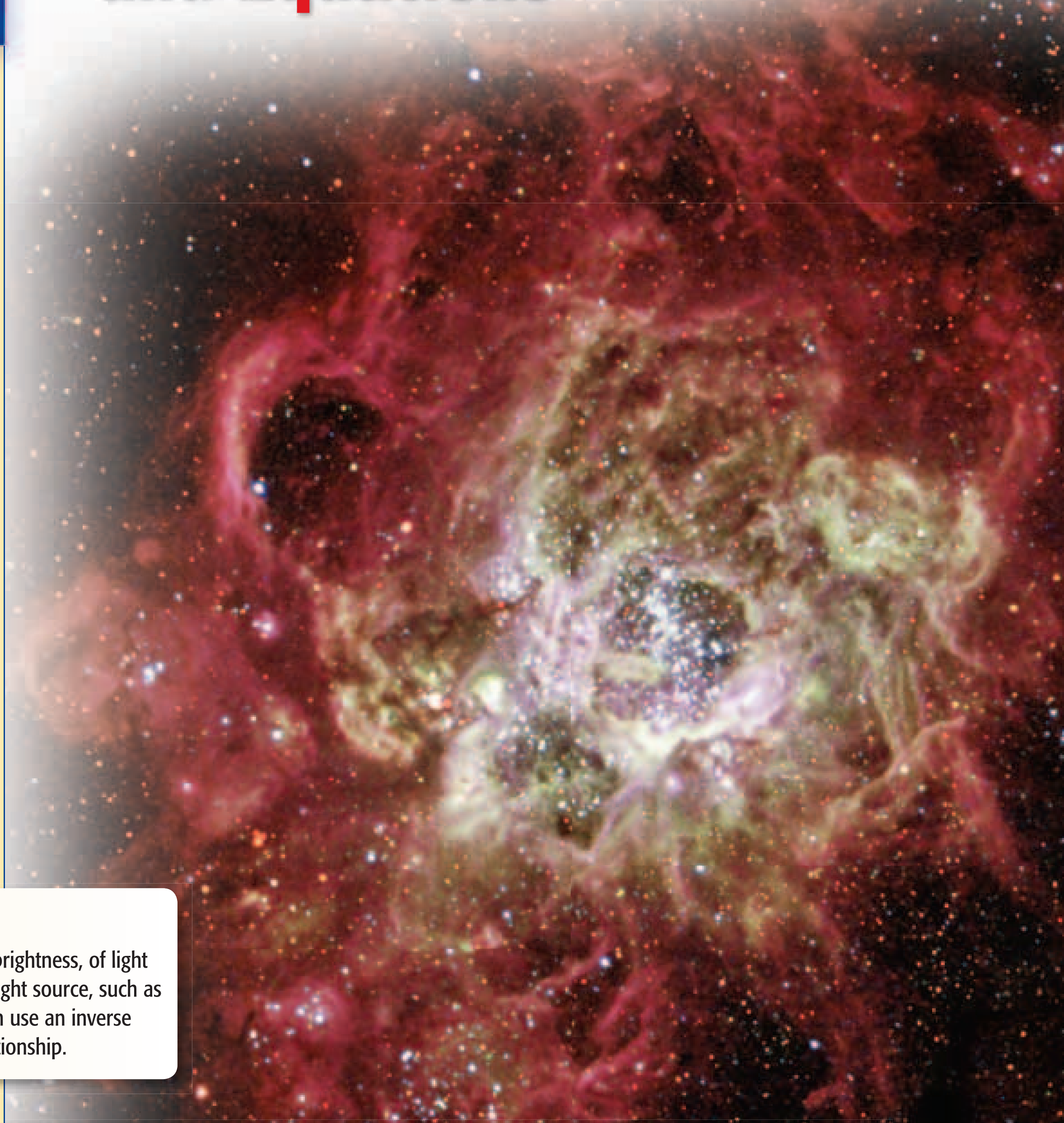
Algebra and Earth Science

Earthquake! Have you ever felt an earthquake? Earthquakes are very frightening and can cause great destruction. An earthquake occurs when the tectonic plates of the Earth split or travel by each other. Some areas of the Earth seem to have more earthquakes than others. San Francisco's "Great Quake" of 1906 almost destroyed the city with a 7.9 magnitude earthquake, 26 aftershocks, and one of the worst urban fires in American history. In this project, you will explore how functions and relations are related to locating, measuring, and classifying earthquakes.

Math Online **Log on to algebra2.com to begin.**

CHAPTER 8

Rational Expressions and Equations



BIG Ideas

- Simplify rational expressions.
- Graph rational functions.
- Solve direct, joint, and inverse variation problems.
- Identify graphs and equations as different types of functions.
- Solve rational equations and inequalities.

Key Vocabulary

continuity (p. 457)

direct variation (p. 465)

inverse variation (p. 467)

rational expression (p. 442)

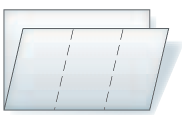
Real-World Link

Intensity of Light The intensity, or brightness, of light decreases as the distance between a light source, such as a star, and a viewer increases. You can use an inverse variation equation to express this relationship.

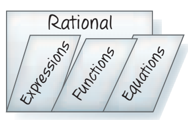
FOLDABLES Study Organizer

Rational Expressions and Equations Make this Foldable to help you organize your notes. Begin with a sheet of plain $8\frac{1}{2}''$ by 11″ paper.

1 **Fold** in half lengthwise leaving a $1\frac{1}{2}''$ margin at the top. Fold again in thirds.



2 **Open.** Cut along the folds on the short tab to make three tabs. Label as shown.



NASA

GET READY for Chapter 8

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

Math Online Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Solve each equation. Write in simplest form. (Lesson 1-3)

1. $\frac{8}{5}x = \frac{4}{15}$
2. $\frac{27}{14}t = \frac{6}{7}$
3. $\frac{3}{10} = \frac{12}{25}a$
4. $\frac{6}{7} = 9m$
5. $\frac{9}{8}b = 18$
6. $\frac{6}{7}s = \frac{3}{4}$
7. $\frac{1}{3}r = \frac{5}{6}$
8. $\frac{2}{3}n = 7$
9. **INCOME** Jamal's allowance is $\frac{7}{8}$ of Syretta's allowance. If Syretta's allowance is $20, how much is Jamal's allowance? (Lesson 1-3)
10. **BAKING** Marc gave away $\frac{2}{3}$ of the cookies he baked. If he gave away 24 cookies, how many cookies did he bake? (Lesson 1-3)

Solve each proportion. (Prerequisite Skill)

11. $\frac{3}{4} = \frac{r}{16}$
12. $\frac{8}{16} = \frac{5}{y}$
13. $\frac{6}{8} = \frac{m}{20}$
14. $\frac{t}{3} = \frac{5}{24}$
15. $\frac{5}{a} = \frac{6}{18}$
16. $\frac{3}{4} = \frac{b}{6}$
17. $\frac{v}{9} = \frac{12}{18}$
18. $\frac{7}{p} = \frac{1}{4}$
19. $\frac{2}{5} = \frac{3}{z}$
20. $\frac{9}{10} = \frac{r}{12}$
21. **REAL ESTATE** A house which is assessed for $200,000 pays $3000 in taxes. What should the taxes be on a house in the same area that is assessed at $350,000? (Prerequisite Skill)

QUICKReview

EXAMPLE 1

Solve $\frac{13}{17} = \frac{41}{43}k$. Write in simplest form.

$(43)\frac{13}{17} = 41k$ Multiply each side by 43.

$\frac{559}{17} = 41k$ Simplify.

$\frac{559}{(41)17} = k$ Divide each side by 41.

$\frac{559}{697} = k$ Simplify.

Since the GCF of 559 and 697 is 1, the solution is in simplest form.

EXAMPLE 2

Solve the proportion $\frac{4}{7} = \frac{u}{15}$.

$\frac{4}{7} = \frac{u}{15}$ Write the equation.

$4(15) = 7u$ Find the cross products.

$60 = 7u$ Simplify.

$\frac{60}{7} = u$ Divide each side by 7.

Since the GCF of 60 and 7 is 1, the answer is in simplified form. $u = \frac{60}{7}$ or $8\frac{4}{7}$.

8-1 Multiplying and Dividing Rational Expressions

Main Ideas

- Simplify rational expressions.
- Simplify complex fractions.

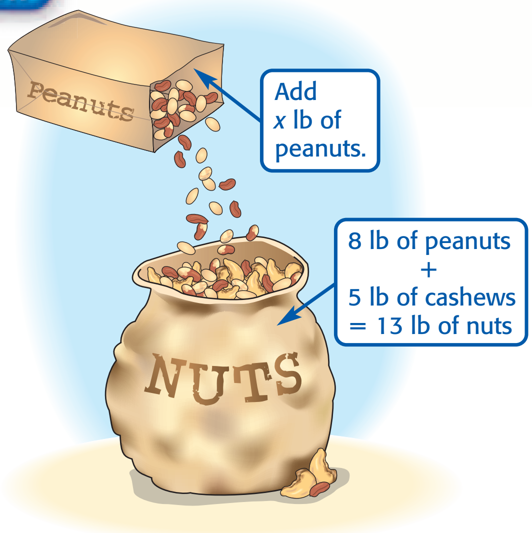
New Vocabulary

rational expression

complex fraction

GET READY for the Lesson

The Goodie Shoppe sells candy and nuts by the pound. One item is a mixture made with 8 pounds of peanuts and 5 pounds of cashews. Therefore, $\frac{8}{8+5}$ or $\frac{8}{13}$ of the mixture is peanuts. If the store manager adds an additional x pounds of peanuts to the mixture, then $\frac{8+x}{13+x}$ of the mixture will be peanuts.



Simplify Rational Expressions A ratio of two polynomial expressions such as $\frac{8+x}{13+x}$ is called a **rational expression**. Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar.

To write a fraction in simplest form, you divide both the numerator and denominator by their greatest common factor (GCF). To simplify a rational expression, you use similar techniques.

EXAMPLE 1 Simplify a Rational Expression

a. Simplify $\frac{2x(x-5)}{(x-5)(x^2-1)}$.

Look for common factors.

$$\frac{2x(x-5)}{(x-5)(x^2-1)} = \frac{2x}{x^2-1} \cdot \frac{\cancel{x-5}^{\,1}}{\cancel{x-5}_{\,1}}$$ How is this similar to simplifying $\frac{10}{15}$?

$$= \frac{2x}{x^2-1}$$ Simplify.

b. Under what conditions is this expression undefined?

Just as with a fraction, a rational expression is undefined if the denominator is equal to 0. To find when this expression is undefined, completely factor the original denominator.

$$\frac{2x(x-5)}{(x-5)(x^2-1)} = \frac{2x(x-5)}{(x-5)(x-1)(x+1)}$$ $x^2-1=(x-1)(x+1)$

The values that would make the denominator equal to 0 are 5, 1, or −1. So the expression is undefined when $x = 5$, $x = 1$, or $x = -1$.

Study Tip

Excluded Values

Numbers that would cause the expression to be undefined are called **excluded values**.

CHECK Your Progress

Simplify each expression. Under what conditions is the expression undefined?

1A. $\dfrac{3y(y+6)}{(y+6)(y^2-8y+12)}$ **1B.** $\dfrac{4x^3(x^2-7x-8)}{12x(x^2-64)}$

Personal Tutor

STANDARDIZED TEST PRACTICE

Use the Process of Elimination

2 **For what value(s) of x is $\dfrac{x^2+x-12}{x^2+7x+12}$ undefined?**

A −4, −3 **B** −4 **C** 0 **D** −4, 3

Read the Test Item

You want to determine which values of x make the denominator equal to 0.

Solve the Test Item

Notice that if x equals 0 or a positive number, $x^2 + 7x + 12$ must be greater than 0. Therefore, you can eliminate choices C and D. Since both A and B contain −4, determine whether the denominator equals 0 when $x = -3$.

$x^2 + 7x + 12 = (-3)^2 + 7(-3) + 12$ $x = -3$

$= 9 - 21 + 12$ or 0 Multiply and simplify.

Since the denominator equals 0 when $x = -3$, the answer is A.

Test-Taking Tip

Eliminating Choices Sometimes you can save time by looking at the possible answers and eliminating choices, rather than actually evaluating an expression or solving an equation.

CHECK Your Progress

2. For what values of x is $\dfrac{x^2+9}{x^2+15x-34}$ undefined?

F −17, −2 **G** −17, 2 **H** −2, 17 **J** 2, 17

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Sometimes you can factor out −1 in the numerator or denominator to help simplify rational expressions.

EXAMPLE Simplify by Factoring Out −1

3 **Simplify $\dfrac{z^2w - z^2}{z^3 - z^3w}$.**

$\dfrac{z^2w - z^2}{z^3 - z^3w} = \dfrac{z^2(w-1)}{z^3(1-w)}$ Factor the numerator and the denominator.

$= \dfrac{\overset{1}{\cancel{z^2}}(-1)\overset{1}{\cancel{(1-w)}}}{\underset{z}{\cancel{z^3}}\underset{1}{\cancel{(1-w)}}}$ $w - 1 = -(-w + 1)$ or $-1(1 - w)$

$= \dfrac{-1}{z}$ or $-\dfrac{1}{z}$ Simplify.

CHECK Your Progress

Simplify each expression.

3A. $\dfrac{xy - 3x}{3x^2 - x^2y}$ **3B.** $\dfrac{2x - x^2}{x^2y - 4y}$

Remember that to multiply two fractions, you multiply the numerators and multiply the denominators. To divide two fractions, you multiply by the multiplicative inverse, or reciprocal, of the divisor.

Multiplication

$$\frac{5}{6}\cdot\frac{4}{15}=\frac{\overset{1}{\cancel{5}}\cdot\overset{1}{\cancel{2}}\cdot 2}{\underset{1}{\cancel{2}}\cdot 3\cdot 3\cdot\underset{1}{\cancel{5}}}=\frac{2}{3\cdot 3}\text{ or }\frac{2}{9}$$

Division

$$\frac{3}{7}\div\frac{9}{14}=\frac{3}{7}\cdot\frac{14}{9}=\frac{\overset{1}{\cancel{3}}\cdot 2\cdot\overset{1}{\cancel{7}}}{\underset{1}{\cancel{7}}\cdot\underset{1}{\cancel{3}}\cdot 3}\text{ or }\frac{2}{3}$$

The same procedures are used for multiplying and dividing rational expressions.

KEY CONCEPT — *Rational Expressions*

Multiplying Rational Expressions

Words To multiply two rational expressions, multiply the numerators and the denominators.

Symbols For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$, if $b\neq 0$ and $d\neq 0$.

Dividing Rational Expressions

Words To divide two rational expressions, multiply by the reciprocal of the divisor.

Symbols For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$, if $b\neq 0$, $c\neq 0$ and $d\neq 0$.

The following examples show how these rules are used with rational expressions.

EXAMPLE Multiply and Divide Rational Expressions

4 **Simplify each expression.**

a. $\frac{4a}{5b}\cdot\frac{15b^2}{16a^3}$

$$\frac{4a}{5b}\cdot\frac{15b^2}{16a^3}=\frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{a}}\cdot 3\cdot\overset{1}{\cancel{5}}\cdot\overset{1}{\cancel{b}}\cdot b}{\underset{1}{\cancel{5}}\cdot\underset{1}{\cancel{b}}\cdot\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{2}}\cdot 2\cdot 2\cdot\underset{1}{\cancel{a}}\cdot a\cdot a}\quad\text{Factor.}$$

$$=\frac{3\cdot b}{2\cdot 2\cdot a\cdot a}\quad\text{Simplify.}$$

$$=\frac{3b}{4a^2}\quad\text{Simplify.}$$

b. $\frac{4x^2y}{15a^3b^3}\div\frac{2xy^2}{5ab^3}$

$$\frac{4x^2y}{15a^3b^3}\div\frac{2xy^2}{5ab^3}=\frac{4x^2y}{15a^3b^3}\cdot\frac{5ab^3}{2xy^2}\quad\text{Multiply by the reciprocal of the divisor.}$$

$$=\frac{\overset{1}{\cancel{2}}\cdot 2\cdot\overset{1}{\cancel{x}}\cdot x\cdot\overset{1}{\cancel{y}}\cdot\overset{1}{\cancel{5}}\cdot\overset{1}{\cancel{a}}\cdot\overset{1}{\cancel{b}}\cdot\overset{1}{\cancel{b}}\cdot\overset{1}{\cancel{b}}}{3\cdot\underset{1}{\cancel{5}}\cdot\underset{1}{\cancel{a}}\cdot a\cdot a\cdot\underset{1}{\cancel{b}}\cdot\underset{1}{\cancel{b}}\cdot\underset{1}{\cancel{b}}\cdot\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{x}}\cdot\underset{1}{\cancel{y}}\cdot y}\quad\text{Factor.}$$

$$=\frac{2\cdot x}{3\cdot a\cdot a\cdot y}\quad\text{Simplify.}$$

$$=\frac{2x}{3a^2y}\quad\text{Simplify.}$$

Study Tip

Alternative Method

When multiplying rational expressions, you can multiply first and then divide by the common factors. For instance, in Example 4, $\frac{4a}{5b}\cdot\frac{15b^2}{16a^3}=\frac{60ab^2}{80a^3b}$. Now divide the numerator and denominator by the common factors.

$$\frac{\overset{3}{\cancel{60}}\overset{1}{\cancel{a}}\overset{b}{\cancel{b^2}}}{\underset{4}{\cancel{80}}\underset{a^2}{\cancel{a^3}}\underset{1}{\cancel{b}}}=\frac{3b}{4a^2}$$

CHECK Your Progress

4A. $\frac{8t^2s}{5r^2} \cdot \frac{15sr}{12t^3s^2}$

4B. $\frac{9m^2n^3}{16ab^4} \cdot \frac{8a^2b}{27m^5n}$

4C. $\frac{18ab^2}{25x^2y^3} \div \frac{9b}{10xy}$

4D. $\frac{14pq^2}{15w^7z^3} \div \frac{21p^3q}{35w^3z^8}$

Sometimes you must factor the numerator and/or the denominator first before you can simplify a product or a quotient of rational expressions.

EXAMPLE 5 Polynomials in the Numerator and Denominator

Simplify each expression.

a. $\frac{x^2 + 2x - 8}{x^2 + 4x + 3} \cdot \frac{3x + 3}{x - 2}$

$$\frac{x^2 + 2x - 8}{x^2 + 4x + 3} \cdot \frac{3x + 3}{x - 2} = \frac{(x + 4)\overset{1}{\cancel{(x - 2)}}}{(x + 3)\underset{1}{\cancel{(x + 1)}}} \cdot \frac{3\overset{1}{\cancel{(x + 1)}}}{\underset{1}{\cancel{(x - 2)}}}$$ Factor.

$$= \frac{3(x + 4)}{(x + 3)}$$ Simplify.

$$= \frac{3x + 12}{x + 3}$$ Distributive Property

b. $\frac{a + 2}{a + 3} \div \frac{a^2 + a - 12}{a^2 - 9}$

$$\frac{a + 2}{a + 3} \div \frac{a^2 + a - 12}{a^2 - 9} = \frac{a + 2}{a + 3} \cdot \frac{a^2 - 9}{a^2 + a - 12}$$ Multiply by the reciprocal of the divisor.

$$= \frac{(a + 2)\overset{1}{\cancel{(a + 3)}}\overset{1}{\cancel{(a - 3)}}}{\underset{1}{\cancel{(a + 3)}}(a + 4)\underset{1}{\cancel{(a - 3)}}}$$ Factor.

$$= \frac{a + 2}{a + 4}$$ Simplify.

CHECK Your Progress

5A. $\frac{y - 1}{5y + 15} \cdot \frac{y^2 + 5y + 6}{y^2 + 4y - 5}$

5B. $\frac{b^2 + 2b - 35}{b^2 - 4} \div \frac{b - 5}{b + 2}$

In Motion

Concepts in Motion
Animation
algebra2.com

Simplify Complex Fractions A **complex fraction** is a rational expression whose numerator and/or denominator contains a rational expression. The expressions below are complex fractions.

$$\frac{\frac{a}{5}}{3b} \qquad \frac{\frac{3}{t}}{t + 5} \qquad \frac{\frac{m^2 - 9}{8}}{\frac{3 - m}{12}} \qquad \frac{\frac{1}{p} + 2}{\frac{3}{p} - 4}$$

To simplify a complex fraction, rewrite it as a division expression, and use the rules for division.

EXAMPLE 6 Simplify a Complex Fraction

6 Simplify $\dfrac{\frac{r^2}{r^2 - 25s^2}}{\frac{r}{5s - r}}$.

$$\dfrac{\frac{r^2}{r^2 - 25s^2}}{\frac{r}{5s - r}} = \frac{r^2}{r^2 - 25s^2} \div \frac{r}{5s - r}$$ Express as a division expression.

$$= \frac{r^2}{r^2 - 25s^2} \cdot \frac{5s - r}{r}$$ Multiply by the reciprocal of the divisor.

$$= \frac{\overset{1}{\cancel{r}} \cdot r(-1)\overset{1}{\cancel{(r - 5s)}}}{(r + 5s)\underset{1}{\cancel{(r - 5s)}}\underset{1}{\cancel{r}}}$$ Factor.

$$= \frac{-r}{r + 5s} \text{ or } -\frac{r}{r + 5s}$$ Simplify.

CHECK Your Progress

Simplify each expression.

6A. $\dfrac{\frac{(x + 3)^2}{x^2 - 16}}{\frac{x + 3}{x + 4}}$

6B. $\dfrac{\frac{y - 7}{y - 3}}{\frac{y^2 - 49}{y^2 + 4y - 21}}$

CHECK Your Understanding

Example 1 (pp. 442–443)

Simplify each expression.

1. $\dfrac{45mn^3}{20n^7}$

2. $\dfrac{a + b}{a^2 - b^2}$

3. $\dfrac{x^2 + 6x + 9}{x + 3}$

4. $\dfrac{36c^3d^2}{54cd^5}$

Example 2 (p. 443)

5. STANDARDIZED TEST PRACTICE Identify all values of y for which $\dfrac{y - 4}{y^2 - 4y - 12}$ is undefined.

A $-2, 4, 6$ **B** $-6, -4, 2$ **C** $-2, 0, 6$ **D** $-2, 6$

Simplify each expression.

Example 3 (p. 443)

6. $\dfrac{9y^2 - 6y^3}{2y^2 + 5y - 12}$

7. $\dfrac{b^3 - a^3}{a^2 - b^2}$

Example 4 (pp. 444–445)

8. $\dfrac{2a^2}{5b^2c} \cdot \dfrac{3bc^3}{8a^2}$

9. $\dfrac{3t + 6}{7t - 7} \cdot \dfrac{14t - 14}{5t + 10}$

10. $\dfrac{35}{16x^2} \div \dfrac{21}{4x}$

11. $\dfrac{20xy^3}{21} \div \dfrac{15x^3y^2}{14}$

Example 5 (p. 445)

12. $\dfrac{12p^2 + 6p - 6}{4(p + 1)^2} \div \dfrac{6p - 3}{2p + 10}$

13. $\dfrac{x^2 + 6x + 9}{x^2 + 7x + 6} \div \dfrac{4x + 12}{3x + 3}$

Example 6 (p. 446)

14. $\dfrac{\frac{c^3d^3}{a}}{\frac{xc^2d}{ax^2}}$

15. $\dfrac{\frac{2y}{y^2 - 4}}{\frac{3}{y^2 - 4y + 4}}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
16–19	1
20, 21	3
22–25	4
26–29	5
30–33	6
34, 35	2

Simplify each expression.

16. $\frac{30bc}{12b^3}$ **17.** $\frac{-3mn^3}{21m^2n^2}$ **18.** $\frac{5t-5}{t^2-1}$

19. $\frac{c+5}{2c+10}$ **20.** $\frac{3t-6}{2-t}$ **21.** $\frac{9-t^2}{t^2+t-12}$

22. $\frac{3xyz}{4xz} \cdot \frac{6x^2}{3y^2}$ **23.** $\frac{-4ab}{21c} \cdot \frac{14c^2}{18a^2}$ **24.** $\frac{3}{5d} \div \left(-\frac{9}{15df}\right)$

25. $\frac{p^3}{2q} \div \frac{-p}{4q}$ **26.** $\frac{3t^2}{t+2} \cdot \frac{t+2}{t^2}$ **27.** $\frac{4w+4}{3} \cdot \frac{1}{w+1}$

28. $\frac{4t^2-4}{9(t+1)^2} \cdot \frac{3t+3}{2t-2}$ **29.** $\frac{3p-21}{p^2-49} \cdot \frac{p^2-7p}{3p}$ **30.** $\frac{\frac{m^3}{3n}}{-\frac{m^4}{9n^2}}$

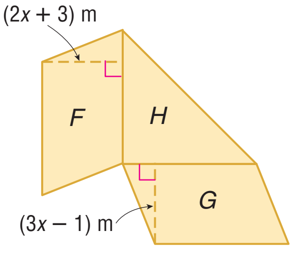
31. $\frac{\frac{p^3}{2q}}{-\frac{p^2}{4q}}$ **32.** $\frac{\frac{m+n}{5}}{\frac{m^2+n^2}{5}}$ **33.** $\frac{\frac{x+y}{2x-y}}{\frac{x+y}{2x+y}}$

34. Under what conditions is $\frac{x-4}{(x+5)(x-1)}$ undefined?

35. For what values is $\frac{2d(d+1)}{(d+1)(d^2-4)}$ undefined?

36. GEOMETRY A parallelogram with an area of $6x^2 - 7x - 5$ square units has a base of $3x - 5$ units. Determine the height of the parallelogram.

37. GEOMETRY Parallelogram F has an area of $8x^2 + 10x - 3$ square meters and a height of $2x + 3$ meters. Parallelogram G has an area of $6x^2 + 13x - 5$ square meters and a height of $3x - 1$ meters. Find the area of right triangle H.



Simplify each expression.

38. $\frac{(-3x^2y)^3}{9x^2y^2}$ **39.** $\frac{(-2rs^2)^2}{12r^2s^3}$ **40.** $\frac{(-5mn^2)^3}{5m^2n^4}$

41. $\frac{y^2+4y+4}{3y^2+5y-2}$ **42.** $\frac{a^2+2a+1}{2a^2+3a+1}$ **43.** $\frac{3x^2-2x-8}{3x^2-12}$

44. $\frac{a^2-4}{6-3a}$ **45.** $\frac{b^2-4b+3}{3-2b-b^2}$ **46.** $\frac{6x^2-6}{14x^2-28x+14}$

47. $\frac{25a^2b^3}{6x^2y} \cdot \frac{8xy^2}{20a^3b^2}$ **48.** $\frac{-9cd}{8xw} \cdot \frac{(-4w)^2}{15c}$

49. $\frac{2x^3y}{z^5} \div \left(\frac{4xy}{z^3}\right)^2$ **50.** $\frac{w^2-11w+24}{w^2-18w+80} \cdot \frac{w^2-15w+50}{w^2-9w+20}$

51. $\frac{r^2+2r-8}{r^2+4r+3} \div \frac{r-2}{3r+3}$ **52.** $\frac{\frac{5x^2-5x-30}{45-15x}}{\frac{6+x-x^2}{4x-12}}$

EXTRA PRACTICE
See pages 907, 933.

53. Under what conditions is $\frac{a^2 + ab + b^2}{a^2 - b^2}$ undefined?

Real-World Link

Ray Allen is a five-time All Star and member of team USA for the 2000 Olympics.

Source: NBA

BASKETBALL For Exercises 54 and 55, use the following information.
At the end of the 2005-2006 season, the Seattle Sonics' Ray Allen had made 5422 field goals out of 12,138 attempts during his NBA career.

54. Write a ratio to represent the ratio of the number of career field goals made to career field goals attempted by Ray Allen at the end of the 2005-2006 season.

55. Suppose Ray Allen attempted a field goals and made m field goals during the 2006-2007 season. Write a rational expression to represent the ratio of the number of career field goals made to the number of career field goals attempted at the end of the 2006-2007 season.

AIRPLANES For Exercises 56–58, use the formula $d = rt$ and the following information.
An airplane is traveling at the rate r of 500 miles per hour for a time t of $(6 + x)$ hours. A second airplane travels at the rate r of $(540 + 90x)$ miles per hour for a time t of 6 hours.

56. Write a rational expression to represent the ratio of the distance d traveled by the first airplane to the distance d traveled by the second airplane.

57. Simplify the rational expression. What does this expression tell you about the distances traveled of the two airplanes?

58. Under what condition is the rational expression undefined? Describe what this condition would tell you about the two airplanes.

Graphing Calculator

For Exercises 59–62, consider $f(x) = \frac{-15x^2 + 10x}{5x}$ and $g(x) = -3x + 2$.

59. Simplify $\frac{-15x^2 + 10x}{5x}$. What do you observe about the expression?

60. Graph $f(x)$ and $g(x)$ on a graphing calculator. How do the graphs appear?

61. Use the table feature to examine the function values for $f(x)$ and $g(x)$. How do the tables compare?

62. How can you use what you have observed with $f(x)$ and $g(x)$ to verify that expressions are equivalent or to identify excluded values?

H.O.T. Problems

63. **OPEN ENDED** Write two rational expressions that are equivalent.

64. **CHALLENGE** Rewrite $\frac{a + \sqrt{b}}{-a^2 + b}$ so it has a numerator of 1.

65. **Which One Doesn't Belong?** Identify the expression that does not belong with the other three. Explain your reasoning.

$\frac{1}{x - 1}$	$\frac{x^2 + 3x + 2}{x - 5}$	$\frac{x + 1}{\sqrt{x + 3}}$	$\frac{x^2 + 1}{3}$

66. **REASONING** Determine whether $\frac{2d + 5}{3d + 5} = \frac{2}{3}$ is *sometimes*, *always*, or *never* true. Explain.

67. ***Writing in Math*** Use the information about rational expressions on page 442 to explain how rational expressions are used in mixtures. Include an example of a mixture problem that could be represented by $\frac{8 + x}{13 + x + y}$.

STANDARDIZED TEST PRACTICE

68. ACT/SAT For what value(s) of x is $\frac{4x}{x^2 - x}$ undefined?

A $-1, 1$

B $-1, 0, 1$

C $0, 1$

D 0

69. REVIEW Which is the simplified form of $\frac{4x^3y^2z^{-1}}{(x^{-2}y^3z^2)^2}$?

F $\frac{4x^7}{y^4z^5}$

G $\frac{4xy}{z^5}$

H $\frac{4}{y^3z^5}$

J $\frac{4}{xy^4z^5}$

Spiral Review

Graph each function. State the domain and range. (Lesson 7-3)

70. $y = \sqrt{x - 2}$

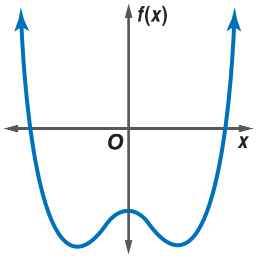
71. $y = \sqrt{x} - 1$

72. $y = 2\sqrt{x} + 1$

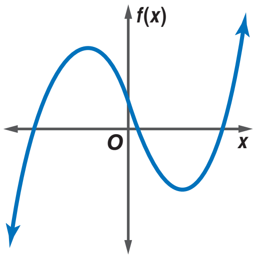
73. Determine whether $f(x) = x - 2$ and $g(x) = 2x$ are inverse functions. (Lesson 7-2)

Determine whether each graph represents an odd-degree or an even-degree polynomial function. Then state how many real zeros each function has. (Lesson 6-3)

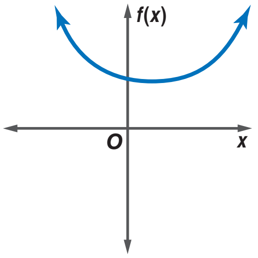
74.



75.



76.



77. ASTRONOMY Earth is an average of 1.496×10^8 kilometers from the Sun. If light travels 3×10^5 kilometers per second, how long does it take sunlight to reach Earth? (Lesson 6-1)

Solve each equation by factoring. (Lesson 5-3)

78. $r^2 - 3r = 4$

79. $18u^2 - 3u = 1$

80. $d^2 - 5d = 0$

Solve each equation. (Lesson 1-4)

81. $|2x + 7| + 5 = 0$

82. $5|3x - 4| = x + 1$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 1-3)

83. $\frac{2}{3} + x = -\frac{4}{9}$

84. $x + \frac{5}{8} = -\frac{5}{6}$

85. $x - \frac{3}{5} = \frac{2}{3}$

86. $x + \frac{3}{16} = -\frac{1}{2}$

87. $x - \frac{1}{6} = -\frac{7}{9}$

88. $x - \frac{3}{8} = -\frac{5}{24}$

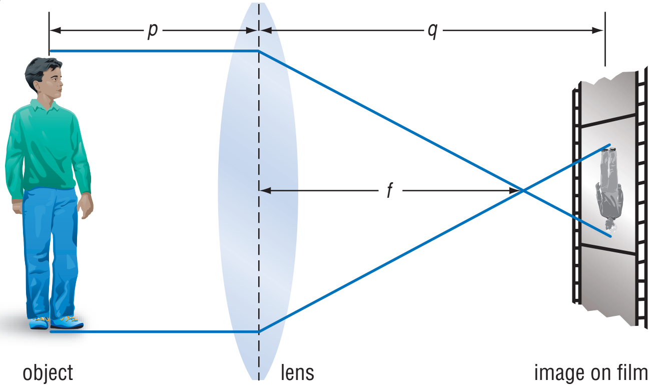
8-2 Adding and Subtracting Rational Expressions

Main Ideas

- Determine the LCM of polynomials.
- Add and subtract rational expressions.

GET READY for the Lesson

In order to produce a picture that is "in focus," the distance between the camera lens and the film q must be controlled so that it satisfies a particular relationship. If the distance from the subject to the lens is p and the focal length of the lens is f, then the formula $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ can be used to determine the correct distance between the lens and the film.



LCM of Polynomials To find $\frac{5}{6} - \frac{1}{4}$ or $\frac{1}{f} - \frac{1}{p}$, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the denominators.

To find the LCM of two or more numbers or polynomials, factor each number or polynomial. The LCM contains each factor the greatest number of times it appears as a factor.

LCM of 6 and 4

$6 = 2 \cdot 3$
$4 = 2^2$
$\text{LCM} = 2^2 \cdot 3$ or 12

LCM of $a^2 - 6a + 9$ and $a^2 + a - 12$

$a^2 - 6a + 9 = (a - 3)^2$
$a^2 + a - 12 = (a - 3)(a + 4)$
$\text{LCM} = (a - 3)^2(a + 4)$

EXAMPLE 1 LCM of Monomials

Find the LCM of $18r^2s^5$, $24r^3st^2$, and $15s^3t$.

$18r^2s^5 = 2 \cdot 3^2 \cdot r^2 \cdot s^5$ — Factor the first monomial.

$24r^3st^2 = 2^3 \cdot 3 \cdot r^3 \cdot s \cdot t^2$ — Factor the second monomial.

$15s^3t = 3 \cdot 5 \cdot s^3 \cdot t$ — Factor the third monomial.

$\text{LCM} = 2^3 \cdot 3^2 \cdot 5 \cdot r^3 \cdot s^5 \cdot t^2$
$= 360r^3s^5t^2$ — Use each factor the greatest number of times it appears as a factor and simplify.

CHECK Your Progress

Find the LCM of each set of monomials.

1A. $12a^2b^4$, $27ac^3$, $18a^5b^2c$

1B. $6m^3n^5$, $42mnp^2$, $36m^3n^4p$

EXAMPLE LCM of Polynomials

2 **Find the LCM of $p^3 + 5p^2 + 6p$ and $p^2 + 6p + 9$.**

$p^3 + 5p^2 + 6p = p(p + 2)(p + 3)$ Factor the first polynomial.

$p^2 + 6p + 9 = (p + 3)^2$ Factor the second polynomial.

$\text{LCM} = p(p + 2)(p + 3)^2$ Use each factor the greatest number of times it appears as a factor.

CHECK Your Progress

Find the LCM of each set of polynomials.

2A. $q^2 - 4q + 4$ and $q^3 - 3q^2 + 2q$

2B. $2k^3 - 5k^2 - 12k$ and $k^3 - 8k^2 + 16k$

Add and Subtract Rational Expressions As with fractions, to add or subtract rational expressions, you must have common denominators.

Specific Case		General Case
$\frac{2}{3} + \frac{3}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{3 \cdot 3}{5 \cdot 3}$	Find equivalent fractions that have a common denominator.	$\frac{a}{c} + \frac{b}{d} = \frac{a \cdot d}{c \cdot d} + \frac{b \cdot c}{d \cdot c}$
$= \frac{10}{15} + \frac{9}{15}$	Simplify each numerator and denominator.	$= \frac{ad}{cd} + \frac{bc}{cd}$
$= \frac{19}{15}$	Add the numerators.	$= \frac{ad + bc}{cd}$

As with fractions, you can use the least common multiple of the denominators to find the least common denominator for two rational expressions.

EXAMPLE Monomial Denominators

3 **Simplify $\frac{7x}{15y^2} + \frac{y}{18xy}$.**

$\frac{7x}{15y^2} + \frac{y}{18xy} = \frac{7x \cdot 6x}{15y^2 \cdot 6x} + \frac{y \cdot 5y}{18xy \cdot 5y}$ The LCD is $90xy^2$. Find the equivalent fractions that have this denominator.

$= \frac{42x^2}{90xy^2} + \frac{5y^2}{90xy^2}$ Simplify each numerator and denominator.

$= \frac{42x^2 + 5y^2}{90xy^2}$ Add the numerators.

CHECK Your Progress

Simplify each expression.

3A. $\frac{8a}{9b} - \frac{1}{7ab^2}$

3B. $\frac{1}{8m^2n} + \frac{2}{mn^2}$

3C. $\frac{2}{3xy} - \frac{3x}{5y}$

3D. $\frac{6c}{7b^2} + \frac{2d}{3ab}$

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EXAMPLE Polynomial Denominators

4 Simplify $\frac{w+12}{4w-16} - \frac{w+4}{2w-8}$.

$\frac{w+12}{4w-16} - \frac{w+4}{2w-8} = \frac{w+12}{4(w-4)} - \frac{w+4}{2(w-4)}$ Factor the denominators.

$= \frac{w+12}{4(w-4)} - \frac{(w+4)(2)}{2(w-4)(2)}$ The LCD is $4(w-4)$.

$= \frac{(w+12)-(2)(w+4)}{4(w-4)}$ Subtract the numerators.

$= \frac{w+12-2w-8}{4(w-4)}$ Distributive Property

$= \frac{-w+4}{4(w-4)}$ Combine like terms.

$= \frac{-1(w-4)}{4(w-4)}$ or $-\frac{1}{4}$ Simplify.

Study Tip

Common Factors

Sometimes when you simplify the numerator, the polynomial contains a factor common to the denominator. Thus the rational expression can be further simplified.

CHECK Your Progress Simplify each expression.

4A. $\frac{x+6}{6x-18} + \frac{x-6}{2x-6}$

4B. $\frac{x-1}{3x^2+8x+5} - \frac{x-1}{12x+20}$

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One way to simplify a complex fraction is to simplify the numerator and the denominator separately, and then simplify the resulting expressions.

EXAMPLE Simplify Complex Fractions

5 Simplify $\frac{\frac{1}{x}-\frac{1}{y}}{1+\frac{1}{x}}$.

$\frac{\frac{1}{x}-\frac{1}{y}}{1+\frac{1}{x}} = \frac{\frac{y}{xy}-\frac{x}{xy}}{\frac{x}{x}+\frac{1}{x}}$ The LCD of the numerator is xy. The LCD of the denominator is x.

$= \frac{\frac{y-x}{xy}}{\frac{x+1}{x}}$ Simplify the numerator and denominator.

$= \frac{y-x}{xy} \div \frac{x+1}{x}$ Write as a division expression.

$= \frac{y-x}{xy} \cdot \frac{x}{x+1}$ Multiply by the reciprocal of the divisor.

$= \frac{y-x}{y(x+1)}$ or $\frac{y-x}{xy+y}$ Simplify.

CHECK Your Progress Simplify each expression.

5A. $\frac{\frac{1}{y}+\frac{1}{x}}{\frac{1}{y}-\frac{1}{x}}$

5B. $\frac{\frac{a}{b}+1}{1-\frac{b}{a}}$

EXAMPLE Use a Complex Fraction to Solve a Problem

6 **COORDINATE GEOMETRY** **Find the slope of the line that passes through $A\left(\frac{2}{p}, \frac{1}{2}\right)$ and $B\left(\frac{1}{3}, \frac{3}{p}\right)$.**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Definition of slope

$$= \frac{\frac{3}{p} - \frac{1}{2}}{\frac{1}{3} - \frac{2}{p}}$$ $y_2 = \frac{3}{p}, y_1 = \frac{1}{2}, x_2 = \frac{1}{3}$, and $x_1 = \frac{2}{p}$

$$= \frac{\frac{6-p}{2p}}{\frac{p-6}{3p}}$$ The LCD of the numerator is $2p$. The LCD of the denominator is $3p$.

$$= \frac{6-p}{2p} \div \frac{p-6}{3p}$$ Write as a division expression.

$$= \frac{\overset{-1}{\cancel{6-p}}}{\underset{1}{2\cancel{p}}} \cdot \frac{\overset{1}{3\cancel{p}}}{\underset{1}{\cancel{p-6}}} \text{ or } -\frac{3}{2}$$ The slope is $-\frac{3}{2}$.

Study Tip

Check Your Solution

You can check your answer by letting p equal any nonzero number, say 1. Use the definition of slope to find the slope of the line through the points.

CHECK Your Progress

Find the slope of the line that passes through each pair of points.

6A. $C\left(\frac{1}{4}, \frac{4}{q}\right)$ and $D\left(\frac{5}{q}, \frac{1}{5}\right)$

6B. $E\left(\frac{7}{w}, \frac{1}{7}\right)$ and $F\left(\frac{1}{7}, \frac{7}{w}\right)$

CHECK Your Understanding

Examples 1, 2 (pp. 450–451)

Find the LCM of each set of polynomials.

1. $12y^2, 6x^2$

2. $16ab^3, 5b^2a^2, 20ac$

3. $x^2 - 2x, x^2 - 4$

4. $x^3 - 4x^2 - 5x, x^2 + 6x + 5$

Simplify each expression.

Example 3 (p. 451)

5. $\frac{2}{x^2y} - \frac{x}{y}$

6. $\frac{7a}{15b^2} - \frac{b}{18ab}$

7. $\frac{5}{3m} - \frac{2}{7m} - \frac{1}{2m}$

8. $\frac{3x}{5} - \frac{1}{2x^2} + \frac{3}{4x}$

Example 4 (p. 452)

9. $\frac{6}{d^2 + 4d + 4} + \frac{5}{d + 2}$

10. $\frac{a}{a^2 - a - 20} + \frac{2}{a + 4}$

11. $\frac{1}{x^2 - 4} + \frac{x}{x + 2}$

12. $\frac{x}{x + 1} + \frac{3}{x^2 - 4x - 5}$

Example 5 (p. 452)

13. $\frac{x + \frac{x}{3}}{x - \frac{x}{6}}$

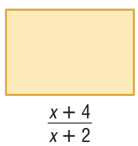
14. $\frac{1 - \frac{1}{x}}{x - \frac{1}{x}}$

15. $\frac{2 - \frac{4}{x}}{x - \frac{4}{x}}$

16. $\frac{x - \frac{x}{2}}{x + \frac{x}{8}}$

Example 6 (p. 453)

17. **GEOMETRY** An expression for the area of a rectangle is $4x + 16$. Find the width of the rectangle. Express in simplest form.



Exercises

HOMEWORK HELP	
For Exercises	See Examples
18, 19	1
20, 21	2
22–25	3
26–31	4
32, 33	5
34, 35	6

Find the LCM of each set of polynomials.

18. $10s^2, 35s^2t^2$

19. $36x^2y, 20xyz$

20. $4w - 12, 2w - 6$

21. $x^2 - y^2, x^3 + x^2y$

Simplify each expression.

22. $\frac{6}{ab} + \frac{8}{a}$

23. $\frac{5}{6v} + \frac{7}{4v}$

24. $\frac{3x}{4y^2} - \frac{y}{6x}$

25. $\frac{5}{a^2b} - \frac{7a}{5a^2}$

26. $\frac{7}{y-8} - \frac{6}{8-y}$

27. $\frac{a}{a-4} - \frac{3}{4-a}$

28. $\frac{m}{m^2-4} + \frac{2}{3m+6}$

29. $\frac{y}{y+3} - \frac{6y}{y^2-9}$

30. $\frac{5}{x^2-3x-28} + \frac{7}{2x-14}$

31. $\frac{d-4}{d^2+2d-8} - \frac{d+2}{d^2-16}$

32. $\dfrac{\frac{1}{b+2} + \frac{1}{b-5}}{\frac{2b^2-b-3}{b^2-3b-10}}$

33. $\dfrac{(x+y)\left(\frac{1}{x} - \frac{1}{y}\right)}{(x-y)\left(\frac{1}{x} + \frac{1}{y}\right)}$

34. GEOMETRY An expression for the length of one rectangle is $\frac{x^2-9}{x-2}$. The length of a similar rectangle is expressed as $\frac{x+3}{x^2-4}$. What is the scale factor of the two rectangles? Write in simplest form.

35. GEOMETRY Find the slope of a line that contains the points $A\left(\frac{1}{p}, \frac{1}{q}\right)$ and $B\left(\frac{1}{q}, \frac{1}{p}\right)$. Write in simplest form.

Find the LCM of each set of polynomials.

36. $14a^3, 15bc^3, 12b^3$

37. $9p^2q^3, 6pq^4, 4p^3$

38. $2t^2 + t - 3, 2t^2 + 5t + 3$

39. $n^2 - 7n + 12, n^2 - 2n - 8$

Simplify each expression.

40. $\frac{5}{r} + 7$

41. $\frac{2x}{3y} + 5$

42. $\frac{3}{4q} - \frac{2}{5q} - \frac{1}{2q}$

43. $\frac{11}{9} - \frac{7}{2w} - \frac{6}{5w}$

44. $\frac{1}{h^2-9h+20} - \frac{5}{h^2-10h+25}$

45. $\frac{x}{x^2+5x+6} - \frac{2}{x^2+4x+4}$

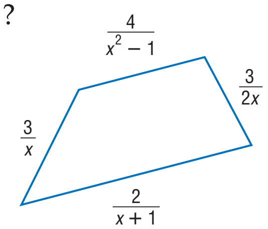
46. $\frac{m^2+n^2}{m^2-n^2} + \frac{m}{n-m} + \frac{n}{m+n}$

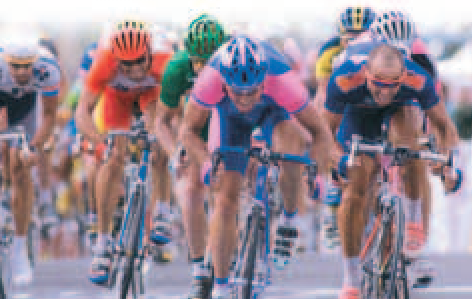
47. $\frac{y+1}{y-1} + \frac{y+2}{y-2} + \frac{y}{y^2-3y+2}$

48. Write $\left(\frac{2s}{2s+1} - 1\right) \div \left(1 + \frac{2s}{1-2s}\right)$ in simplest form.

49. What is the simplest form of $\left(3 + \frac{5}{a+2}\right) \div \left(3 - \frac{10}{a+7}\right)$?

50. GEOMETRY Find the perimeter of the quadrilateral. Express in simplest form.





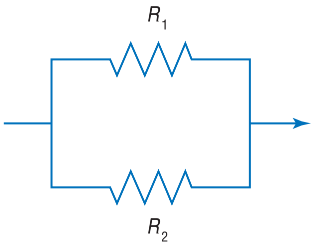
Real-World Link

The Tour de France is the most popular bicycle road race. It lasts 21 days and covers 2500 miles.

Source: *World Book Encyclopedia*

ELECTRICITY **For Exercises 51 and 52, use the following information.**

In an electrical circuit, if two resistors with resistance R_1 and R_2 are connected in parallel as shown, the relationship between these resistances and the resulting combination resistance R is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



51. If R_1 is x ohms and R_2 is 4 ohms less than twice x ohms, write an expression for $\frac{1}{R}$.

52. A circuit with two resistors connected in parallel has an effective resistance of 25 ohms. One of the resistors has a resistance of 30 ohms. Find the resistance of the other resistor.

BICYCLING **For Exercises 53–55, use the following information.**

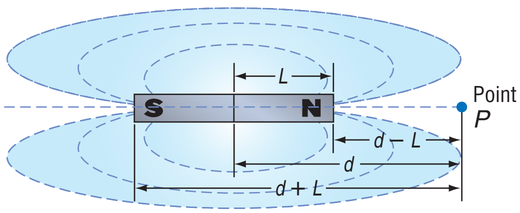
Jalisa is competing in a 48-mile bicycle race. She travels half the distance at one rate. The rest of the distance, she travels 4 miles per hour slower.

53. If x represents the faster pace in miles per hour, write an expression that represents the time spent at that pace.

54. Write an expression for the time spent at the slower pace.

55. Write an expression for the time Jalisa needed to complete the race.

56. **MAGNETS** For a bar magnet, the magnetic field strength H at a point P along the axis of the magnet is $H = \frac{m}{2L(d - L)^2} - \frac{m}{2L(d + L)^2}$. Write a simpler expression for H.



EXTRA PRACTICE

See pages 908, 933.

Math Online

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H.O.T. Problems

57. **OPEN ENDED** Write two polynomials that have a LCM of $d^3 - d$.

58. **FIND THE ERROR** Lorena and Yong-Chan are simplifying $\frac{x}{a} - \frac{x}{b}$. Who is correct? Explain your reasoning.

Lorena

$$\frac{x}{a} - \frac{x}{b} = \frac{bx}{ab} - \frac{ax}{ab}$$
$$= \frac{bx - ax}{ab}$$

Yong-Chan

$$\frac{x}{a} - \frac{x}{b} = \frac{x}{a - b}$$

59. **CHALLENGE** Find two rational expressions whose sum is $\frac{2x - 1}{(x + 1)(x - 2)}$.

60. **REASONING** In the expression $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, a, b, and c are nonzero real numbers. Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your answer.

a. abc is a common denominator.

b. abc is the LCD.

c. ab is the LCD.

d. b is the LCD.

e. The sum is $\frac{bc + ac + ab}{abc}$.

61. *Writing in Math* Use the information on page 450 to explain how subtraction of rational expressions is used in photography. Include an equation that could be used to find the distance between the lens and the film if the focal length of the lens is 50 millimeters and the distance between the lens and the object is 1000 millimeters.

STANDARDIZED TEST PRACTICE

62. **ACT/SAT** What is the sum of $\frac{x-y}{5}$ and $\frac{x+y}{4}$?

A $\frac{x+9y}{20}$

B $\frac{9x+y}{20}$

C $\frac{9x-y}{20}$

D $\frac{x-9y}{20}$

63. **REVIEW**

Given: Two angles are complementary. The measure of one angle is 15° more than the measure of the other angle.
Conclusion: The measures of the angles are 30° and 45°.

This conclusion —

F is contradicted by the first statement given.

G is verified by the first statement given.

H invalidates itself because a 45° angle cannot be complementary to another.

J verifies itself because 30° is 15° less than 45°.

Spiral Review

Simplify each expression. (Lesson 8-1)

64. $\frac{9x^2y^3}{(5xyz)^2} \div \frac{(3xy)^3}{20x^2y}$

65. $\frac{5a^2-20}{2a+2} \cdot \frac{4a}{10a-20}$

66. Graph $y \leq \sqrt{x+1}$. (Lesson 7-7)

Find all of the zeros of each function. (Lesson 6-9)

67. $g(x) = x^4 - 8x^2 - 9$

68. $h(x) = 3x^3 - 5x^2 + 13x - 5$

69. **GARDENS** Helene Jonson has a rectangular garden 25 feet by 50 feet. She wants to increase the garden on all sides by an equal amount. If the area of the garden is to be increased by 400 square feet, by how much should each dimension be increased? (Lesson 5-5)

70. Three times a number added to four times a second number is 22. The second number is two more than the first number. Find the numbers. (Lesson 3-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Factor each polynomial.** (Lesson 5-3)

71. $x^2 + 3x + 2$

72. $x^2 - 6x + 5$

73. $x^2 + 11x - 12$

74. $x^2 - 16$

75. $3x^2 - 75$

76. $x^3 - 3x^2 + 4x - 12$

8-3 Graphing Rational Functions

Main Ideas

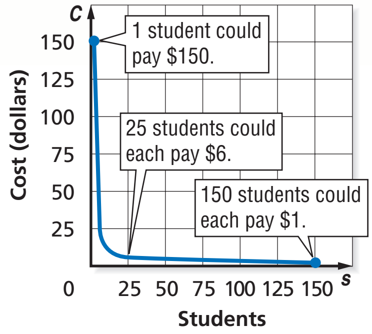
- Determine the limitations on the domains and ranges of the graphs of rational functions.
- Graph rational functions.

New Vocabulary

rational function
continuity
asymptote
point discontinuity

GET READY for the Lesson

A group of students want to get their favorite teacher, Mr. Salgado, a retirement gift. They plan to get him a gift certificate for a weekend package at a lodge in a state park. The certificate costs \$150. If c represents the cost for each student and s represents the number of students, then $c = \frac{150}{s}$.



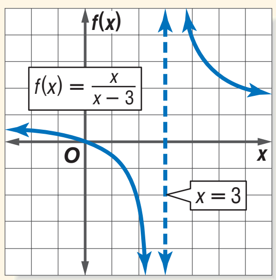
Domain and Range The function $c = \frac{150}{s}$ is a rational function. A **rational function** has an equation of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. Here are other rational functions.

$$f(x) = \frac{x}{x+3} \qquad g(x) = \frac{5}{x-6} \qquad h(x) = \frac{x+4}{(x-1)(x+4)}$$

No denominator in a rational function can be zero because division by zero is not defined. The functions above are not defined at $x = -3$, $x = 6$, and $x = 1$ and $x = -4$, respectively. The domain of a rational function is limited to values for which the function is defined.

The graphs of rational functions may have breaks in **continuity**. This means that, unlike polynomial functions, which can be traced with a pencil never leaving the paper, not all rational functions are traceable. Breaks in continuity occur at values that are excluded from the domain. They can appear as vertical asymptotes or as point discontinuity. An **asymptote** is a line that the graph of the function approaches, but never touches. **Point discontinuity** is like a hole in a graph.

KEY CONCEPT — Vertical Asymptotes

Property	Words	Example	Model
Vertical Asymptote	If the rational expression of a function is written in simplest form and the function is undefined for $x = a$, then the line $x = a$ is a vertical asymptote.	For $f(x) = \frac{x}{x-3}$, the line $x = 3$ is a vertical asymptote.	

Concepts in Motion
Animation
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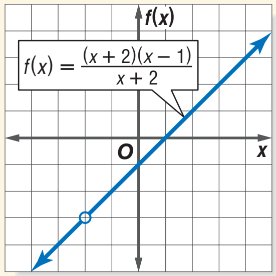
KEY CONCEPT — Point Discontinuity

Property	Words	Example	Model
Point Discontinuity	If the original function is undefined for $x = a$ but the rational expression of the function in simplest form is defined for $x = a$, then there is a hole in the graph at $x = a$.	$f(x) = \frac{(x+2)(x-1)}{x+2}$ can be simplified to $f(x) = x - 1$. So, $x = -2$ represents a hole in the graph.	

EXAMPLE Limitations on Domain

1 **Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of $f(x) = \frac{x^2 - 1}{x^2 - 6x + 5}$.**

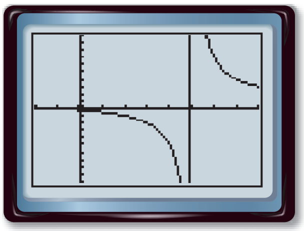
First factor the numerator and denominator of the rational expression.

$$\frac{x^2 - 1}{x^2 - 6x + 5} = \frac{(x-1)(x+1)}{(x-1)(x-5)}$$

The function is undefined for $x = 1$ and $x = 5$. Since $\frac{\cancel{(x-1)}^{1}(x+1)}{\cancel{(x-1)}_{1}(x-5)} = \frac{x+1}{x-5}$,

$x = 5$ is a vertical asymptote, and $x = 1$ represents a hole in the graph.

CHECK You can use a graphing calculator to check this solution. The graphing calculator screen at the right shows the graph of $f(x)$. The graph shows the vertical asymptote at $x = 5$. It is not clear from the graph that the function is not defined at $x = 1$. However, if you use the value function of the CALC menu and enter 1 at the X= prompt, you will see that no value is returned for Y=. This shows that $f(x)$ is not defined at $x = 1$.



[−2, 8] scl: 1 by [−10, 10] scl: 1

Study Tip

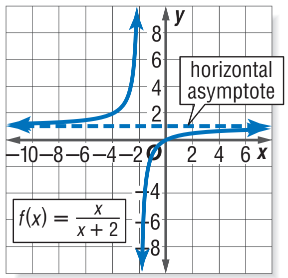
Parent Function

The parent function for the family of rational, or reciprocal, functions is $y = \frac{1}{x}$.

CHECK Your Progress

1. Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of $f(x) = \frac{x^2 + 6x + 8}{x^2 - 16}$.

For some rational functions, the values of the range are limited. Often a horizontal asymptote occurs where a value is excluded from the range. For example, 1 is excluded from the range of $f(x) = \frac{x}{x+2}$. The graph of $f(x)$ gets increasingly close to a horizontal asymptote as x increases or decreases.



Graph Rational Functions You can use what you know about vertical asymptotes and point discontinuity to graph rational functions.

Personal Tutor

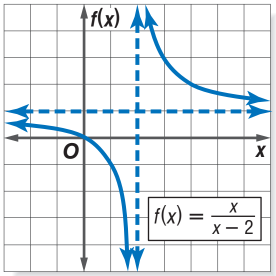
EXAMPLE 2 Graph with Vertical and Horizontal Asymptotes

Graph $f(x) = \frac{x}{x-2}$.

The function is undefined for $x = 2$. Since $\frac{x}{x-2}$ is in simplest form, $x = 2$ is a vertical asymptote. Draw the vertical asymptote. Make a table of values. Plot the points and draw the graph.

As $|x|$ increases, it appears that the y-values of the function get closer and closer to 1. The line with the equation $f(x) = 1$ is a horizontal asymptote of the function.

x	$f(x)$
−50	0.96154
−20	0.90909
−10	0.83333
−2	0.5
−1	0.33333
0	0
1	−1
3	3
4	2
5	1.6667
10	1.25
20	1.1111
50	1.0417



Study Tip

Graphing Rational Functions
Finding the x- and y-intercepts is often useful when graphing rational functions.

CHECK Your Progress

2. Graph $f(x) = \frac{x+1}{x-1}$.

Online Personal Tutor at algebra2.com

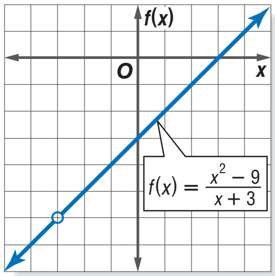
As you have learned, graphs of rational functions may have point discontinuity rather than vertical asymptotes. The graphs of these functions appear to have holes. These holes are usually shown as circles on graphs.

EXAMPLE 3 Graph with Point Discontinuity

Graph $f(x) = \frac{x^2-9}{x+3}$.

Notice that $\frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3}$ or $x - 3$.

Therefore, the graph of $f(x) = \frac{x^2-9}{x+3}$ is the graph of $f(x) = x - 3$ with a hole at $x = -3$.



CHECK Your Progress

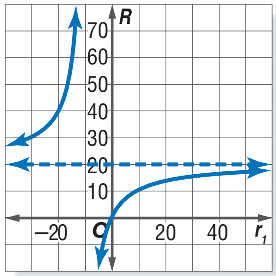
3. Graph $f(x) = \frac{x^2+4x-5}{x+5}$.

In the real world, sometimes values on the graph of a rational function are not meaningful.

Real-World EXAMPLE Use Graphs of Rational Functions

4 AVERAGE SPEED **A boat traveled upstream at r_1 miles per hour. During the return trip to its original starting point, the boat traveled at r_2 miles per hour. The average speed for the entire trip R is given by the formula $R = \frac{2r_1r_2}{r_1 + r_2}$.**

a. Let r_1 be the independent variable and let R be the dependent variable. Draw the graph if $r_2 = 10$ miles per hour.



The function is $R = \frac{2r_1(10)}{r_1 + 10}$ or $R = \frac{20r_1}{r_1 + 10}$. The vertical asymptote is $r_1 = -10$. Graph the vertical asymptote and the function. Notice that the horizontal asymptote is $R = 20$.

b. What is the R-intercept of the graph? The R-intercept is 0.

c. What domain and range values are meaningful in the context of the problem?

In the problem context, the speeds are nonnegative values. Therefore, only values of r_1 greater than or equal to 0 and values of R between 0 and 20 are meaningful.

CHECK Your Progress

4. SALARIES A company uses the formula $S(x) = \frac{45x + 25}{x + 1}$ to determine the salary in thousands of dollars of an employee during his xth year. Draw the graph of $S(x)$. What domain and range values are meaningful in the context of the problem? What is the meaning of the horizontal asymptote for the graph?

CHECK Your Understanding

Example 1 (p. 458)

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

1. $f(x) = \frac{3}{x^2 - 4x + 4}$

2. $f(x) = \frac{x - 1}{x^2 + 4x - 5}$

Graph each rational function.

Example 2 (p. 459)

3. $f(x) = \frac{x}{x + 1}$

4. $f(x) = \frac{6}{(x - 2)(x + 3)}$

5. $f(x) = \frac{4}{(x - 1)^2}$

6. $f(x) = \frac{x - 5}{x + 1}$

Example 3 (p. 459)

7. $f(x) = \frac{x^2 - 25}{x - 5}$

8. $f(x) = \frac{x + 2}{x^2 - x - 6}$

Example 4 (p. 460)

ELECTRICITY **For Exercises 9–12, use the following information.**
The current I in amperes in an electrical circuit with three resistors in series is given by the equation $I = \frac{V}{R_1 + R_2 + R_3}$, where V is the voltage in volts in the circuit and R_1, R_2, and R_3 are the resistances in ohms of the three resistors.

9. Let R_1 be the independent variable, and let I be the dependent variable. Graph the function if $V = 120$ volts, $R_2 = 25$ ohms, and $R_3 = 75$ ohms.

10. Give the equation of the vertical asymptote and the R_1- and I-intercepts of the graph.

11. Find the value of I when the value of R_1 is 140 ohms.

12. What domain and range values are meaningful in the context of the problem?

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–16	1
17–26	2
27, 28	3
29–36	4

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

13. $f(x) = \frac{2}{x^2 - 5x + 6}$

14. $f(x) = \frac{4}{x^2 + 2x - 8}$

15. $f(x) = \frac{x + 3}{x^2 + 7x + 12}$

16. $f(x) = \frac{x - 5}{x^2 - 4x - 5}$

Graph each rational function.

17. $f(x) = \frac{1}{x}$

18. $f(x) = \frac{3}{x}$

19. $f(x) = \frac{1}{x + 2}$

20. $f(x) = \frac{-5}{x + 1}$

21. $f(x) = \frac{x}{x - 3}$

22. $f(x) = \frac{5x}{x + 1}$

23. $f(x) = \frac{-3}{(x - 2)^2}$

24. $f(x) = \frac{1}{(x + 3)^2}$

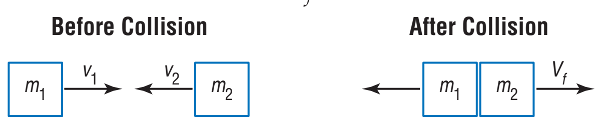
25. $f(x) = \frac{x + 4}{x - 1}$

26. $f(x) = \frac{x - 1}{x - 3}$

27. $f(x) = \frac{x^2 - 36}{x + 6}$

28. $f(x) = \frac{x^2 - 1}{x - 1}$

PHYSICS **For Exercises 29–32, use the following information.**
Under certain conditions, when two objects collide, the objects are repelled from each other with velocity given by the equation $V_f = \frac{2m_1v_1 + v_2(m_2 - m_1)}{m_1 + m_2}$.
In this equation m_1 and m_2 are the masses of the two objects, v_1 and v_2 are the initial speeds of the two objects, and V_f is the final speed of the second object.



29. Let m_2 be the independent variable, and let V_f be the dependent variable. Graph the function if $m_1 = 5$ kilograms and $v_1 = 15$ meters per second, and $v_2 = 20$ meters per second.

30. Use the equation and the values in Exercise 29 to determine the final speed if $m_2 = 20$ kilograms.

31. Give the equation of any asymptotes and the m_2- and V_f-intercepts of the graph.

32. What domain and range values are meaningful in the context of the problem?

BASKETBALL **For Exercises 33–36, use the following information.**
Zonta plays basketball for Centerville High School. So far this season, she has made 6 out of 10 free throws. She is determined to improve her free-throw percentage. If she can make x consecutive free throws, her free-throw percentage can be determined using $P(x) = \frac{6 + x}{10 + x}$.

33. Graph the function.

34. What part of the graph is meaningful in the context of the problem?

35. Describe the meaning of the y-intercept.

36. What is the equation of the horizontal asymptote? Explain its meaning with respect to Zonta's shooting percentage.

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

37. $f(x) = \frac{x^2 - 8x + 16}{x - 4}$

38. $f(x) = \frac{x^2 - 3x + 2}{x - 1}$

Graph each rational function.

39. $f(x) = \frac{3}{(x - 1)(x + 5)}$

40. $f(x) = \frac{-1}{(x + 2)(x - 3)}$

41. $f(x) = \frac{x}{x^2 - 1}$

42. $f(x) = \frac{x - 1}{x^2 - 4}$

43. $f(x) = \frac{6}{(x - 6)^2}$

44. $f(x) = \frac{1}{(x + 2)^2}$

45. $f(x) = \frac{x^2 + 6x + 5}{x + 1}$

46. $f(x) = \frac{x^2 - 4x}{x - 4}$

HISTORY **For Exercises 47–49, use the following information.**
In Maria Gaetana Agnesi's book *Analytical Institutions*, Agnesi discussed the characteristics of the equation $x^2y = a^2(a - y)$, the graph of which is called the "curve of Agnesi." This equation can be expressed as $y = \frac{a^3}{x^2 + a^2}$.

47. Graph $f(x) = \frac{a^3}{x^2 + a^2}$ if $a = 4$.

48. Describe the graph. What are the limitations on the domain and range?

49. Make a conjecture about the shape of the graph of $f(x) = \frac{a^3}{x^2 + a^2}$ if $a = -4$. Explain your reasoning.

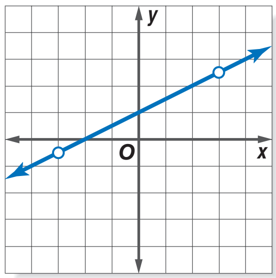
EXTRA PRACTICE
See pages 908, 933.
Math Online
Self-Check Quiz at
algebra2.com

H.O.T. Problems

50. **OPEN ENDED** Write a function the graph of which has vertical asymptotes located at $x = -5$ and $x = 2$.

51. **REASONING** Compare and contrast the graphs of $f(x) = \frac{(x - 1)(x + 5)}{x - 1}$ and $g(x) = x + 5$.

52. **CHALLENGE** Write a rational function for the graph at the right.



53. **CHALLENGE** Write three rational functions that have a vertical asymptote at $x = 3$ and a hole at $x = -2$.

54. ***Writing in Math*** Use the information on page 457 to explain how rational functions can be used when buying a group gift. Explain why only part of the graph of the rational function is meaningful in the context of the problem.

STANDARDIZED TEST PRACTICE

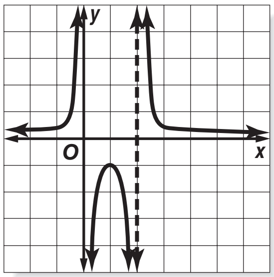
55. **ACT/SAT** Which set is the domain of the function graphed below?

A $\{x \mid x \neq 0, 2\}$

B $\{x \mid x \neq -2, 0\}$

C $\{x \mid x < 4\}$

D $\{x > -4\}$



56. **REVIEW** $\frac{x+2}{x+3} + \frac{4}{x^2+x-6} =$

F $\frac{-3x-9}{x^2+x-6}$

G $\frac{x^2-3x-24}{x^2+x-6}$

H $\frac{x^2}{x^2+x-6}$

J $\frac{x^2+x-1}{x^2+x-6}$

Spiral Review

Simplify each expression. (Lessons 8-1 and 8-2)

57. $\frac{3m+2}{m+n} + \frac{4}{2m+2n}$

58. $\frac{5}{x+3} - \frac{2}{x-2}$

59. $\frac{2w-4}{w+3} \div \frac{2w+6}{5}$

Find all of the rational zeros for each function. (Lesson 6-8)

60. $f(x) = x^3 + 5x^2 + 2x - 8$

61. $g(x) = 2x^3 - 9x^2 + 7x + 6$

62. **ART** Joyce Jackson purchases works of art for an art gallery. Two years ago she bought a painting for \$20,000, and last year she bought one for \$35,000. If paintings appreciate 14% per year, how much are the two paintings worth now? (Lesson 6-5)

Solve each equation by completing the square. (Lesson 5-5)

63. $x^2 + 8x + 20 = 0$

64. $x^2 + 2x - 120 = 0$

65. $x^2 + 7x - 17 = 0$

66. Write the slope-intercept form of the equation for the line that passes through $(1, -2)$ and is perpendicular to the line with equation $y = -\frac{1}{5}x + 2$. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each proportion.**

67. $\frac{16}{v} = \frac{32}{9}$

68. $\frac{7}{25} = \frac{a}{5}$

69. $\frac{6}{15} = \frac{8}{s}$

70. $\frac{b}{9} = \frac{40}{30}$

EXTEND 8-3

Graphing Calculator Lab
Graphing Rational Functions

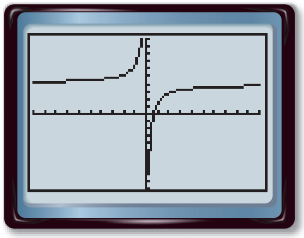
A TI-83/84 Plus graphing calculator can be used to explore the graphs of rational functions. These graphs have some features that never appear in the graphs of polynomial functions.

ACTIVITY 1

Graph $y = \frac{8x - 5}{2x}$ in the standard viewing window. Find the equations of any asymptotes.

Enter the equation in the Y= list.

KEYSTROKES: [Y=] [(] 8 [X,T,θ,n] [–] 5 [)] [÷] [(] 2 [X,T,θ,n] [)] [ZOOM] 6


[−10, 10] scl: 1 by [−10, 10] scl: 1

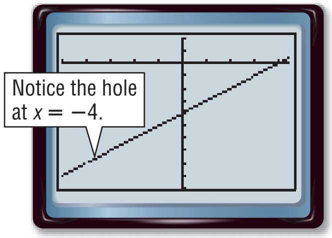
By looking at the equation, we can determine that if $x = 0$, the function is undefined. The equation of the vertical asymptote is $x = 0$. Notice what happens to the y-values as x grows larger and as x gets smaller. The y-values approach 4. So, the equation for the horizontal asymptote is $y = 4$.

ACTIVITY 2

Graph $y = \frac{x^2 - 16}{x + 4}$ in the window [−5, 4.4] by [−10, 2] with scale factors of 1.

Because the function is not continuous, put the calculator in dot mode.

KEYSTROKES: [MODE] [▼] [▼] [▼] [▼] [▶] [ENTER]



[−5, 4.4] scl: 1 by [−10, 2] scl: 1

This graph looks like a line with a break in continuity at $x = -4$. This happens because the denominator is 0 when $x = -4$. Therefore, the function is undefined when $x = -4$.

If you TRACE along the graph, when you come to $x = -4$, you will see that there is no corresponding y-value.

EXERCISES

Use a graphing calculator to graph each function. Be sure to show a complete graph. Draw the graph on a sheet of paper. Write the x-coordinates of any points of discontinuity and/or the equations of any asymptotes.

1. $f(x) = \frac{1}{x}$
2. $f(x) = \frac{x}{x + 2}$
3. $f(x) = \frac{2}{x - 4}$
4. $f(x) = \frac{2x}{3x - 6}$
5. $f(x) = \frac{4x + 2}{x - 1}$
6. $f(x) = \frac{x^2 - 9}{x + 3}$
7. Which graph(s) has point discontinuity?
8. Describe functions that have point discontinuity.

Math Online Other Calculator Keystrokes at algebra2.com

8-4 Direct, Joint, and Inverse Variation

Main Ideas

- Recognize and solve direct and joint variation problems.
- Recognize and solve inverse variation problems.

New Vocabulary

direct variation

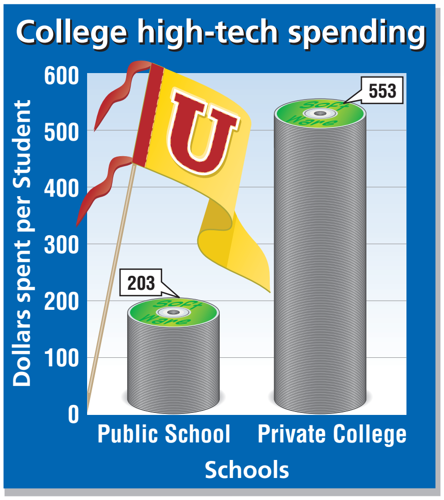
constant of variation

joint variation

inverse variation

GET READY for the Lesson

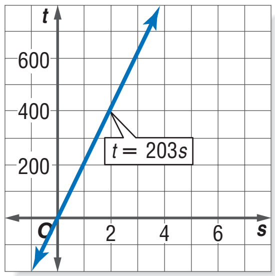
The total high-tech spending t of an average public college can be found by using the equation $t = 203s$, where s is the number of students.



Source: dthonline.com

Direct Variation and Joint Variation The relationship given by $t = 203s$ is an example of direct variation. A **direct variation** can be expressed in the form $y = kx$. The k in this equation is called the **constant of variation**.

Notice that the graph of $t = 203s$ is a straight line through the origin. An equation of a direct variation is a special case of an equation written in slope-intercept form, $y = mx + b$. When $m = k$ and $b = 0$, $y = mx + b$ becomes $y = kx$. So the slope of a direct variation equation is its constant of variation.



To express a direct variation, we say that y varies directly as x. In other words, as x increases, y increases or decreases at a constant rate.

KEY CONCEPT *Direct Variation*

y varies directly as x if there is some nonzero constant k such that $y = kx$. k is called the constant of variation.

If you know that y varies directly as x and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1 \quad \text{and} \quad y_2 = kx_2$$

$$\frac{y_1}{x_1} = k \qquad \frac{y_2}{x_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Using the properties of equality, you can find many other proportions that relate these same x- and y-values.

EXAMPLE 1 Direct Variation

If y varies directly as x and $y = 12$ when $x = -3$, find y when $x = 16$.

Use a proportion that relates the values.

$\frac{y_1}{x_1} = \frac{y_2}{x_2}$ Direct proportion

$\frac{12}{-3} = \frac{y_2}{16}$ $y_1 = 12$, $x_1 = -3$, and $x_2 = 16$

$16(12) = -3(y_2)$ Cross multiply.

$192 = -3y_2$ Simplify.

$-64 = y_2$ Divide each side by -3.

When $x = 16$, the value of y is -64.

CHECK Your Progress

1. If r varies directly as s and $r = -20$ when $s = 4$, find r when $s = -6$.

Another type of variation is joint variation. **Joint variation** occurs when one quantity varies directly as the product of two or more other quantities.

KEY CONCEPT — Joint Variation

y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.

If you know that y varies jointly as x and z and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1z_1 \quad \text{and} \quad y_2 = kx_2z_2$$

$$\frac{y_1}{x_1z_1} = k \qquad \frac{y_2}{x_2z_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}.$$

EXAMPLE 2 Joint Variation

Suppose y varies jointly as x and z. Find y when $x = 8$ and $z = 3$, if $y = 16$ when $z = 2$ and $x = 5$.

Use a proportion that relates the values.

$\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}$ Joint variation

$\frac{16}{5(2)} = \frac{y_2}{8(3)}$ $y_1 = 16$, $x_1 = 5$, $z_1 = 2$, $x_2 = 8$, and $z_2 = 3$

$8(3)(16) = 5(2)(y_2)$ Cross multiply.

$384 = 10y_2$ Simplify.

$38.4 = y_2$ Divide each side by 10.

When $x = 8$ and $z = 3$, the value of y is 38.4.

CHECK Your Progress

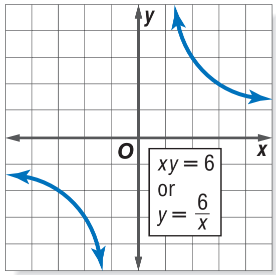
2. Suppose r varies jointly as s and t. Find r when $s = 2$ and $t = 8$, if $r = 70$ when $s = 10$ and $t = 4$.

Inverse Variation Another type of variation is inverse variation. For two quantities with **inverse variation**, as one quantity increases, the other quantity decreases. For example, speed and time for a fixed distance vary inversely with each other. When you travel to a particular location, as your speed increases, the time it takes to arrive at that location decreases.

KEY CONCEPT *Direct Variation*

y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Suppose y varies inversely as x such that $xy = 6$ or $y = \frac{6}{x}$. The graph of this equation is shown at the right. Since k is a positive value, as the values of x increase, the values of y decrease.



A proportion can be used with inverse variation to solve problems where some quantities are known. The following proportion is only one of several that can be formed.

$$x_1y_1 = k \text{ and } x_2y_2 = k$$

$$x_1y_1 = x_2y_2 \quad \text{Substitution Property of Equality}$$

$$\frac{x_1}{y_2} = \frac{x_2}{y_1} \quad \text{Divide each side by } y_1y_2.$$

EXAMPLE Inverse Variation

3 **If r varies inversely as t and $r = 18$ when $t = -3$, find r when $t = -11$.**

$$\frac{r_1}{t_2} = \frac{r_2}{t_1} \quad \text{Use a proportion that relates the values.}$$

$$\frac{18}{-11} = \frac{r_2}{-3} \quad r_1 = 18,\ t_1 = -3, \text{ and } t_2 = -11$$

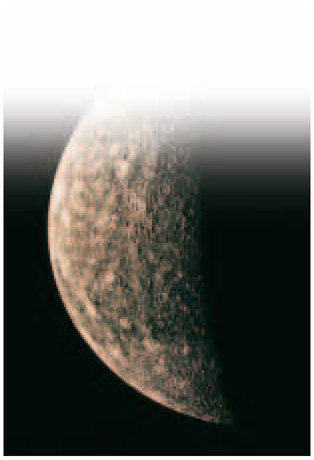
$$18(-3) = -11(r_2) \quad \text{Cross multiply.}$$

$$-54 = -11r_2 \quad \text{Simplify.}$$

$$4\frac{10}{11} = r_2 \quad \text{Divide each side by } -11.$$

CHECK Your Progress

3. If x varies inversely as y and $x = 24$ when $y = 4$, find x when $y = 12$.



Real-World Link

Mercury is about 36 million miles from the Sun, making it the closest planet to the Sun. Its proximity to the Sun causes its temperature to be as high as 800°F.

Source: *World Book Encyclopedia*

Real-World EXAMPLE

Personal Tutor

4 **SPACE The apparent length of an object is inversely proportional to one's distance from the object. Earth is about 93 million miles from the Sun. Use the information at the left to find how many times as large the diameter of the Sun would appear on Mercury than on Earth.**

Explore The apparent diameter of the Sun varies inversely with the distance from the Sun. You know Mercury's distance from the Sun and Earth's distance from the Sun. You want to know how much larger the diameter of the Sun appears on Mercury than on Earth.

Plan Let the apparent diameter of the Sun from Earth equal 1 unit and the apparent diameter of the Sun from Mercury equal m. Then use a proportion that relates the values.

Solve

$$\frac{\text{distance from Mercury}}{\text{apparent diameter from Earth}} = \frac{\text{distance from Earth}}{\text{apparent diameter from Mercury}}$$ Inverse variation

$$\frac{36 \text{ million miles}}{1 \text{ unit}} = \frac{93 \text{ million miles}}{m \text{ units}}$$ Substitution

$$(36 \text{ million miles})(m \text{ units}) = (93 \text{ million miles})(1 \text{ unit})$$ Cross multiply.

$$m = \frac{(93 \text{ million miles})(1 \text{ unit})}{36 \text{ million miles}}$$ Divide each side by 36 million miles.

$$m \approx 2.58 \text{ units}$$ Simplify.

Check Since the distance between the Sun and Earth is between 2 and 3 times the distance between the Sun and Mercury, the answer seems reasonable. From Mercury, the diameter of the Sun will appear about 2.58 times as large as it appears from Earth.

CHECK Your Progress

4. SPACE Jupiter is about 483.6 million miles from the Sun. Use the information above to find how many times as large the diameter of the Sun would appear on Earth as on Jupiter.

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CHECK Your Understanding

Examples 1–3 (pp. 466–467)

1. If y varies directly as x and $y = 18$ when $x = 15$, find y when $x = 20$.

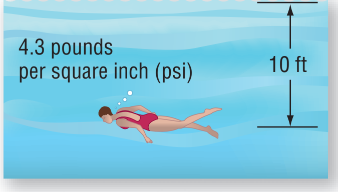
2. Suppose y varies jointly as x and z. Find y when $x = 9$ and $z = -5$, if $y = -90$ when $z = 15$ and $x = -6$.

3. If y varies inversely as x and $y = -14$ when $x = 12$, find x when $y = 21$.

Example 4 (pp. 467–468)

SWIMMING For Exercises 4–7, use the following information.

When a person swims underwater, the pressure in his or her ears varies directly with the depth at which he or she is swimming.



4. Write a direct variation equation that represents this situation.

5. Find the pressure at 60 feet.

6. It is unsafe for amateur divers to swim where the water pressure is more than 65 pounds per square inch. How deep can an amateur diver safely swim?

7. Make a table showing the number of pounds of pressure at various depths of water. Use the data to draw a graph of pressure versus depth.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8, 9	1
10, 11	2
12, 13	3
14, 15	4

8. If y varies directly as x and $y = 15$ when $x = 3$, find y when $x = 12$.

9. If y varies directly as x and $y = 8$ when $x = 6$, find y when $x = 15$.

10. Suppose y varies jointly as x and z. Find y when $x = 2$ and $z = 27$, if $y = 192$ when $x = 8$ and $z = 6$.

11. If y varies jointly as x and z and $y = 80$ when $x = 5$ and $z = 8$, find y when $x = 16$ and $z = 2$.

12. If y varies inversely as x and $y = 5$ when $x = 10$, find y when $x = 2$.

13. If y varies inversely as x and $y = 16$ when $x = 5$, find y when $x = 20$.

14. **GEOMETRY** How does the circumference of a circle vary with respect to its radius? What is the constant of variation?

15. **TRAVEL** A map of Alaska is scaled so that 3 inches represents 93 miles. How far apart are Anchorage and Fairbanks if they are 11.6 inches apart on the map?

Real-World Career

Travel Agent

Travel agents give advice and make arrangements for transportation, accommodations, and recreation. For international travel, they also provide information on customs and currency exchange.

Math Online

For more information, go to algebra2.com.

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation.

16. $\frac{n}{m} = 1.5$

17. $3 = \frac{a}{b}$

18. $a = 5bc$

19. $V = \frac{1}{3}Bh$

20. $p = \frac{12}{q}$

21. $\frac{2.5}{t} = s$

22. $vw = -18$

23. $y = -7x$

24. $V = \pi r^2 h$

25. If y varies directly as x and $y = 9$ when x is -15, find y when $x = 21$.

26. If y varies directly as x and $x = 6$ when $y = 0.5$, find y when $x = 10$.

27. Suppose y varies jointly as x and z. Find y when $x = \frac{1}{2}$ and $z = 6$, if $y = 45$ when $x = 6$ and $z = 10$.

28. If y varies jointly as x and z and $y = \frac{1}{8}$ when $x = \frac{1}{2}$ and $z = 3$, find y when $x = 6$ and $z = \frac{1}{3}$.

29. If y varies inversely as x and $y = 2$ when $x = 25$, find x when $y = 40$.

30. If y varies inversely as x and $y = 4$ when $x = 12$, find y when $x = 5$.

31. **CHEMISTRY** Boyle's Law states that when a sample of gas is kept at a constant temperature, the volume varies inversely with the pressure exerted on it. Write an equation for Boyle's Law that expresses the variation in volume V as a function of pressure P.

32. **CHEMISTRY** Charles' Law states that when a sample of gas is kept at a constant pressure, its volume V will increase directly as the temperature t. Write an equation for Charles' Law that expresses volume as a function.

LAUGHTER **For Exercises 33–35, use the following information.**

A newspaper reported that the average American laughs 15 times per day.

33. Write an equation to represent the average number of laughs produced by m household members during a period of d days.

34. Is your equation in Exercise 33 a *direct*, *joint*, or *inverse* variation?

35. Assume that members of your household laugh the same number of times each day as the average American. How many times would the members of your household laugh in a week?

Real-World Link

In order to sustain itself in its cold habitat, a Siberian tiger requires 20 pounds of meat per day.

Source: *Wildlife Fact File*

BIOLOGY **For Exercises 36–38, use the information at the left.**

36. Write an equation to represent the amount of meat needed to sustain *s* Siberian tigers for *d* days.

37. Is your equation in Exercise 36 a *direct*, *joint*, or *inverse* variation?

38. How much meat do three Siberian tigers need for the month of January?

39. **WORK** Paul drove from his house to work at an average speed of 40 miles per hour. The drive took him 15 minutes. If the drive home took him 20 minutes and he used the same route in reverse, what was his average speed going home?

40. **WATER SUPPLY** Many areas of Northern California depend on the snowpack of the Sierra Nevada Mountains for their water supply. If 250 cubic centimeters of snow will melt to 28 cubic centimeters of water, how much water does 900 cubic centimeters of snow produce?

41. **RESEARCH** According to Johannes Kepler's third law of planetary motion, the ratio of the square of a planet's period of revolution around the Sun to the cube of its mean distance from the Sun is constant for all planets. Verify that this is true for at least three planets.

ASTRONOMY **For Exercises 42–44, use the following information.**
Astronomers can use the brightness of two light sources, such as stars, to compare the distances from the light sources. The intensity, or brightness, of light I is inversely proportional to the square of the distance from the light source d.

42. Write an equation that represents this situation.

43. If d is the independent variable and I is the dependent variable, graph the equation from Exercise 42 when $k = 16$.

44. If two people are viewing the same light source, and one person is three times the distance from the light source as the other person, compare the light intensities that the two people observe.

GRAVITY **For Exercises 45–47, use the following information.**
According to the Law of Universal Gravitation, the attractive force F in Newtons between any two bodies in the universe is directly proportional to the product of the masses m_1 and m_2 in kilograms of the two bodies and inversely proportional to the square of the distance d in meters between the bodies. That is, $F = G\frac{m_1m_2}{d_2}$. G is the universal gravitational constant. Its value is 6.67×10^{-11} Nm^2/kg^2.

45. The distance between Earth and the Moon is about 3.84×10^8 meters. The mass of the Moon is 7.36×10^{22} kilograms. The mass of Earth is 5.97×10^{24} kilograms. What is the gravitational force that the Moon and Earth exert upon each other?

46. The distance between Earth and the Sun is about 1.5×10^{11} meters. The mass of the Sun is about 1.99×10^{30} kilograms. What is the gravitational force that the Sun and Earth exert upon each other?

47. Find the gravitational force exerted on each other by two 1000-kilogram iron balls a distance of 0.1 meter apart.

EXTRA PRACTICE
See pages 908, 933.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

48. OPEN ENDED Describe two real life quantities that vary directly with each other and two quantities that vary inversely with each other.

49. CHALLENGE Write a real-world problem that involves a joint variation. Solve the problem.

50. *Writing in Math* Use the information about variation on page 465 to explain how variation is used to determine the total cost if you know the unit cost.

STANDARDIZED TEST PRACTICE

51. ACT/SAT Suppose b varies inversely as the square of a. If a is multiplied by 9, which of the following is true for the value of b?

A It is multiplied by $\frac{1}{3}$.

B It is multiplied by $\frac{1}{9}$.

C It is multiplied by $\frac{1}{81}$.

D It is multiplied by 3.

52. REVIEW If $ab = 1$ and a is less than 0, which of the following statements cannot be true?

F b is negative.

G b is less than a.

H As a increases, b decreases.

J As a increases, b increases.

Spiral Review

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function. (Lesson 8-3)

53. $f(x) = \dfrac{x+1}{x^2-1}$ **54.** $f(x) = \dfrac{x+3}{x^2+x-12}$ **55.** $f(x) = \dfrac{x^2+4x+3}{x+3}$

Simplify each expression. (Lesson 8-2)

56. $\dfrac{3x}{x-y} + \dfrac{4x}{y-x}$ **57.** $\dfrac{t}{t+2} - \dfrac{2}{t^2-4}$ **58.** $\dfrac{m - \frac{1}{m}}{1 + \frac{4}{m} - \frac{5}{m^2}}$

59. BIOLOGY One estimate for the number of cells in the human body is 100,000,000,000,000. Write this number in scientific notation. (Lesson 6-1)

State the slope and the y-intercept of the graph of each equation. (Lesson 2-4)

60. $y = 0.4x + 1.2$ **61.** $2y = 6x + 14$ **62.** $3x + 5y = 15$

GET READY for the Next Lesson

PREREQUISITE SKILL **Identify each function as S for step, C for constant, A for absolute value, or P for piecewise.** (Lesson 2-6)

63. $h(x) = \frac{2}{3}$ **64.** $g(x) = 3|x|$ **65.** $f(x) = [\![2x]\!]$

66. $f(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x \leq 0 \end{cases}$ **67.** $h(x) = |x - 2|$ **68.** $g(x) = -3$

CHAPTER 8

Mid-Chapter Quiz

Lessons 8-1 through 8-4

Simplify each expression. (Lesson 8-1)

1. $\frac{t^2 - t - 6}{t^2 - 6t + 9}$

2. $\frac{3ab^3}{8a^2b} \cdot \frac{4ac}{9b^4}$

3. $\frac{-4}{8x} \div \frac{16}{2xy^2}$

4. $\frac{48}{6a + 42} \cdot \frac{7a + 49}{16}$

5. $\frac{w^2 + 5w + 4}{6} \div \frac{w + 1}{18w + 24}$

6. $\frac{\frac{x^2 + x}{x + 1}}{\frac{x}{x - 1}}$

7. MULTIPLE CHOICE For all $t \neq 5$, $\frac{t^2 - 25}{3t - 15} =$ (Lesson 8-2)

A $\frac{t - 5}{3}$.

B $\frac{t + 5}{3}$.

C $t - 5$.

D $t + 5$.

Simplify each expression. (Lesson 8-2)

8. $\frac{4a + 2}{a + b} + \frac{1}{-b - a}$

9. $\frac{2x}{5ab^3} + \frac{4y}{3a^2b^2}$

10. $\frac{5}{n + 6} - \frac{4}{n - 1}$

11. $\frac{x - 5}{2x - 6} - \frac{x - 7}{4x - 12}$

For Exercises 12–14, use the following information.

Lucita is going to a beach 100 miles away. She travels half the distance at one rate. The rest of the distance, she travels 15 miles per hour slower. (Lesson 8-2)

12. If x represents the faster pace in miles per hour, write an expression that represents the time spent at that pace.

13. Write an expression for the amount of time spent at the slower pace.

14. Write an expression for the amount of time Lucita needed to complete the trip.

Graph each rational function. (Lesson 8-3)

15. $f(x) = \frac{x - 1}{x - 4}$

16. $f(x) = \frac{-2}{x^2 - 6x + 9}$

17. MULTIPLE CHOICE What is the range of the function $y = \frac{x^2 + 8}{2}$? (Lesson 8-3)

F $\{y \mid y \neq \pm 2\sqrt{2}\}$

G $\{y \mid y \geq 4\}$

H $\{y \mid y \geq 0\}$

J $\{y \mid y \leq 0\}$

WORK For Exercises 18 and 19, use the following information. (Lesson 8-3)

Andy is a new employee at Quick Oil Change. The company's goal is to change every customer's oil in 10 minutes. So far, he has changed 13 out of 20 customers' oil in 10 minutes. Suppose Andy changes the next x customers' oil in 10 minutes. His 10-minute oil changing percentage can be determined using $P(x) = \frac{13 + x}{20 + x}$.

18. Graph the function.

19. What domain and range values are meaningful in the context of the problem?

Find each value. (Lesson 8-4)

20. If y varies inversely as x and $x = 14$ when $y = 7$, find x when $y = 2$.

21. If y varies directly as x and $y = 1$ when $x = 5$, find y when $x = 22$.

22. If y varies jointly as x and z and $y = 80$ when $x = 25$ and $z = 4$, find y when $x = 20$ and $z = 7$.

For Exercises 23–25, use the following information.

In order to remain healthy, a horse requires 10 pounds of hay a day. (Lesson 8-4)

23. Write an equation to represent the amount of hay needed to sustain x horses for d days.

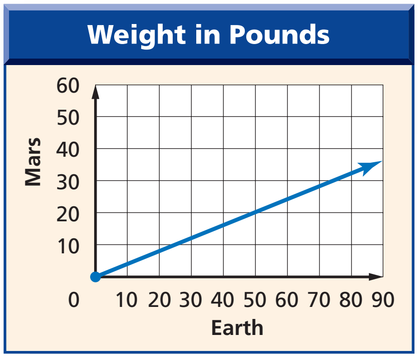
24. Is your equation a *direct*, *joint*, or *inverse* variation? Explain.

25. How much hay do three horses need for the month of July?

8-5 Classes of Functions

GET READY for the Lesson

The purpose of the Mars Exploration Program is to study conditions on Mars. The findings will help NASA prepare for a possible mission with human explorers. The graph at the right compares a person's weight on Earth with his or her weight on Mars. This graph represents a direct variation, which you studied in the previous lesson.



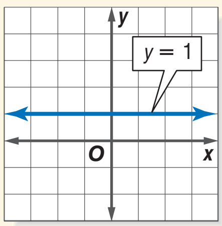
Main Ideas

- Identify graphs as different types of functions.
- Identify equations as different types of functions.

Identify Graphs In this book, you have studied several types of graphs representing special functions. The following is a summary of these graphs.

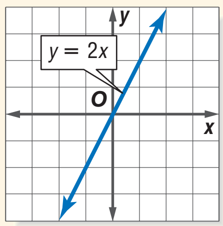
CONCEPT SUMMARY — *Special Functions*

Constant Function



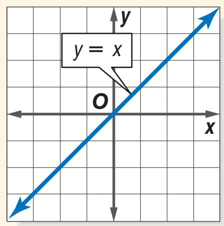
The general equation of a constant function is $y = a$, where a is any number. Its graph is a horizontal line that crosses the y-axis at a.

Direct Variation Function



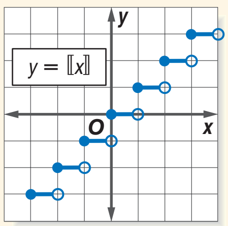
The general equation of a direct variation function is $y = ax$, where a is a nonzero constant. Its graph is a line that passes through the origin and is neither horizontal nor vertical.

Identity Function



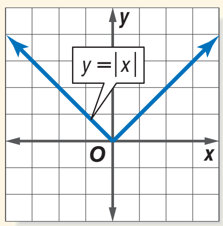
The identity function $y = x$ is a special case of the direct variation function in which the constant is 1. Its graph passes through all points with coordinates (a, a).

Greatest Integer Function



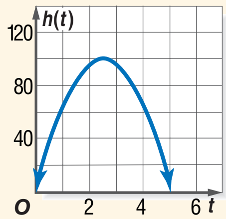
If an equation includes an expression inside the greatest integer symbol, the function is a greatest integer function. Its graph looks like steps.

Absolute Value Function



An equation with the independent variable inside absolute value symbols is an absolute value function. Its graph is in the shape of a V.

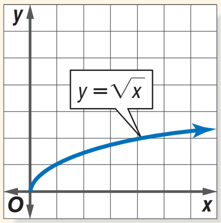
Quadratic Function



The general equation of a quadratic function is $y = ax^2 + bx + c$, where $a \neq 0$. Its graph is a parabola.

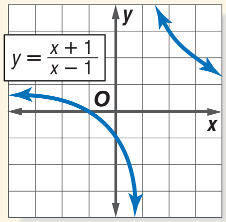
CONCEPT SUMMARY — Special Functions

Square Root Function



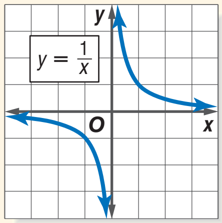
If an equation includes the independent variable inside the radical sign, the function is a square root function. Its graph is a curve that starts at a point and continues in only one direction.

Rational Function



The general equation for a rational function is $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions. Its graph may have one or more asymptotes and/or holes.

Inverse Variation Function



The inverse variation function $y = \frac{a}{x}$ is a special case of the rational function where $p(x)$ is a constant and $q(x) = x$. Its graph has two asymptotes, $x = 0$ and $y = 0$.

You can use the shape of the graphs of each type of function to identify the type of function that is represented by a given graph. To do so, keep in mind the graph of the parent function of each function type.

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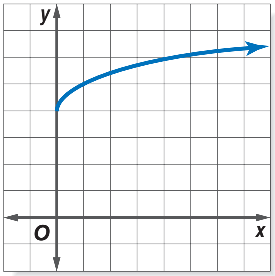
EXAMPLE 1 Identify a Function Given the Graph

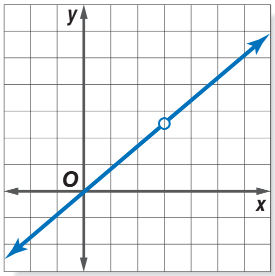
Identify the type of function represented by each graph.

a.



The graph has a starting point and curves in one direction. The graph represents a square root function.

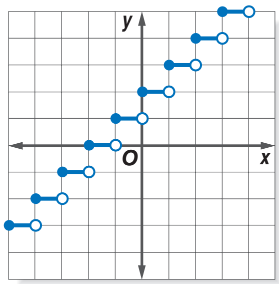
b.



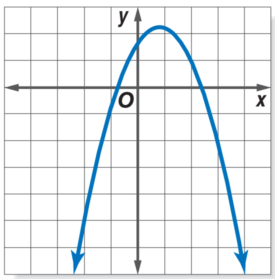
The graph appears to be a direct variation since it is a straight line passing through the origin. However, the hole indicates that it represents a rational function.

CHECK Your Progress

1A.



1B.



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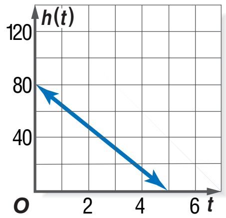
Identify Equations If you can identify an equation as a type of function, you can determine the shape of the graph.

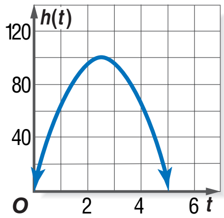
EXAMPLE Match Equation with Graph

2 ROCKETRY Emily launched a toy rocket from ground level. The height above the ground level h, in feet, after t seconds is given by the formula $h(t) = -16t^2 + 80t$. Which graph depicts the height of the rocket during its flight?

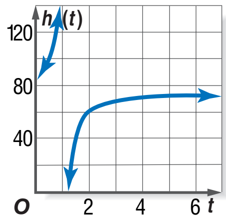
a.



b.



c.



The function includes a second-degree polynomial. Therefore, it is a quadratic function, and its graph is a parabola. Graph **b** is on the only parabola. Therefore, the answer is graph **b**.

CHECK Your Progress

2. Which graph above could represent an elevator moving from a height of 80 feet to ground level in 5 seconds?

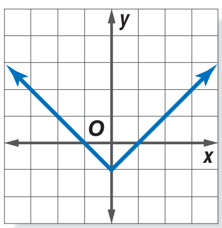
Sometimes recognizing an equation as a specific type of function can help you graph the function.

EXAMPLE Identify a Function Given its Equation

3 Identify the type of function represented by each equation. Then graph the equation.

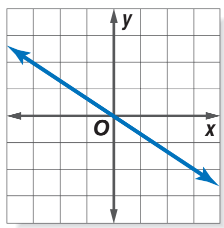
a. $y = |x| - 1$

Since the equation includes an expression inside absolute value symbols, it is an absolute value function. Therefore, the graph will be in the shape of a V. Plot some points and graph the absolute value function.



b. $y = -\frac{2}{3}x$

The function is in the form $y = ax$, where $a = -\frac{2}{3}$. Therefore, it is a direct variation function. The graph passes through the origin and has a slope of $-\frac{2}{3}$.



CHECK Your Progress

3A. $y = [\![x - 1]\!]$

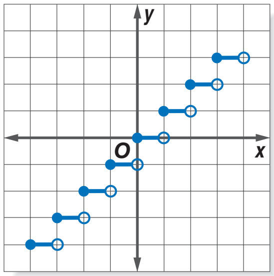
3B. $y = \frac{-1}{x + 1}$

CHECK Your Understanding

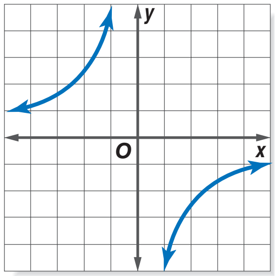
Example 1 (p. 474)

Identify the type of function represented by each graph.

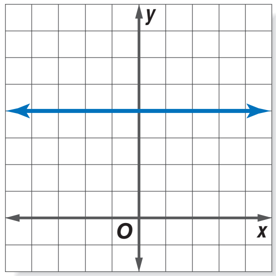
1.



2.



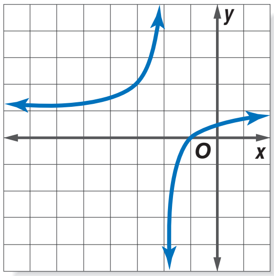
3.



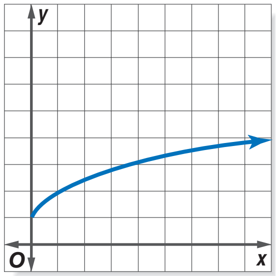
Example 2 (p. 475)

Match each graph with an equation at the right.

4.



5.



a. $y = x^2 + 2x + 3$
b. $y = \sqrt{x} + 1$
c. $y = \dfrac{x + 1}{x + 2}$
d. $y = [\![2x]\!]$

6. **GEOMETRY** Write the equation for the area of a circle. Identify the equation as a type of function. Describe the graph of the function.

Example 3 (p. 475)

Identify the type of function represented by each equation. Then graph the equation.

7. $y = x$ **8.** $y = -x^2 + 2$ **9.** $y = |x + 2|$

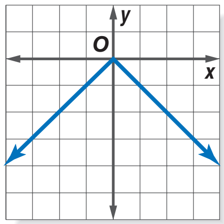
Exercises

HOMEWORK HELP

For Exercises	See Examples
10–15	1
16–23	3
24–31	2

Identify the function represented by each graph.

10.



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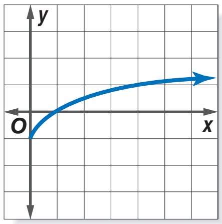

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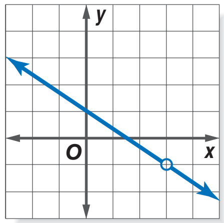

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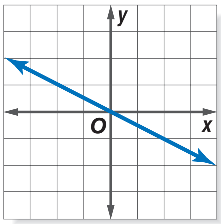

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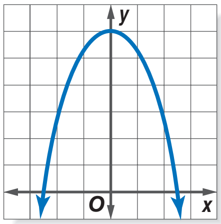

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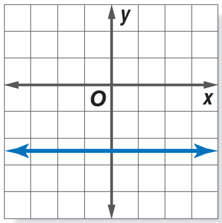


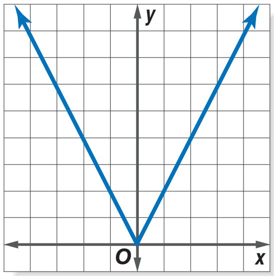
Identify the type of function represented by each equation. Then graph the equation.

16. $y = -1.5$ **17.** $y = 2.5x$ **18.** $y = \sqrt{9x}$ **19.** $y = \frac{4}{x}$

20. $y = \dfrac{x^2 - 1}{x - 1}$ **21.** $y = 3[\![x]\!]$ **22.** $y = |2x|$ **23.** $y = 2x^2$

Match each graph with an equation at the right.

24.



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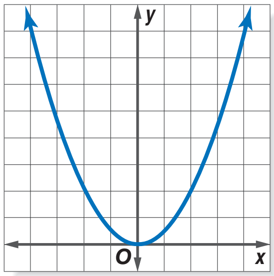

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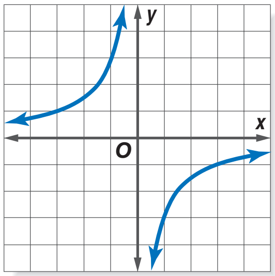

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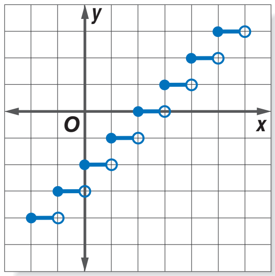


a. $y = [\![x]\!] - 2$
b. $y = 2|x|$
c. $y = 2\sqrt{x}$
d. $y = -3x$
e. $y = 0.5x^2$
f. $y = -\frac{3}{x + 1}$
g. $y = -\frac{3}{x}$

Real-World Link

When the Hope Diamond was shipped from New York to the Smithsonian Institution in Washington, D.C., it was mailed in a plain brown paper package.

Source: usps.com

HEALTH **For Exercises 28–30, use the following information.**
A woman painting a room will burn an average of 4.5 Calories per minute.

28. Write an equation for the number of Calories burned in m minutes.

29. Identify the equation in Exercise 28 as a type of function.

30. Describe the graph of the function.

31. **ARCHITECTURE** The shape of the Gateway Arch of the Jefferson National Expansion Memorial in St. Louis, Missouri, resembles the graph of the function $f(x) = -0.00635x^2 + 4.0005x - 0.07875$, where x is in feet. Describe the shape of the Gateway Arch.

MAIL **For Exercises 32 and 33, use the following information.**
In 2006, the cost to mail a first-class letter was 39¢ for any weight up to and including 1 ounce. Each additional ounce or part of an ounce added 24¢ to the cost.

32. Make a graph showing the postal rates to mail any letter from 0 to 8 ounces.

33. Compare your graph in Exercise 32 to the graph of the greatest integer function.

EXTRA PRACTICE
See pages 909, 933.

Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

34. **OPEN ENDED** Find a counterexample to the statement *All functions are continuous*. Describe your function.

35. **CHALLENGE** Identify each table of values as a type of function.

a.

x	f(x)
−5	7
−3	5
−1	3
0	2
1	3
3	5
5	7
7	9

b.

x	f(x)
−5	24
−3	8
−1	0
0	−1
1	0
3	8
5	24
7	48

c.

x	f(x)
−1.3	−1
−1.7	−1
0	1
0.8	1
0.9	1
1	2
1.5	2
2.3	3

d.

x	f(x)
−5	undefined
−3	undefined
−1	undefined
0	0
1	1
4	2
9	3
16	4

36. CHALLENGE Without graphing either function, explain how the graph of $y = [\![x + 2]\!] - 3$ is related to the graph of $y = [\![x + 1]\!] - 1$.

37. *Writing in Math* Use the information on page 473 to explain how the graph of a function can be used to determine the type of relationship that exists between the quantities represented by the domain and the range.

STANDARDIZED TEST PRACTICE

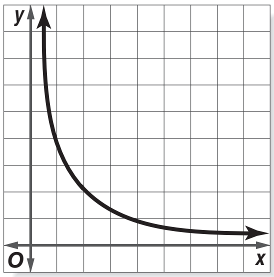
38. ACT/SAT The curve below could be part of the graph of which function?

A $y = \sqrt{x}$

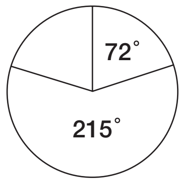
B $y = x^2 - 5x + 4$

C $xy = 4$

D $y = -x + 20$



39. REVIEW A paper plate with a 12-inch diameter is divided into 3 sections.



What is the approximate length of the arc of the largest section?

F 20.3 inches

G 22.5 inches

H 24.2 inches

J 26.5 inches

Spiral Review

40. If x varies directly as y and $y = \frac{1}{5}$ when $x = 11$, find x when $y = \frac{2}{5}$. (Lesson 8-4)

Graph each rational function. (Lesson 8-3)

41. $f(x) = \frac{3}{x + 2}$

42. $f(x) = \frac{8}{(x - 1)(x + 3)}$

43. $f(x) = \frac{x^2 - 5x + 4}{x - 4}$

Solve each equation by factoring. (Lesson 5-2)

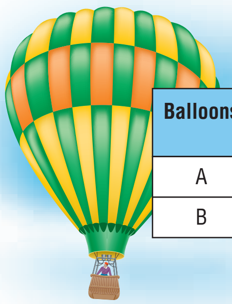
44. $x^2 + 6x + 8 = 0$

45. $2q^2 + 11q = 21$

HOT-AIR BALLOONS For Exercises 46 and 47, use the table. (Lesson 3-2)

46. If both balloons are launched at the same time, how long will it take for them to be the same distance from the ground?

47. What is the distance of the balloons from the ground at that time?



Balloons	Distance from Ground (m)	Rate of Ascension (m/min)
A	60	15
B	40	20

GET READY for the Next Lesson

PREREQUISITE SKILL Find the LCM of each set of polynomials. (Lesson 8-2)

48. $15ab^2c, 6a^3, 4bc^2$

49. $9x^3, 5xy^2, 15x^2y^3$

50. $5d - 10, 3d - 6$

51. $x^2 - y^2, 3x + 3y$

52. $a^2 - 2a - 3, a^2 - a - 6$

53. $2t^2 - 9t - 5, t^2 + t - 30$

8-6 Solving Rational Equations and Inequalities

Main Ideas

- Solve rational equations.
- Solve rational inequalities.

New Vocabulary

rational equation
rational inequality

GET READY for the Lesson

A music download service advertises downloads for \$1 per song. The service also charges a monthly access fee of \$15. If a customer downloads x songs in one month, the bill in dollars will be $15 + x$. The actual cost per song is $\frac{15 + x}{x}$. To find how many songs a person would need to download to make the actual cost per song \$1.25, you would need to solve the equation $\frac{15 + x}{x} = 1.25$.

Solve Rational Equations The equation $\frac{15 + x}{x} = 6$ is an example of a rational equation. In general, any equation that contains one or more rational expressions is called a **rational equation**.

Rational equations are easier to solve if the fractions are eliminated. You can eliminate the fractions by multiplying each side of the equation by the least common denominator (LCD). Remember that when you multiply each side by the LCD, each term on each side must be multiplied by the LCD.

EXAMPLE 1 Solve a Rational Equation

Solve $\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4}$. Check your solution.

The LCD for the terms is $28(z + 2)$.

$$\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4}$$ Original equation

$$28(z + 2)\left(\frac{9}{28} + \frac{3}{z + 2}\right) = 28(z + 2)\left(\frac{3}{4}\right)$$ Multiply each side by $28(z + 2)$.

$$\overset{1}{\cancel{28}}(z + 2)\left(\frac{9}{\underset{1}{\cancel{28}}}\right) + 28\overset{1}{\cancel{(z + 2)}}\left(\frac{3}{\underset{1}{\cancel{z + 2}}}\right) = \overset{7}{\cancel{28}}(z + 2)\left(\frac{3}{\underset{1}{\cancel{4}}}\right)$$ Distributive Property

$$(9z + 18) + 84 = 21z + 42$$ Simplify.

$$9z + 102 = 21z + 42$$ Simplify.

$$60 = 12z$$ Subtract $9z$ and 42 from each side.

$$5 = z$$ Divide each side by 12.

(continued on the next page)

CHECK $\frac{9}{28} + \frac{3}{z+2} = \frac{3}{4}$ Original equation

$\frac{9}{28} + \frac{3}{5+2} \stackrel{?}{=} \frac{3}{4}$ $z = 5$

$\frac{9}{28} + \frac{3}{7} \stackrel{?}{=} \frac{3}{4}$ Simplify.

$\frac{9}{28} + \frac{12}{28} \stackrel{?}{=} \frac{3}{4}$ Simplify.

$\frac{3}{4} = \frac{3}{4}$ ✓ The solution is correct.

CHECK Your Progress

Solve each equation. Check your solution.

1A. $\frac{5}{6} + \frac{2}{x-6} = \frac{1}{2}$

1B. $\frac{7}{12} + \frac{9}{x-4} = \frac{55}{48}$

When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from your list of solutions.

EXAMPLE 2 Elimination of a Possible Solution

Solve $r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$. Check your solution.

$r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$ Original equation; the LCD is $(r^2 - 1)$.

$(r^2-1)\left(r + \frac{r^2-5}{r^2-1}\right) = (r^2-1)\left(\frac{r^2+r+2}{r+1}\right)$ Multiply each side by the LCD.

$(r^2-1)r + \overset{1}{\cancel{(r^2-1)}}\left(\frac{r^2-5}{\underset{1}{\cancel{r^2-1}}}\right) = \cancel{(r^2-1)}\left(\frac{r^2+r+2}{\underset{1}{\cancel{r+1}}}\right)$ Distributive Property

$(r^3 - r) + (r^2 - 5) = (r-1)(r^2+r+2)$ Simplify.

$r^3 + r^2 - r - 5 = r^3 + r - 2$ Simplify.

$r^2 - 2r - 3 = 0$ Subtract $(r^3 + r - 2)$ from each side.

$(r-3)(r+1) = 0$ Factor.

$r - 3 = 0$ or $r + 1 = 0$ Zero Product Property

$r = 3$ $\quad$ $r = -1$

CHECK $r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$

$3 + \frac{3^2-5}{3^2-1} \stackrel{?}{=} \frac{3^2+3+2}{3+1}$

$3 + \frac{4}{8} \stackrel{?}{=} \frac{14}{4}$

$\frac{7}{2} = \frac{7}{2}$ ✓

$r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$

$-1 + \frac{(-1)^2-5}{(-1)^2-1} \stackrel{?}{=} \frac{(-1)^2+(-1)+2}{-1+1}$

$-1 + \frac{-4}{0} \stackrel{?}{=} \frac{2}{0}$

Since $r = -1$ results in a zero in the denominator, eliminate -1 from the list of solutions. The solution is 3.

Study Tip

Extraneous Solutions

Multiplying each side of an equation by the LCD of rational expressions can yield results that are not solutions of the original equation. These solutions are called *extraneous solutions*.

CHECK Your Progress Solve each equation. Check your solution.

2A. $\frac{2}{r+1} - \frac{1}{r-1} = \frac{-2}{r^2-1}$ **2B.** $\frac{7n}{3n+3} - \frac{5}{4n-4} = \frac{3n}{2n+2}$

Real-World EXAMPLE

Personal Tutor

Real-World Link

The Loetschberg Tunnel is 21 miles long. It was created for train travel and cut travel time between the locations in half.

Source: usatoday.com

3 **TUNNELS** **The Loetschberg tunnel was built to connect Bern, Switzerland, with the ski resorts in the southern Swiss Alps. The Swiss used one company that started at the north end and another company that started at the south end. Suppose the company at the north end could drill the entire tunnel in 22.2 years and the south company could do it in 21.8 years. How long would it have taken the two companies to drill the tunnel?**

In 1 year, the north company could complete $\frac{1}{22.2}$ of the tunnel.

In 2 years, the north company could complete $\frac{1}{22.2} \cdot 2$ or $\frac{2}{22.2}$ of the tunnel.

In t years, the north company could complete $\frac{1}{22.2} \cdot t$ or $\frac{t}{22.2}$ of the tunnel.

Likewise, in t years, the south company could complete $\frac{1}{21.8} \cdot t$ or $\frac{t}{21.8}$ of the tunnel.

Together, they completed the whole tunnel.

Part completed by the north company	plus	part completed by the south company	equals	entire tunnel.
$\frac{t}{22.2}$	$+$	$\frac{t}{21.8}$	$=$	1

$\frac{t}{22.2} + \frac{t}{21.8} = 1$ Original equation

$483.96\left(\frac{t}{22.2} + \frac{t}{21.8}\right) = 483.96(1)$ Multiply each side by 483.96.

$21.8t + 22.2t = 483.96$ Simplify.

$44t = 483.96$ Simplify.

$t \approx 11$ Divide each side by 44.

It would have taken about 11 years to build the tunnel.

This answer is reasonable. Working alone, either company could have drilled the tunnel in about 22 years. Working together, they must be able to do it in about half that time.

CHECK Your Progress

3. WORK Breanne and Owen paint houses together. If Breanne can paint a particular house in 6 days and Owen can paint the same house in 5 days, how long would it take the two of them if they work together?

Online Personal Tutor at algebra2.com

Rate problems frequently involve rational equations.

Real-World EXAMPLE

4 **NAVIGATION** **The speed of the current in the Puget sound is 5 miles per hour. A barge travels 26 miles with the current and returns in $10\frac{2}{3}$ hours. What is the speed of the barge in still water?**

Words The formula that relates distance, time, and rate is $d = rt$ or $\frac{d}{r} = t$.

Variables Let r = the speed of the barge in still water. Then the speed of the barge with the current is $r + 5$, and the speed of the barge against current is $r - 5$.

	Time going with the current	plus	time going against the current	equals	total time.
Equation	$\frac{26}{r+5}$	$+$	$\frac{26}{r-5}$	$=$	$10\frac{2}{3}$

$$\frac{26}{r+5} + \frac{26}{r-5} = 10\frac{2}{3}$$ Original equation

$$3(r^2 - 25)\left(\frac{26}{r+5} + \frac{26}{r-5}\right) = 3(r^2 - 25)10\frac{2}{3}$$ Multiply each side by $3(r^2 - 25)$.

$$3(r^2 - 25)\left(\frac{26}{r+5}\right) + 3(r^2 - 25)\left(\frac{26}{r-5}\right) = 3(r^2 - 25)\left(\frac{32}{3}\right)$$ Distributive Property

$$(78r - 390) + (78r + 390) = 32r^2 - 800$$ Simplify.

$$156r = 32r^2 - 800$$ Simplify.

$$0 = 32r^2 - 156r - 800$$ Subtract $156r$ from each side.

$$0 = 8r^2 - 39r - 200$$ Divide each side by 4.

Use the Quadratic Formula to solve for r.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$r = \frac{-(-39) \pm \sqrt{(-39)^2 - 4(8)(-200)}}{2(8)}$$ $x = r$, $a = 8$, $b = -39$, and $c = -200$

$$r = \frac{39 \pm \sqrt{7921}}{16}$$ Simplify.

$$r = \frac{39 \pm 89}{16}$$ Simplify.

$$r = 8 \text{ or } -3.125$$ Simplify.

Since speed must be positive, it is 8 miles per hour. Is this answer reasonable?

Study Tip

Look Back

To review the **Quadratic Formula**, see Lesson 5-6.

In Motion

In Motion for MAC

Concepts in Motion

Interactive Lab
algebra2.com

CHECK Your Progress

4. **SWIMMING** The speed of the current in a body of water is 1 mile per hour. Juan swims 2 miles against the current and 2 miles with the current in a total time of $2\frac{2}{3}$ hours. How fast can Juan swim in still water?

Solve Rational Inequalities Inequalities that contain one or more rational expressions are called **rational inequalities**. To solve rational inequalities, complete the following steps.

Step 1 State the excluded values.

Step 2 Solve the related equation.

Step 3 Use the values determined in Steps 1 and 2 to divide a number line into intervals. Test a value in each interval to determine which intervals contain values that satisfy the original inequality.

EXAMPLE 5 Solve a Rational Inequality

Solve $\frac{1}{4a} + \frac{5}{8a} > \frac{1}{2}$.

Step 1 Values that make a denominator equal to 0 are excluded from the domain. For this inequality, the excluded value is 0.

Step 2 Solve the related equation.

$$\frac{1}{4a} + \frac{5}{8a} = \frac{1}{2} \quad \text{Related equation}$$

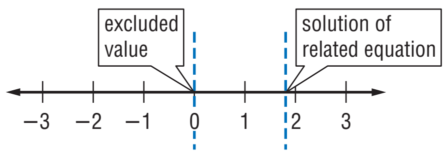
$$8a\left(\frac{1}{4a} + \frac{5}{8a}\right) = 8a\left(\frac{1}{2}\right) \quad \text{Multiply each side by } 8a.$$

$$2 + 5 = 4a \quad \text{Simplify.}$$

$$7 = 4a \quad \text{Add.}$$

$$1\tfrac{3}{4} = a \quad \text{Divide each side by 4.}$$

Step 3 Draw vertical lines at the excluded value and at the solution to separate the number line into intervals.



Now test a sample value in each interval to determine if the values in the interval satisfy the inequality.

Test $a = -1$.

$$\frac{1}{4(-1)} + \frac{5}{8(-1)} \overset{?}{>} \frac{1}{2}$$

$$-\frac{1}{4} - \frac{5}{8} \overset{?}{>} \frac{1}{2}$$

$$-\frac{7}{8} \not> \frac{1}{2}$$

$a < 0$ is *not* a solution.

Test $a = 1$.

$$\frac{1}{4(1)} + \frac{5}{8(1)} \overset{?}{>} \frac{1}{2}$$

$$\frac{1}{4} + \frac{5}{8} \overset{?}{>} \frac{1}{2}$$

$$\frac{7}{8} > \frac{1}{2} \checkmark$$

$0 < a < 1\frac{3}{4}$ is a solution.

Test $a = 2$.

$$\frac{1}{4(2)} + \frac{5}{8(2)} \overset{?}{>} \frac{1}{2}$$

$$\frac{1}{8} + \frac{5}{16} \overset{?}{>} \frac{1}{2}$$

$$\frac{7}{16} \not> \frac{1}{2}$$

$a > 1\frac{3}{4}$ is *not* a solution.

The solution is $0 < a < 1\frac{3}{4}$.

CHECK Your Progress

Solve each inequality.

5A. $\frac{1}{3b} - \frac{2}{5b} < \frac{1}{15}$

5B. $1 + \frac{5}{x-1} \leq \frac{7}{6}$

CHECK Your Understanding

Example 1 (pp. 479–480)

Solve each equation. Check your solutions.

1. $\frac{2}{d} + \frac{1}{4} = \frac{11}{12}$

2. $t + \frac{12}{t} - 8 = 0$

3. $\frac{1}{x-1} + \frac{2}{x} = 0$

4. $\frac{12}{v^2 - 16} - \frac{24}{v-4} = 3$

Example 2 (p. 480)

5. $\frac{w}{w-1} + w = \frac{4w-3}{w-1}$

6. $\frac{4n^2}{n^2-9} - \frac{2n}{n+3} = \frac{3}{n-3}$

Examples 3, 4 (pp. 481, 482)

7. **WORK** A worker can powerwash a wall of a certain size in 5 hours. Another worker can do the same job in 4 hours. If the workers work together, how long would it take to do the job? Determine whether your answer is reasonable.

Example 5 (p. 483)

Solve each inequality.

8. $\frac{4}{c+2} > 1$

9. $\frac{1}{3v} + \frac{1}{4v} < \frac{1}{2}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–15	1
16, 17	2
18–21	5
22, 23	3, 4

Solve each equation or inequality. Check your solutions.

10. $\frac{y}{y+1} = \frac{2}{3}$

11. $\frac{p}{p-2} = \frac{2}{5}$

12. $s + 5 = \frac{6}{s}$

13. $a + 1 = \frac{6}{a}$

14. $\frac{9}{t-3} = \frac{t-4}{t-3} + \frac{1}{4}$

15. $\frac{5}{x+1} - \frac{1}{3} = \frac{x+2}{x+1}$

16. $\frac{2}{y+2} - \frac{y}{2-y} = \frac{y^2+4}{y^2-4}$

17. $\frac{1}{d+4} = \frac{2}{d^2+3d-4} - \frac{1}{1-d}$

18. $\frac{7}{a+1} > 7$

19. $\frac{10}{m+1} > 5$

20. $5 + \frac{1}{t} > \frac{16}{t}$

21. $7 - \frac{2}{b} < \frac{5}{b}$

22. **NUMBER THEORY** The ratio of 16 more than a number to 12 less than that number is 1 to 3. What is the number?

23. **NUMBER THEORY** The sum of a number and 8 times its reciprocal is 6. Find the number(s).

Solve each equation or inequality. Check your solutions.

24. $\frac{b-4}{b-2} = \frac{b-2}{b+2} + \frac{1}{b-2}$

25. $\frac{1}{n-2} = \frac{2n+1}{n^2+2n-8} + \frac{2}{n+4}$

26. $\frac{2q}{2q+3} - \frac{2q}{2q-3} = 1$

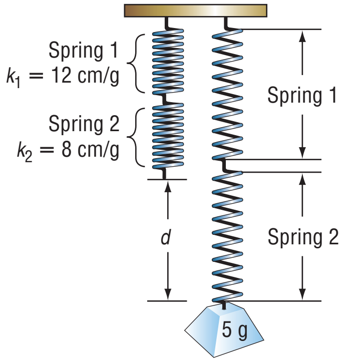
27. $\frac{4}{z-2} - \frac{z+6}{z+1} = 1$

28. $\frac{2}{3y} + \frac{5}{6y} > \frac{3}{4}$

29. $\frac{1}{2p} + \frac{3}{4p} < \frac{1}{2}$

30. **ACTIVITIES** The band has 30 more members than the school chorale. If each group had 10 more members, the ratio of their membership would be 3:2. How many members are in each group?

PHYSICS **For Exercises 31 and 32, use the following information.**
The distance a spring stretches is related to the mass attached to the spring. This is represented by $d = km$, where d is the distance, m is the mass, and k is the spring constant. When two springs with spring constants k_1 and k_2 are attached in a series, the resulting spring constant k is found by the equation $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$.



31. If one spring with constant of 12 centimeters per gram is attached in a series with another spring with constant of 8 centimeters per gram, find the resultant spring constant.

32. If a 5-gram object is hung from the series of springs, how far will the springs stretch? Is this answer reasonable in this context?

Real-World Career
Chemist
Many chemists work for manufacturers developing products or doing quality control to ensure the products meet industry and government standards.

Math Online
For more information, go to algebra2.com.

33. CYCLING On a particular day, the wind added 3 kilometers per hour to Alfonso's rate when he was cycling with the wind and subtracted 3 kilometers per hour from his rate on his return trip. Alfonso found that in the same amount of time he could cycle 36 kilometers with the wind, he could go only 24 kilometers against the wind. What is his normal bicycling speed with no wind? Determine whether your answer is reasonable.

34. CHEMISTRY Kiara adds an 80% acid solution to 5 milliliters of solution that is 20% acid. The function that represents the percent of acid in the resulting solution is $f(x) = \frac{5(0.20) + x(0.80)}{5 + x}$, where x is the amount of 80% solution added. How much 80% solution should be added to create a solution that is 50% acid?

35. NUMBER THEORY The ratio of 3 more than a number to the square of 1 more than that number is less than 1. Find the numbers which satisfy this statement.

EXTRA PRACTICE
See pages 909, 933.
Math Online
Self-Check Quiz at algebra2.com

STATISTICS **For Exercises 36 and 37, use the following information.**
A number x is the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$.

36. Eight is the harmonic mean of 20 and what number?

37. What is the harmonic mean of 5 and 8?

H.O.T. Problems

38. OPEN ENDED Write a rational equation that can be solved by first multiplying each side by $5(a + 2)$.

39. FIND THE ERROR Jeff and Dustin are solving $2 - \frac{3}{a} = \frac{2}{3}$. Who is correct? Explain your reasoning.

Jeff	Dustin
$2 - \frac{3}{a} = \frac{2}{3}$	$2 - \frac{3}{a} = \frac{2}{3}$
$6a - 9 = 2a$	$2 - 9 = 2a$
$4a = 9$	$-7 = 2a$
$a = 2.25$	$-3.5 = a$

40. CHALLENGE Solve for a if $\frac{1}{a} - \frac{1}{b} = c$.

41. *Writing in Math* Use the information about music downloads on page 479 to explain how rational equations are used to solve problems involving unit price. Include an explanation of why the actual price per download could never be $1.00.

STANDARDIZED TEST PRACTICE

42. **ACT/SAT** Amanda wanted to determine the average of her 6 test scores. She added the scores correctly to get T, but divided by 7 instead of 6. The result was 12 less than her actual average. Which equation could be used to determine the value of T?

A $6T + 12 = 7T$

B $\frac{T}{7} = \frac{T - 12}{6}$

C $\frac{T}{7} + 12 = \frac{T}{6}$

D $\frac{T}{6} = \frac{T - 12}{7}$

43. **REVIEW**
What is $\frac{10a^{-3}}{29b^4} \div \frac{5a^{-5}}{16b^{-7}}$?

F $\frac{25b^3}{232a^8}$

G $\frac{25}{232a^2b^3}$

H $\frac{32b^3}{29a^8}$

J $\frac{32a^2}{29b^{11}}$

Spiral Review

Identify the type of function represented by each equation. Then graph the equation. (Lesson 8-5)

44. $y = 2x^2 + 1$

45. $y = 2\sqrt{x}$

46. $y = 0.8x$

47. If y varies inversely as x and $y = 24$ when $x = 9$, find y when $x = 6$. (Lesson 8-4)

Solve each inequality. (Lesson 5-8)

48. $(x + 11)(x - 3) > 0$

49. $x^2 - 4x \leq 0$

50. $2b^2 - b < 6$

Find each product, if possible. (Lesson 4-3)

51. $\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 & -3 \\ 8 & -4 & 9 \end{bmatrix}$

52. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

53. **HEALTH** The prediction equation $y = 205 - 0.5x$ relates a person's maximum heart rate for exercise y and age x. Use the equation to find the maximum heart rate for an 18-year-old. (Lesson 2-5)

Determine the value of r so that a line through the points with the given coordinates has the given slope. (Lesson 2-3)

54. $(r, 2), (4, -6)$; slope $= -\frac{8}{3}$

55. $(r, 6), (8, 4)$; slope $= \frac{1}{2}$

56. Evaluate $[(-7 + 4) \times 5 - 2] \div 6$. (Lesson 1-1)

EXTEND 8-6

Graphing Calculator Lab

Solving Rational Equations and Inequalities with Graphs and Tables

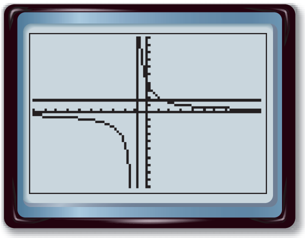
You can use a TI-83/84 graphing calculator to solve rational equations by graphing or by using the table feature. Graph both sides of the equation and locate the point(s) of intersection.

ACTIVITY 1 Solve $\frac{4}{x+1} = \frac{3}{2}$.

Step 1 **Graph each side of the equation.**

Graph each side of the equation as a separate function. Enter $\frac{4}{x+1}$ as Y1 and $\frac{3}{2}$ as Y2. Then graph the two equations.

KEYSTROKES: [Y=] 4 [÷] [(] [X,T,θ,n] [+] 1 [)] [ENTER] 3 [÷] 2 [ZOOM] 6



[−10, 10] scl: 1 by [−10, 10] scl: 1

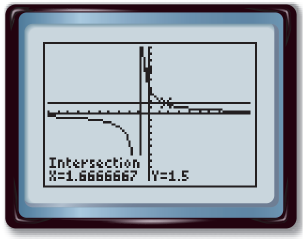
Because the calculator is in connected mode, a vertical line may appear connecting the two branches of the graph. This is not part of the graph.

Step 2 **Use the intersect feature.**

The intersect feature on the [CALC] menu allows you to approximate the ordered pair of the point at which the graphs cross.

KEYSTROKES: [2nd] [CALC] 5

Select one graph and press [ENTER]. Select the other graph, press [ENTER], and press [ENTER] again.



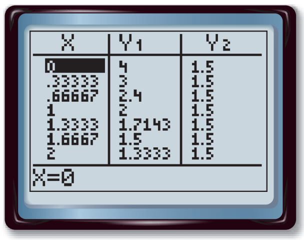
[−10, 10] scl: 1 by [−10, 10] scl: 1

The solution is $1\frac{2}{3}$.

Step 3 **Use the table feature.**

Verify the solution using the table feature. Set up the table to show x-values in increments of $\frac{1}{3}$.

KEYSTROKES: [2nd] [TBLSET] 0 [ENTER] 1 [÷] 3 [ENTER] [2nd] [TABLE]



The table displays x-values and corresponding y-values for each graph. At $x = 1\frac{2}{3}$, both functions have a y-value of 1.5. Thus, the solution of the equation is $1\frac{2}{3}$.

You can use a similar procedure to solve rational inequalities using a graphing calculator.

ACTIVITY 2 **Solve $\frac{3}{x} + \frac{7}{x} > 9$.**

Step 1 **Enter the inequalities.**

Rewrite the problem as a system of inequalities.

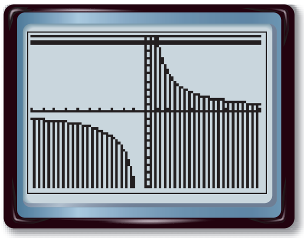
The first inequality is $\frac{3}{x} + \frac{7}{x} > y$ or $y < \frac{3}{x} + \frac{7}{x}$. Since this inequality includes the *less than* symbol, shade below the curve. First, enter the boundary and then use the arrow and [ENTER] keys to choose the shade below icon, .

The second inequality is $y > 9$. Shade above the curve since this inequality contains *greater than*.

KEYSTROKES: [◀] [◀] [ENTER] [ENTER] [ENTER] [▶] [▶] 3 [÷] [X,T,θ,n] [+] 7 [÷] [X,T,θ,n] [ENTER] [◀] [◀] [ENTER] [ENTER] [▶] [▶] 9 [GRAPH]

Step 2 **Graph the system.**

KEYSTROKES: [GRAPH]



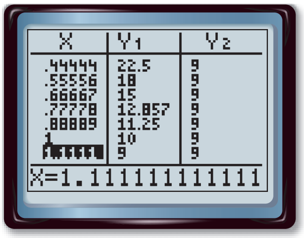
[−10, 10] scl: 1 by [−10, 10] scl: 1

The solution set of the original inequality is the set of x-values of the points in the region where the shadings overlap. Using the calculator's intersect feature, you can conclude that the solution set is $\left\{x \middle| 0 > x > 1\frac{1}{9}\right\}$.

Step 3 **Use the table feature.**

Verify using the table feature. Set up the table to show x-values in increments of $\frac{1}{9}$.

KEYSTROKES: [2nd] [TBLSET] 0 [ENTER] 1 [÷] 9 [ENTER] [2nd] [TABLE]



Scroll through the table. Notice that for x-values greater than 0 and less than $1\frac{1}{9}$, Y1 > Y2. This confirms that the solution of the inequality is $\left\{x \middle| 0 > x > 1\frac{1}{9}\right\}$.

Exercises

Solve each equation or inequality.

1. $\frac{1}{x} + \frac{1}{2} = \frac{2}{x}$

2. $\frac{1}{x-4} = \frac{2}{x-2}$

3. $\frac{4}{x} = \frac{6}{x^2}$

4. $\frac{1}{1-x} = 1 - \frac{x}{x-1}$

5. $\frac{1}{x+4} = \frac{2}{x^2+3x-4} - \frac{1}{1-x}$

6. $\frac{1}{x} + \frac{1}{2x} > 5$

7. $\frac{1}{x-1} + \frac{2}{x} < 0$

8. $1 + \frac{5}{x-1} \leq 0$

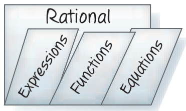
9. $2 + \frac{1}{x-1} \geq 0$

CHAPTER 8 Study Guide and Review

Download Vocabulary Review from algebra2.com

FOLDABLES™ Study Organizer — GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Rational Expressions (Lessons 8-1 and 8-2)

- Multiplying and dividing rational expressions is similar to multiplying and dividing fractions.
- To simplify complex fractions, simplify the numerator and the denominator separately, and then simplify the resulting expression.

Direct, Joint, and Inverse Variation (Lesson 8-4)

- Direct Variation: There is a nonzero constant k such that $y = kx$.
- Joint Variation: There is a number k such that $y = kxz$, where $x \neq 0$ and $z \neq 0$.
- Inverse Variation: There is a nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$.

Classes of Functions (Lesson 8-5)

- The following functions can be classified as special functions: constant function, direct variation function, identity function, greatest integer function, absolute value function, quadratic function, square root function, rational function, inverse variation function.

Rational Equations and Inequalities (Lesson 8-6)

- Eliminate fractions in rational equations by multiplying each side of the equation by the LCD.
- Possible solutions of a rational equation must exclude values that result in zero in the denominator.

Key Vocabulary

asymptote (p. 457)
complex fraction (p. 445)
constant of variation (p. 465)
continuity (p. 457)
direct variation (p. 465)
inverse variation (p. 467)
joint variation (p. 466)
point discontinuity (p. 457)
rational equation (p. 479)
rational expression (p. 442)
rational function (p. 457)
rational inequality (p. 483)

Vocabulary Check

State whether each sentence is *true* or *false*. If *false*, replace the underlined word or number to make a true sentence.

1. The equation $y = \frac{x^2 - 1}{x + 1}$ has a(n) <u>asymptote</u> at $x = -1$.
2. The equation $y = 3x$ is an example of a(n) <u>direct</u> variation equation.
3. The equation $y = \frac{x^2}{x + 1}$ is a(n) <u>polynomial</u> equation.
4. The graph of $y = \frac{4}{x - 4}$ has a(n) <u>variation</u> at $x = 4$.
5. The equation $b = \frac{2}{a}$ is a(n) <u>inverse</u> variation equation.
6. On the graph of $y = \frac{x - 5}{x + 2}$, there is a break in continuity at $x = \underline{2}$.
7. The expression $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$ is an example of a <u>complex fraction</u>.
8. In the direct variation $y = 6x$, 6 is the <u>degree</u>.

Vocabulary Review at algebra2.com

Lesson-by-Lesson Review

8–1 Multiplying and Dividing Rational Expressions (pp. 442–449)

Simplify each expression.

9. $\frac{-4ab}{21c} \cdot \frac{14c^2}{22a^2}$

10. $\frac{a^2 - b^2}{6b} \div \frac{a + b}{36b^2}$

11. $\frac{\frac{x^2 + 7x + 10}{x + 2}}{\frac{x^2 + 2x - 15}{x + 2}}$

12. $\frac{\frac{1}{n^2 - 6n + 9}}{\frac{n + 3}{2n^2 - 18}}$

13. $\frac{y^2 - y - 12}{y + 2} \div \frac{y - 4}{y^2 - 4y - 12}$

14. $\frac{x^2 + 3x - 10}{x^2 + 8x + 15} \cdot \frac{x^2 + 5x + 6}{x^2 + 4x + 4}$

15. **GEOMETRY** A triangle has an area of $2x^2 + 4x - 16$ square meters. If the base is $x - 2$ meters, find the height.

Example 1 Simplify $\frac{3x}{2y} \cdot \frac{8y^3}{6x^2}$.

$$\frac{3x}{2y} \cdot \frac{8y^3}{6x^2} = \frac{\cancel{3} \cdot \cancel{x} \cdot \cancel{2} \cdot \cancel{2} \cdot 2 \cdot \cancel{y} \cdot y \cdot y}{\cancel{2} \cdot \cancel{y} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{x} \cdot x}$$

$$= \frac{2y^2}{x}$$

Example 2 Simplify $\frac{p^2 + 7p}{3p} \div \frac{49 - p^2}{3p - 21}$.

$$\frac{p^2 + 7p}{3p} \div \frac{49 - p^2}{3p - 21} = \frac{p^2 + 7p}{3p} \cdot \frac{3p - 21}{49 - p^2}$$

$$= \frac{\cancel{p}\cancel{(7 + p)}}{\cancel{3p}} \cdot \frac{-\cancel{3}\cancel{(7 - p)}}{\cancel{(7 + p)}\cancel{(7 - p)}}$$

$$= -1$$

8–2 Adding and Subtracting Rational Expressions (pp. 450–456)

Simplify each expression.

16. $\frac{x + 2}{x - 5} + 6$

17. $\frac{x - 1}{x^2 - 1} + \frac{2}{5x + 5}$

18. $\frac{7}{y} - \frac{2}{3y}$

19. $\frac{7}{y - 2} - \frac{11}{2 - y}$

20. $\frac{3}{4b} - \frac{2}{5b} - \frac{1}{2b}$

21. $\frac{m + 3}{m^2 - 6m + 9} - \frac{8m - 24}{9 - m^2}$

BIOLOGY For Exercises 22 and 23, use the following information.

After a person eats something, the pH or acid level A of their mouth can be determined by the formula $A = -\frac{20.4t}{t^2 + 36} + 6.5$, where t is the number of minutes that have elapsed since the food was eaten.

22. Simplify the equation.

23. What would the acid level be after 30 minutes?

Example 3 Simplify $\frac{14}{x + y} - \frac{9x}{x^2 - y^2}$.

$$\frac{14}{x + y} - \frac{9x}{x^2 - y^2} = \frac{14}{x + y} - \frac{9x}{(x + y)(x - y)}$$

$$= \frac{14(x - y)}{(x + y)(x - y)} - \frac{9x}{(x + y)(x - y)}$$

$$= \frac{14(x - y) - 9x}{(x + y)(x - y)}$$ Subtract the numerators.

$$= \frac{14x - 14y - 9x}{(x + y)(x - y)}$$ Distributive Property

$$= \frac{5x - 14y}{(x + y)(x - y)}$$ Simplify.

Mixed Problem Solving
For mixed problem-solving practice, see page 801.

8–3 Graphing Rational Functions (pp. 457–463)

Graph each rational function.

24. $f(x) = \frac{4}{x - 2}$

25. $f(x) = \frac{x}{x + 3}$

26. $f(x) = \frac{2}{x}$

27. $f(x) = \frac{x^2 + 2x + 1}{x + 1}$

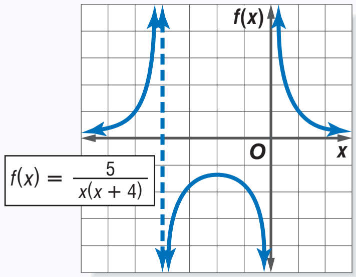
28. $f(x) = \frac{x - 4}{x + 3}$

29. $f(x) = \frac{5}{(x + 1)(x - 3)}$

30. SANDWICHES A group makes 45 sandwiches to take on a picnic. The number of sandwiches a person can eat depends on how many people go on the trip. Write and graph a function to illustrate this situation.

Example 4 **Graph $f(x) = \frac{5}{x(x + 4)}$.**

The function is undefined for $x = 0$ and $x = -4$. Since $\frac{5}{x(x + 4)}$ is in simplest form, $x = 0$ and $x = -4$ are vertical asymptotes. Draw the two asymptotes and sketch the graph.



8–4 Direct, Joint, and Inverse Variation (pp. 465–471)

31. If y varies directly as x and $y = 21$ when $x = 7$, find x when $y = -5$.

32. If y varies inversely as x and $y = 9$ when $x = 2.5$, find y when $x = -0.6$.

33. If y varies inversely as x and $y = -4$ when $x = 8$, find y when $x = -121$.

34. If y varies jointly as x and z and $x = 2$ and $z = 4$ when $y = 16$, find y when $x = 5$ and $z = 8$.

35. If y varies jointly as x and z and $y = 14$ when $x = 10$ and $z = 7$, find y when $x = 11$ and $z = 8$.

36. EMPLOYMENT Chris's pay varies directly with how many lawns he mows. If his pay is \$65 for 5 yards, find his pay after he has mowed 13 yards.

Example 5 **If y varies inversely as x and $x = 14$ when $y = -6$, find x when $y = -11$.**

$\frac{x_1}{y_2} = \frac{x_2}{y_1}$ Inverse variation

$\frac{14}{-11} = \frac{x_2}{-6}$ $x_1 = 14, y_1 = -6, y_2 = -11$

$14(-6) = -11(x_2)$ Cross multiply.

$-84 = -11x_2$ Simplify.

$7\frac{7}{11} = x_2$ Divide each side by -11.

When $y = -11$, the value of x is $7\frac{7}{11}$.

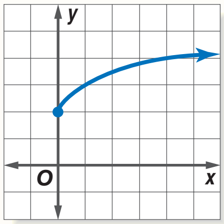
CHAPTER 8

Study Guide and Review

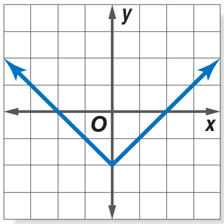
8-5 Classes of Functions (pp. 473–478)

Identify the type of function represented by each graph.

37.

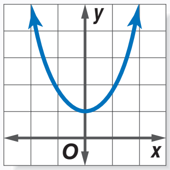


38.



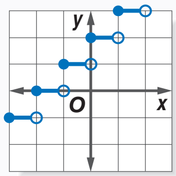
Example 6 **Identify the type of function represented by each graph.**

a.



The graph has a parabolic shape; therefore, it is a quadratic function.

b.



The graph has a stair-step pattern; therefore, it is a greatest integer function.

8-6 Solving Rational Equations and Inequalities (pp. 479–486)

Solve each equation or inequality. Check your solutions.

39. $\frac{3}{y} + \frac{7}{y} = 9$

40. $\frac{3x + 2}{4} = \frac{9}{4} - \frac{3 - 2x}{6}$

41. $\frac{1}{r^2 - 1} = \frac{2}{r^2 + r - 2}$

42. $\frac{x}{x^2 - 1} + \frac{2}{x + 1} = 1 + \frac{1}{2x - 2}$

43. $\frac{1}{3b} - \frac{3}{4b} > \frac{1}{6}$

44. **PUZZLES** Danielle can put a puzzle together in three hours. Aidan can put the same puzzle together in five hours. How long will it take them if they work together?

Example 7 **Solve** $\frac{1}{x - 1} + \frac{2}{x} = 0$.

The LCD is $x(x - 1)$.

$$\frac{1}{x - 1} + \frac{2}{x} = 0$$

$$x(x - 1)\left(\frac{1}{x - 1} + \frac{2}{x}\right) = x(x - 1)(0)$$

$$x(x - 1)\left(\frac{1}{x - 1}\right) + x(x - 1)\left(\frac{2}{x}\right) = x(x - 1)(0)$$

$$1(x) + 2(x - 1) = 0$$

$$x + 2x - 2 = 0$$

$$3x - 2 = 0$$

$$3x = 2$$

$$x = \frac{2}{3}$$

Chapter 8 Practice Test

Simplify each expression.

1. $\frac{a^2 - ab}{3a} \div \frac{a - b}{15b^2}$

2. $\frac{x^2 - y^2}{y^2} \cdot \frac{y^3}{y - x}$

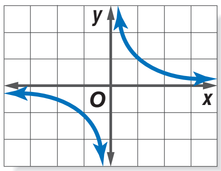
3. $\frac{x^2 - 2x + 1}{y - 5} \div \frac{x - 1}{y^2 - 25}$

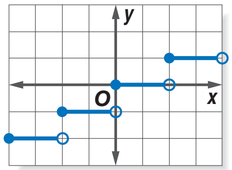
4. $\dfrac{\frac{x^2 - 1}{x^2 - 3x - 10}}{\frac{x^2 + 3x + 2}{x^2 - 12x + 35}}$

5. $\frac{x - 2}{x - 1} + \frac{6}{7x - 7}$

6. $\frac{x}{x^2 - 9} + \frac{1}{2x + 6}$

Identify the type of function represented by each graph.

7. 

8. 

Graph each rational function.

9. $f(x) = \frac{-4}{x - 3}$

10. $f(x) = \frac{2}{(x - 2)(x + 1)}$

Solve each equation or inequality.

11. $\frac{2}{x - 1} = 4 - \frac{x}{x - 1}$

12. $\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4}$

13. $5 + \frac{3}{t} > -\frac{2}{t}$

14. $x + \frac{12}{x} - 8 = 0$

15. $\frac{5}{6} - \frac{2m}{2m + 3} = \frac{19}{6}$

16. $\frac{x - 3}{2x} = \frac{x - 2}{2x + 1} - \frac{1}{2}$

17. If y varies inversely as x and $y = 9$ when $x = -\frac{2}{3}$, find x when $y = -7$.

18. If g varies directly as w and $g = 10$ when $w = -3$, find w when $g = 4$.

19. Suppose y varies jointly as x and z. If $x = 10$ when $y = 250$ and $z = 5$, find x when $y = 2.5$ and $z = 4.5$.

20. **AUTO MAINTENANCE** When air is pumped into a tire, the pressure required varies inversely as the volume of the air. If the pressure is 30 pounds per square inch when the volume is 140 cubic inches, find the pressure when the volume is 100 cubic inches.

21. **WORK** Sofia and Julie must pick up all of the apples in the yard so the lawn can be mowed. Working alone, Julie could complete the job in 1.7 hours. Sofia could complete it alone in 2.3 hours. How long will it take them to complete the job when they work together?

ELECTRICITY For Exercises 22 and 23, use the following information.

The current I in a circuit varies inversely with the resistance R.

22. Use the table below to write an equation relating the current and the resistance.

I	0.5	1.0	1.5	2.0	2.5	3.0	5.0
R	12.0	6.0	4.0	3.0	2.4	2.0	1.2

23. What is the constant of variation?

24. **MULTIPLE CHOICE** If $m = \frac{1}{x}$, $n = 7m$, $p = \frac{1}{n}$, $q = 14p$, and $r = \frac{1}{\frac{1}{2}q}$, find x.

A r **B** q **C** p **D** $\frac{1}{r}$

Standardized Test Practice

Chapter 8

Cumulative, Chapters 1–8

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Jamal is putting a stone walkway around a circular pond. He has enough stones to make a walkway 144 feet long. If he uses all of the stones to surround the pond, what is the radius of the pond?

A $\frac{144}{\pi}$ ft

B $\frac{72}{\pi}$ ft

C 144π ft

D 72π ft

2. Hooke's Law states that the force needed to keep a spring stretched x units is directly proportional to x. If a spring is stretched 5 centimeters, and a force of 40 N is required to maintain the spring stretched to 5 centimeters, what force is needed to keep the spring stretched 14 centimeters?

F 8 N

G 19 N

H 112 N

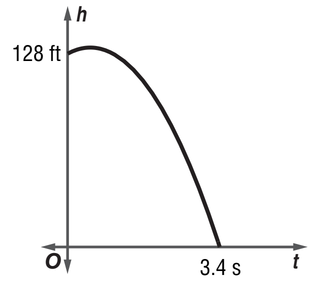
J 1600 N

3. GRIDDABLE Perry drove to the gym at an average rate of 30 miles per hour. It took him 45 minutes. Going home, he took the same route, but drove at a rate of 45 miles per hour. How many miles is it to his house from the gym?

TEST-TAKING TIP

Question 3 When answering questions, make sure you know exactly what the question is asking you to find. For example, if you find the time that it takes him to drive home from the gym in question 3, you have not solved the problem. You need to find the distance the gym is from his home.

4. A ball was thrown upward with an initial velocity of 16 feet per second from the top of a building 128 feet high. Its height h in feet above the ground t seconds later will be $h = 128 + 16t - 16t^2$.



Which is the best conclusion about the ball's action?

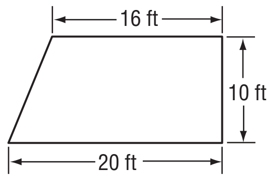
A The ball stayed above 128 feet for more than 3 seconds.

B The ball returned to the ground in less than 4 seconds.

C The ball traveled more slowly going up than it did going down.

D The ball traveled less than 128 feet in 3.4 seconds.

5. Martha is putting a stone walkway around the garden pictured below.



About how many feet of stone are needed?

F 36.0 ft **G** 46.0 ft

H 46.8 ft **J** 56.8 ft

6. Which of these equations describes a relationship in which every negative real number x corresponds to a nonnegative real number y?

A $y = -x$ **C** $y = x^2$

B $y = x$ **D** $y = x^3$

Preparing for Standardized Tests
For test-taking strategies and more practice, see pages 941–956.

7. Miller's General Store needs at least 120 of their employees to oppose the building of a new cafeteria in order for the cafeteria not to be built. Miller's employs 1532 people. Jay surveyed a random sample of employees and asked which facility the employees want built.

Survey Results

Facility	Employees
Gym	50
Cafeteria	80
Park	65
Parking Garage	140

Based on the data in the survey, about how many employees are likely to choose building a new cafeteria?

F 77 **H** 230
G 80 **J** 366

8. Mario purchased a pair of shoes that were on sale for \$85. The shoes were originally \$110. Which expression can be used to determine the percent of the original price that Mario saved on the purchase of his shoes?

A $\frac{85}{100} \times 100$ **C** $\frac{110}{85} \times 100$

B $\frac{110 - 85}{110} \times 100$ **D** $\frac{110 - 85}{85} \times 100$

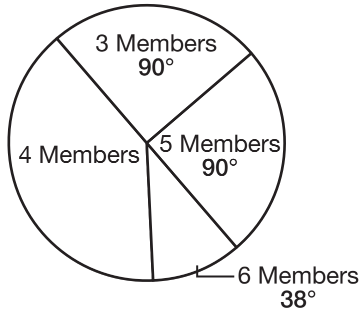
9. Lisa is 6 years younger than Petra. Stella is twice as old as Petra. The total of their ages is 54. Which equation can be used to find Petra's age?

F $x + (x - 6) + 2(x - 6) = 54$
G $x - 6x + (x + 2) = 54$
H $x - 6 + 2x = 54$
J $x + (x - 6) + 2x = 54$

10. $\angle M$ and $\angle N$ are supplementary angles. If $m\angle M$ is x, which equation can be used to find y, the measure of $\angle N$?

A $y = 90 + x$
B $y = 90 - x$
C $y = 180 - x$
D $y = x + 180$

11. GRIDDABLE The graph that shows the sizes of families in Gretchen's class is shown below. The diameter of the circle is 2 inches.



What is the approximate length in inches of the arc with the section that contains 4 family members? Round to the nearest hundredth.

Pre-AP

Record your answers on a sheet of paper. Show your work.

12. A gear that is 8 inches in diameter turns a smaller gear that is 3 inches in diameter.

a. Does this situation represent a *direct* or *inverse* variation? Explain your reasoning.

b. If the larger gear makes 36 revolutions, how many revolutions does the smaller gear make in that time?

NEED EXTRA HELP?												
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12
Go to Lesson...	8-4	8-4	8-4	5-7	1-4	2-4	12-1	1-3	2-4	1-3	10-3	8-4