CHAPTER 9

Exponential and Logarithmic Relations

BIG Ideas

- Simplify exponential and logarithmic expressions.
- Solve exponential and logarithmic equations and inequalitites.
- Solve problems involving exponential growth and decay.

Key Vocabulary

common logarithm (p. 528)
exponential function (p. 499)
logarithm (p. 510)
natural base, *e* (p. 536)
natural logarithm (p. 537)

Real-World Link

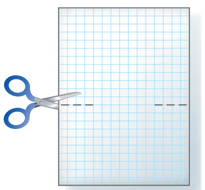
Seismograph A seismograph is an instrument used to detect and record the forces caused by earthquakes. The Richter Scale, which rates the intensity of earthquakes, is a logarithmic scale.

FOLDABLES Study Organizer

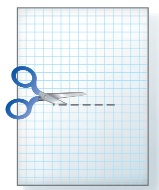
Exponential and Logarithmic Relations Make this Foldable to help you organize your notes. Begin with two sheets of grid paper.

1. **Fold** in half along the width. On the first sheet, cut 5 cm along the fold at the ends. On the second sheet, cut in the center, stopping 5 cm from the ends.

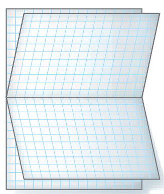
First Sheet



Second Sheet



2. **Insert** the first sheet through the second sheet and align the folds. Label the pages with lesson numbers.



Grant Smith/CORBIS

GET READY for Chapter 9

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

Math Online Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Simplify. Assume that no variable equals zero. (Lesson 6-1)

1. $x^5 \cdot x \cdot x^6$

2. $(3ab^4c^2)^3$

3. $\dfrac{-36x^7y^4z^3}{21x^4y^9z^4}$

4. $\left(\dfrac{4ab^2}{64b^3c}\right)^2$

5. **CHEMISTRY** The density D of an object in grams per milliliter is found by dividing the mass m of the substance by the volume V of the object. A sample of gold has a mass of 4.2×10^{-2} kilograms and a volume of 2.2×10^{-6} cubic meters. Find the density of gold. (Lesson 6-1)

QUICKReview

EXAMPLE 1 **Simplify $\dfrac{(x^2y^3z^4)^2}{x^2x^3y^3y^4z^4z^5}$. Assume that no variable equals zero.**

$$\frac{(x^2y^3z^4)^2}{x^2x^3y^3y^4z^4z^5}$$

$$= \frac{x^4y^6z^8}{x^5y^7z^9}$$ Simplify the numerator by using the Power of a Power Rule and the denominator by using the Product of Powers Rule.

$$= \frac{1}{xyz} \text{ or } x^{-1}y^{-1}z^{-1}$$ Simplify by using the Quotient of Powers Rule.

QUICKCheck

Find the inverse of each function. Then graph the function and its inverse. (Lesson 7-2)

6. $f(x) = -2x$

7. $f(x) = 3x - 2$

8. $f(x) = -x + 1$

9. $f(x) = \dfrac{x-4}{3}$

REMODELING **For Exercises 10 and 11, use the following information.**

Marc is wallpapering a 23-foot by 9-foot wall. The wallpaper costs $11.99 per square yard. The formula $f(x) = 9x$ converts square yards to square feet. (Lesson 7-2)

10. Find the inverse $f^{-1}(x)$. What is the significance of $f^{-1}(x)$?

11. What will the wallpaper cost?

QUICKReview

EXAMPLE 2 **Find the inverse of $f(x) = 2x + 3$.**

Step 1 Replace $f(x)$ with y in the original equation. $f(x) = 2x + 3 \rightarrow y = 2x + 3$

Step 2 Interchange x and y: $x = 2y + 3$.

Step 3 Solve for y.

$x = 2y + 3$ Inverse

$x - 3 = 2y$ Subtract 3 from each side.

$\dfrac{x-3}{2} = y$ Divide each side by 2.

$\dfrac{1}{2}x - \dfrac{3}{2} = y$ Simplify.

Step 4 Replace y with $f^{-1}(x)$.

$$y = \frac{1}{2}x - \frac{3}{2} \rightarrow f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$$

9-1 Exponential Functions

Main Ideas

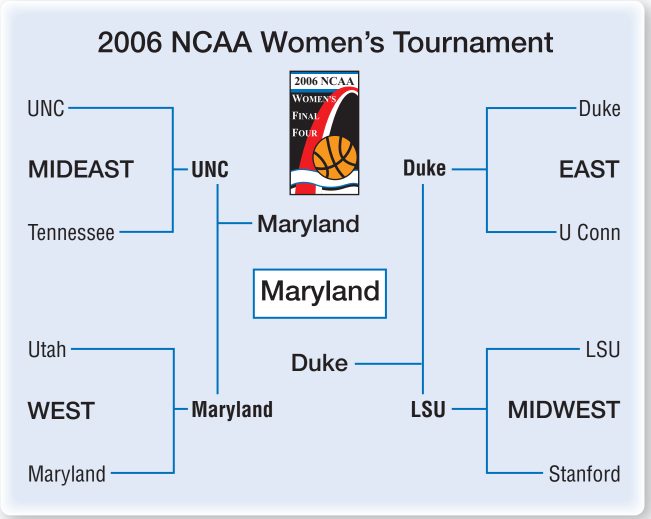
- Graph exponential functions.
- Solve exponential equations and inequalities.

New Vocabulary

exponential function
exponential growth
exponential decay
exponential equation
exponential inequality

GET READY for the Lesson

The NCAA women's basketball tournament begins with 64 teams and consists of 6 rounds of play. The winners of the first round play against each other in the second round. The winners then move from the Sweet Sixteen to the Elite Eight to the Final Four and finally to the Championship Game. The number of teams y that compete in a tournament of x rounds is $y = 2^x$.



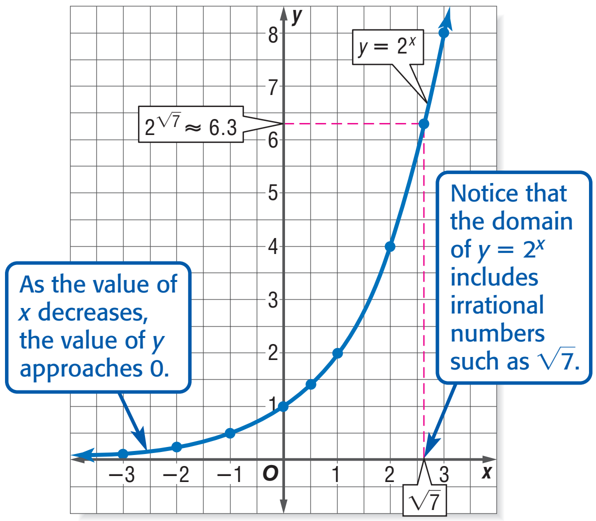
Exponential Functions In an exponential function like $y = 2^x$, the base is a constant, and the exponent is a variable. Let's examine the graph of $y = 2^x$.

EXAMPLE Graph an Exponential Function

1 **Sketch the graph of $y = 2^x$. Then state the function's domain and range.**

Make a table of values. Connect the points to sketch a smooth curve.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2}$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



The domain is all real numbers, and the range is all positive numbers.

CHECK Your Progress

1. Sketch the graph of $y = \left(\frac{1}{2}\right)^x$. Then state the function's domain and range.

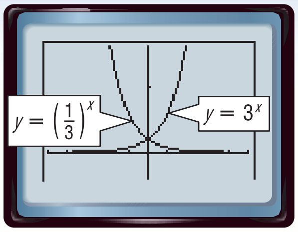
You can use a TI-83/84 Plus graphing calculator to look at the graphs of two other exponential functions, $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.

GRAPHING CALCULATOR LAB

Families of Exponential Functions

The calculator screen shows the graphs of parent functions $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.

THINK AND DISCUSS



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1. How do the shapes of the graphs compare?
2. How do the asymptotes and y-intercepts of the graphs compare?
3. Describe the relationship between the graphs.
4. Graph each group of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.
   - **a.** $y = 2^x$, $y = 3^x$, and $y = 4^x$
   - **b.** $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$
   - **c.** $y = -3(2)^x$ and $y = 3(2)^x$; $y = -1(2)^x$ and $y = 2^x$.
5. Describe the relationship between the graphs of $y = -1(2)^x$ and $y = 2^x$. Then graph the functions on a graphing calculator to verify your conjecture.

Study Tip

Common Misconception

Be sure not to confuse polynomial functions and exponential functions. While $y = x^3$ and $y = 3^x$ each have an exponent, $y = x^3$ is a polynomial function and $y = 3^x$ is an exponential function.

The Graphing Calculator Lab allowed you to discover many characteristics of the graphs of exponential functions. In general, an equation of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, is called an **exponential function** with base b. Exponential functions have the following characteristics.

1. The function is continuous and one-to-one.
2. The domain is the set of all real numbers.
3. The x-axis is an asymptote of the graph.
4. The range is the set of all positive numbers if $a > 0$ and all negative numbers if $a < 0$.
5. The graph contains the point $(0, a)$. That is, the y-intercept is a.
6. The graphs of $y = ab^x$ and $y = a\left(\frac{1}{b}\right)^x$ are reflections across the y-axis.

Study Tip

Look Back

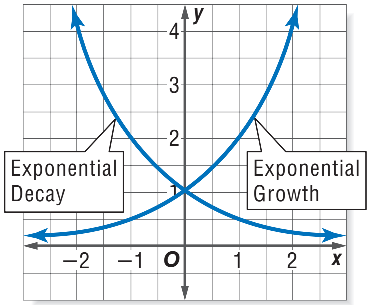
To review **continuous functions** and **one-to-one functions**, see Lessons 2-1 and 7-2.

Study Tip

Exponential Growth and Decay

Notice that the graph of an exponential growth function *rises* from left to right. The graph of an exponential decay function *falls* from left to right.

There are two types of exponential functions: **exponential growth** and **exponential decay**. The base of an exponential growth function is a number greater than one. The base of an exponential decay function is a number between 0 and 1.



KEY CONCEPT *Exponential Growth and Decay*

Symbols If $a > 0$ and $b > 1$, the function $y = ab^x$ represents exponential growth.

Example If $a > 0$ and $0 < b < 1$, the function $y = ab^x$ represents exponential decay.

EXAMPLE Identify Exponential Growth and Decay

2 **Determine whether each function represents exponential *growth* or *decay*.**

a. $y = \left(\frac{1}{5}\right)^x$

The function represents exponential decay, since the base, $\frac{1}{5}$, is between 0 and 1.

b. $y = 7(1.2)^x$

The function represents exponential growth, since the base, 1.2, is greater than 1.

CHECK Your Progress

2A. $y = 2(5)^x$

2B. $y = \left(\frac{2}{3}\right)^x$

Study Tip

Checking Reasonableness

In Example 2, you learned that if $a > 1$ and $b > 1$, then the function represents growth. Here, $a = 1{,}321{,}045$ and $b = 1.002$, and the population representing growth increased.

Exponential functions are frequently used to model the growth or decay of a population. You can use the *y*-intercept and one other point on the graph to write the equation of an exponential function.

Real-World EXAMPLE Write an Exponential Function

Personal Tutor

3 **POPULATION** **In 2000, the population of Phoenix was 1,321,045, and it increased to 1,331,391 in 2004.**

a. Write an exponential function of the form $y = ab^x$ that could be used to model the population y of Phoenix. Write the function in terms of x, the number of years since 2000.

For 2000, the time x equals 0, and the initial population y is 1,321,045. Thus, the y-intercept, and value of a, is 1,321,045.

For 2004, the time x equals 2004 – 2000 or 4, and the population y is 1,331,391. Substitute these values and the value of a into an exponential function to approximate the value of b.

$y = ab^x$	Exponential function
$1{,}331{,}391 = 1{,}321{,}045b^4$	Replace x with 4, y with 1,331,391, and a with 1,321,045.
$1.008 \approx b^4$	Divide each side by 1,321,045.
$\sqrt[4]{1.008} \approx b$	Take the 4th root of each side.

To find the 4th root of 1.008, use selection 5: $\sqrt[x]{\ }$ under the MATH menu on the TI–83/84 Plus.

KEYSTROKES: 4 [MATH] 5 1.008 [ENTER] 1.001994028

An equation that models the population growth of Phoenix from 2000 to 2004 is $y = 1{,}321{,}045(1.002)^x$.

b. Suppose the population of Pheonix continues to increase at the same rate. Estimate the population in 2015.

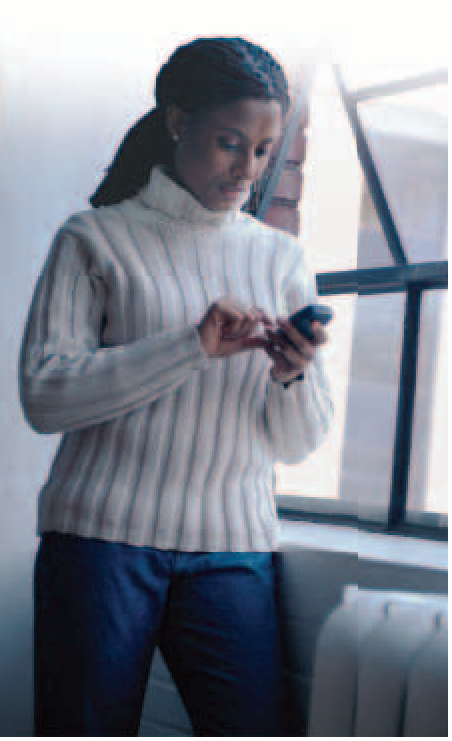
For 2015, the time equals 2015 – 2000 or 15.

$y = 1{,}321{,}045(1.002)^x$ Modeling equation

$= 1{,}321{,}045(1.002)^{15}$ Replace x with 15.

$\approx 1{,}360{,}262$ Use a calculator.

The population in Phoenix will be about 1,360,262 in 2015.



Real-World Link

The first virus that spread via cell phone networks was discovered in June 2004.

Source: internetnews.com

CHECK Your Progress

3. **SPAM** In 2003, the amount of annual cell phone spam messages totaled about ten million. In 2005, the total grew exponentially to 500 million. Write an exponential function of the form $y = ab^x$ that could be used to model the increase of spam messages y. Write the function in terms of x, the number of years since 2003. If the number of spam messages continues increasing at the same rate, estimate the annual number of spam messages in 2010.

Online Personal Tutor at algebra2.com

Exponential Equations and Inequalities

Exponential equations are equations in which variables occur as exponents.

KEY CONCEPT *Property of Equality for Exponential Functions*

Symbols If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example If $2^x = 2^8$, then $x = 8$.

EXAMPLE 4 Solve Exponential Equations

Solve each equation.

a. $3^{2n+1} = 81$

$3^{2n+1} = 81$ Original equation

$3^{2n+1} = 3^4$ Rewrite 81 as 3^4 so each side has the same base.

$2n + 1 = 4$ Property of Equality for Exponential Functions

$2n = 3$ Subtract 1 from each side.

$n = \frac{3}{2}$ Divide each side by 2.

(continued on the next page)

Math Online Extra Examples at algebra2.com

CHECK

$3^{2n+1} = 81$	Original equation
$3^{2\left(\frac{3}{2}\right)+1} \stackrel{?}{=} 81$	Substitute $\frac{3}{2}$ for n.
$3^4 \stackrel{?}{=} 81$	Simplify.
$81 = 81$ ✓	Simplify.

b. $4^{2x} = 8^{x-1}$

$4^{2x} = 8^{x-1}$	Original equation
$(2^2)^{2x} = (2^3)^{x-1}$	Rewrite each side with a base of 2.
$2^{4x} = 2^{3(x-1)}$	Power of a Power
$4x = 3(x-1)$	Property of Equality for Exponential Functions
$4x = 3x - 3$	Distributive Property
$x = -3$	Subtract $3x$ from each side.

CHECK

$4^{2x} = 8^{x-1}$	Original equation
$4^{2(-3)} \stackrel{?}{=} 8^{-3-1}$	Substitute -3 for x.
$4^{-6} \stackrel{?}{=} 8^{-4}$	Simplify.
$\frac{1}{4096} = \frac{1}{4096}$ ✓	Simplify.

CHECK Your Progress

Solve each equation.

4A. $4^{2n-1} = 64$ **4B.** $5^{5x} = 125^{x+2}$

The following property is useful for solving inequalities involving exponential functions or **exponential inequalities**.

KEY CONCEPT — *Property of Inequality for Exponential Functions*

Symbols If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Example If $5^x < 5^4$, then $x < 4$.

This property also holds true for $\leq$ and $\geq$.

EXAMPLE Solve Exponential Inequalities

5 Solve $4^{3p-1} > \frac{1}{256}$.

$4^{3p-1} > \frac{1}{256}$	Original inequality
$4^{3p-1} > 4^{-4}$	Rewrite $\frac{1}{256}$ as $\frac{1}{4^4}$ or 4^{-4} so each side has the same base.
$3p - 1 > -4$	Property of Inequality for Exponential Functions
$3p > -3$	Add 1 to each side.
$p > -1$	Divide each side by 3.

Study Tip

Look Back

You can review **negative exponents** in Lesson 6-1.

CHECK Test a value of p greater than -1; for example, $p = 0$.

$$4^{3p-1} > \frac{1}{256} \quad \text{Original inequality}$$

$$4^{3(0)-1} \stackrel{?}{>} \frac{1}{256} \quad \text{Replace } p \text{ with } 0.$$

$$4^{-1} \stackrel{?}{>} \frac{1}{256} \quad \text{Simplify.}$$

$$\frac{1}{4} > \frac{1}{256} \checkmark \quad a^{-1} = \frac{1}{a}$$

CHECK Your Progress

Solve each inequality.

5A. $3^{2x-1} \geq \frac{1}{243}$ **5B.** $2^{x+2} > \frac{1}{32}$

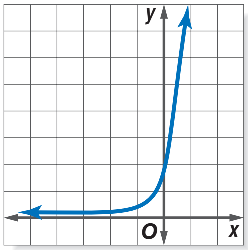
CHECK Your Understanding

Example 1 (p. 498–499)

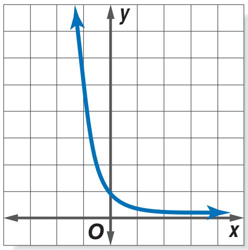
Match each function with its graph.

1. $y = 5^x$ **2.** $y = 2(5)^x$ **3.** $y = \left(\frac{1}{5}\right)^x$

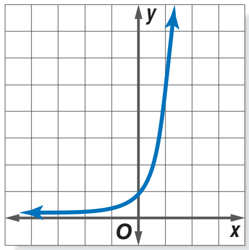
a.



b.



c.



Sketch the graph of each function. Then state the function's domain and range.

4. $y = 3(4)^x$ **5.** $y = 2\left(\frac{1}{3}\right)^x$

Example 2 (p. 500)

Determine whether each function represents exponential *growth* or *decay*.

6. $y = (0.5)^x$ **7.** $y = 0.3(5)^x$

Example 3 (pp. 500–501)

Write an exponential function for the graph that passes through the given points.

8. $(0, 3)$ and $(-1, 6)$ **9.** $(0, -18)$ and $(-2, -2)$

MONEY For Exercises 10 and 11, use the following information.
In 1993, My-Lien inherited \$1,000,000 from her grandmother. She invested all of the money, and by 2005, the amount had grown to \$1,678,000.

10. Write an exponential function that could be used to model the money y. Write the function in terms of x, the number of years since 1993.

11. Assume that the amount of money continues to grow at the same rate. Estimate the amount of money in 2015. Is this estimate reasonable? Explain your reasoning.

Example 4 (pp. 501–502)

Solve each equation. Check your solution.

12. $2^{n+4} = \frac{1}{32}$ **13.** $9^{2y-1} = 27^y$ **14.** $4^{3x+2} = \frac{1}{256}$

Example 5 (pp. 502–503)

Solve each inequality. Check your solution.

15. $5^{2x+3} \leq 125$ **16.** $3^{3x-2} > 81$ **17.** $4^{4a+6} \leq 16^a$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
18–21	1
22–27	2
28–38	3
39–44	4
45–48	5

Sketch the graph of each function. Then state the function's domain and range.

18. $y = 2(3)^x$ **19.** $y = 5(2)^x$

20. $y = 0.5(4)^x$ **21.** $y = 4\left(\frac{1}{3}\right)^x$

Determine whether each function represents exponential *growth* or *decay*.

22. $y = 10(3.5)^x$ **23.** $y = 2(4)^x$ **24.** $y = 0.4\left(\frac{1}{3}\right)^x$

25. $y = 3\left(\frac{5}{2}\right)^x$ **26.** $y = 30^{-x}$ **27.** $y = 0.2(5)^{-x}$

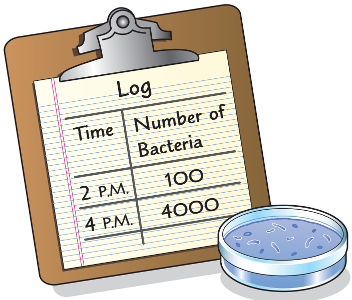
Write an exponential function for the graph that passes through the given points.

28. $(0, -2)$ and $(-2, -32)$ **29.** $(0, 3)$ and $(1, 15)$

30. $(0, 7)$ and $(2, 63)$ **31.** $(0, -5)$ and $(-3, -135)$

32. $(0, 0.2)$ and $(4, 51.2)$ **33.** $(0, -0.3)$ and $(5, -9.6)$

BIOLOGY **For Exercises 34 and 35, use the following information.**
The number of bacteria in a colony is growing exponentially.



34. Write an exponential function to model the population y of bacteria x hours after 2 P.M.

35. How many bacteria were there at 7 P.M. that day?

Cross-Curricular Project

Math Online The magnitude of an earthquake can be represented by an exponential equation. Visit algebra2.com to continue work on your project.

MONEY **For Exercises 36–38, use the following information.**
Suppose you deposit a principal amount of P dollars in a bank account that pays compound interest. If the annual interest rate is r (expressed as a decimal) and the bank makes interest payments n times every year, the amount of money A you would have after t years is given by $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

36. If the principal, interest rate, and number of interest payments are known, what type of function is $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$? Explain your reasoning.

37. Write an equation giving the amount of money you would have after t years if you deposit \$1000 into an account paying 4% annual interest compounded quarterly (four times per year).

38. Find the account balance after 20 years.

EXTRA PRACTICE
See pages 909, 934.

Math Online
Self-Check Quiz at algebra2.com

Solve each equation. Check your solution.

39. $2^{3x+5} = 128$

40. $5^{n-3} = \frac{1}{25}$

41. $\left(\frac{1}{9}\right)^{m} = 81^{m+4}$

42. $\left(\frac{1}{7}\right)^{y-3} = 343$

43. $10^{x-1} = 100^{2x-3}$

44. $36^{2p} = 216^{p-1}$

Solve each inequality. Check your solution.

45. $3^{n-2} > 27$

46. $2^{2n} \leq \frac{1}{16}$

47. $16^{n} < 8^{n+1}$

48. $32^{5p+2} \geq 16^{5p}$

Sketch the graph of each function. Then state the function's domain and range.

49. $y = -\left(\frac{1}{5}\right)^{x}$

50. $y = -2.5(5)^{x}$

Real-World Link
Since computers were invented, computational speed has multiplied by a factor of 4 about every three years.

Source: wired.com

COMPUTERS **For Exercises 51 and 52, use the information at the left.**

51. If a typical computer operates with a computational speed s today, write an expression for the speed at which you can expect an equivalent computer to operate after x three-year periods.

52. Suppose your computer operates with a processor speed of 2.8 gigahertz and you want a computer that can operate at 5.6 gigahertz. If a computer with that speed is currently unavailable for home use, how long can you expect to wait until you can buy such a computer?

POPULATION **For Exercises 53–55, use the following information.**
Every ten years, the Bureau of the Census counts the number of people living in the United States. In 1790, the population of the U.S. was 3.93 million. By 1800, this number had grown to 5.31 million.

53. Write an exponential function that could be used to model the U.S. population y in millions for 1790 to 1800. Write the equation in terms of x, the number of decades x since 1790.

54. Assume that the U.S. population continued to grow at least that fast. Estimate the population for the years 1820, 1840, and 1860. Then compare your estimates with the actual population for those years, which were 9.64, 17.06, and 31.44 million, respectively.

55. **RESEARCH** Estimate the population of the U.S. in the most recent census. Then use the Internet or other reference to find the actual population of the U.S. in the most recent census. Has the population of the U.S. continued to grow at the same rate at which it was growing in the early 1800s? Explain.

Graphing Calculator

Graph each pair of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.

	Parent Function	New Function		Parent Function	New Function
56.	$y = 2^{x}$	$y = 2^{x} + 3$	**57.**	$y = 3^{x}$	$y = 3^{x+1}$
58.	$y = \left(\frac{1}{5}\right)^{x}$	$y = \left(\frac{1}{5}\right)^{x-2}$	**59.**	$y = \left(\frac{1}{4}\right)^{x}$	$y = \left(\frac{1}{4}\right)^{x} - 1$

60. Describe the effect of changing the values of h and k in the equation $y = 2^{x-h} + k$.

H.O.T. Problems

61. **OPEN ENDED** Give an example of a value of b for which $y = b^{x}$ represents exponential decay.

62. REASONING Identify each function as *linear*, *quadratic*, or *exponential*.

a. $y = 3x^2$ **b.** $y = 4(3)^x$ **c.** $y = 2x + 4$ **d.** $y = 4(0.2)^x + 1$

63. CHALLENGE Decide whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For a positive base b other than 1, $b^x > b^y$ if and only if $x > y$.

64. *Writing in Math* Use the information about women's basketball on page 498 to explain how an exponential function can be used to describe the teams in a tournament. Include an explanation of how you could use the equation $y = 2^x$ to determine the number of rounds of tournament play for 128 teams and an example of an inappropriate number of teams for a tournament.

STANDARDIZED TEST PRACTICE

65. ACT/SAT If $4^{x+2} = 48$, then $4^x =$

A 3.0

B 6.4

C 6.9

D 12.0

66. REVIEW If the equation $y = 3^x$ is graphed, which of the following values of x would produce a point closest to the x-axis?

F $\frac{3}{4}$

G $\frac{1}{4}$

H 0

J $-\frac{3}{4}$

Spiral Review

Solve each equation. Check your solutions. (Lesson 8-6)

67. $\frac{15}{p} + p = 16$

68. $\frac{s-3}{s+4} = \frac{6}{s^2 - 16}$

69. $\frac{2a-5}{a-9} + \frac{a}{a+9} = \frac{-6}{a^2 - 81}$

Identify each equation as a type of function. Then graph the equation. (Lesson 8-5)

70. $y = \sqrt{x-2}$

71. $y = -2[\![x]\!]$

72. $y = 8$

Find the inverse of each matrix, if it exists. (Lesson 4-7)

73. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

74. $\begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$

75. $\begin{bmatrix} -5 & 6 \\ -11 & 3 \end{bmatrix}$

76. ENERGY A circular cell must deliver 18 watts of energy. If each square centimeter of the cell that is in sunlight produces 0.01 watt of energy, how long must the radius of the cell be? (Lesson 7-4)

GET READY for the Next Lesson

Find $g[h(x)]$ and $h[g(x)]$. (Lesson 7-5)

77. $h(x) = 2x - 1$
$g(x) = x - 5$

78. $h(x) = x + 3$
$g(x) = x^2$

79. $h(x) = 2x + 5$
$g(x) = -x + 3$

EXTEND 9-1

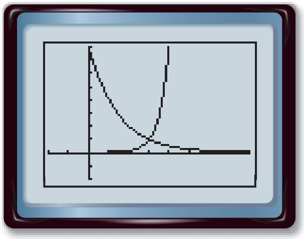
Graphing Calculator Lab
Solving Exponential Equations and Inequalities

You can use a TI-83/84 Plus graphing calculator to solve exponential equations by graphing or by using the table feature. To do this, you will write the equations as systems of equations.

ACTIVITY 1 **Solve** $2^{3x-9} = \left(\frac{1}{2}\right)^{x-3}$.

Step 1 **Graph each side of the equation.**

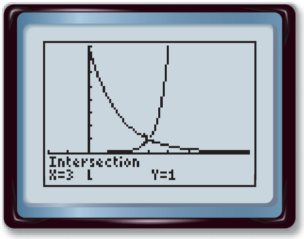
Graph each side of the equation as a separate function. Enter $2^{(3x-9)}$ as Y1. Enter $\left(\frac{1}{2}\right)^{(x-3)}$ as Y2. Be sure to include the added parentheses around each exponent. Then graph the two equations.


[−2, 8] scl: 1 by [−2, 8] scl: 1

KEYSTROKES: *See pages 92–94 to review graphing equations.*

Step 2 **Use the intersect feature.**

You can use the intersect feature on the CALC menu to approximate the ordered pair of the point at which the curves cross.



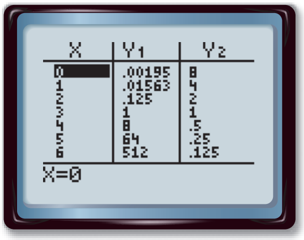
[−2, 8] scl: 1 by [−2, 8] scl: 1

KEYSTROKES: *See page 121 to review how to use the intersect feature.*

The calculator screen shows that the x-coordinate of the point at which the curves cross is 3. Therefore, the solution of the equation is 3.

Step 3 **Use the TABLE feature.**

You can also use the TABLE feature to locate the point at which the curves cross.



KEYSTROKES: 2nd [TABLE]

The table displays x-values and corresponding y-values for each graph. Examine the table to find the x-value for which the y-values for the graphs are equal. At $x = 3$, both functions have a y-value of 1. Thus, the solution of the equation is 3.

CHECK Substitute 3 for x in the original equation.

$2^{3x-9} = \left(\frac{1}{2}\right)^{x-3}$ Original equation

$2^{3(3)-9} \stackrel{?}{=} \left(\frac{1}{2}\right)^{3-3}$ Substitute 3 for x.

$2^0 \stackrel{?}{=} \left(\frac{1}{2}\right)^0$ Simplify.

$1 = 1$ ✓ The solution checks.

A similar procedure can be used to solve exponential inequalities using a graphing calculator.

ACTIVITY 2 **Solve $2^{x-2} \geq 0.5^{x-3}$.**

Step 1 Enter the related inequalities.

Rewrite the problem as a system of inequalities.

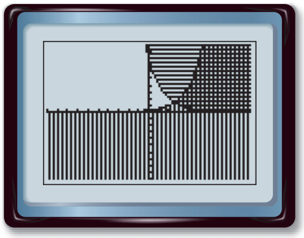
The first inequality is $2^{x-2} \geq y$ or $y \leq 2^{x-2}$. Since this last inequality includes the *less than or equal to* symbol, shade below the curve. First enter the boundary and then use the arrow and ENTER keys to choose the shade below icon, .

The second inequality is $y \geq 0.5^{x-3}$. Shade above the curve since this inequality contains *greater than or equal to.*

KEYSTROKES: Y= 2 ^ (X,T,θ,n − 2) ▼ ◀ ◀ ▲ ENTER ENTER ENTER ▼ ENTER ENTER ▶ ▶ .5 ^ (X,T,θ,n − 3)

Step 2 Graph the system.

KEYSTROKES: GRAPH



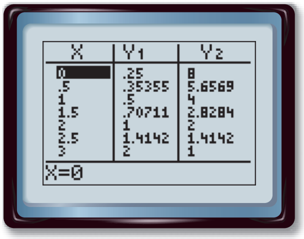
[−2, 8] scl: 1 by [−2, 8] scl: 1

The x-values of the points in the region where the shadings overlap are the solutions of the original inequality. Using the calculator's intersect feature, you can conclude that the solution set is $\{x | x \geq 2.5\}$.

Step 3 Use the TABLE feature.

Verify using the TABLE feature. Set up the table to show x-values in increments of 0.5.

KEYSTROKES: 2nd [TBLSET] 0 ENTER .5 ENTER 2nd [TABLE]



Notice that for x-values greater than $x = 2.5$, Y1 > Y2. This confirms the solution of the inequality is $\{x | x \geq 2.5\}$.

Exercises

Solve each equation or inequality.

1. $9^{x-1} = \frac{1}{81}$
2. $4^{x+3} = 2^{5x}$
3. $5^{x-1} = 2^x$
4. $3.5^{x+2} = 1.75^{x+3}$
5. $-3^{x+4} = -0.5^{2x+3}$
6. $6^{2-x} - 4 > -0.25^{x-2.5}$
7. $16^{x-1} > 2^{2x+2}$
8. $3^x - 4 \leq 5^{\frac{x}{2}}$
9. $5^{x+3} \leq 2^{x+4}$
10. $12^x - 5 > 6^{1-x}$
11. Explain why this technique of graphing a system of equations or inequalities works to solve exponential equations and inequalities.

9-2 Logarithms and Logarithmic Functions

Main Ideas

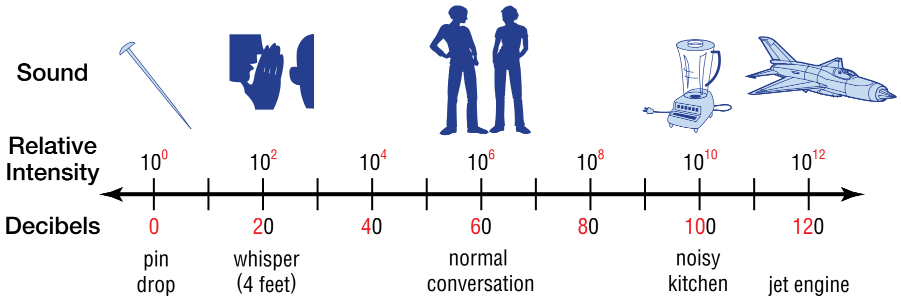
- Evaluate logarithmic expressions.
- Solve logarithmic equations and inequalities.

New Vocabulary

logarithm
logarithmic function
logarithmic equation
logarithmic inequality

GET READY for the Lesson

Many scientific measurements have such an enormous range of possible values that it makes sense to write them as powers of 10 and simply keep track of their exponents. For example, the loudness of sound is measured in units called decibels. The graph shows the relative intensities and decibel measures of common sounds.



The decibel measure of the loudness of a sound is the exponent or logarithm of its relative intensity multiplied by 10.

Review Vocabulary

Inverse Relation
when one relation contains the element (a, b), the other relation contains the element (b, a) (Lesson 7-6)

Inverse Function
The inverse function of $f(x)$ is $f^{-1}(x)$. (Lesson 7-6)

Concepts in Motion
Animation
algebra2.com

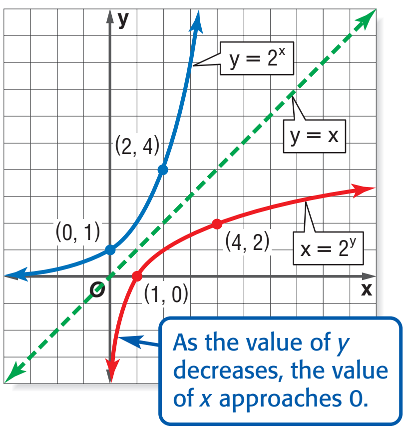
Logarithmic Functions and Expressions To better understand what is meant by a logarithm, consider the graph of $y = 2^x$ and its inverse. Since exponential functions are one-to-one, the inverse of $y = 2^x$ exists and is also a function. Recall that you can graph the inverse of a function by interchanging the x- and y-values in the ordered pairs of the function. Consider the exponential function $y = 2^x$.

$y = 2^x$

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$x = 2^y$

x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



The inverse of $y = 2^x$ can be defined as $x = 2^y$. Notice that the graphs of these two functions are reflections of each other over the line $y = x$.

In general, the inverse of $y = b^x$ is $x = b^y$. In $x = b^y$, y is called the **logarithm** of x. It is usually written as $y = \log_b x$ and is read *y equals log base b of x*.

KEY CONCEPT — *Logarithm with Base b*

Words Let b and x be positive numbers, $b \neq 1$. The *logarithm of x with base b* is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.

Symbols Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.

EXAMPLE 1 Logarithmic to Exponential Form

Study Tip

Zero Exponent
Recall that for any $b \neq 0$, $b^0 = 1$.

Write each equation in exponential form.

a. $\log_8 1 = 0$

$\log_8 1 = 0 \rightarrow 1 = 8^0$

b. $\log_2 \frac{1}{16} = -4$

$\log_2 \frac{1}{16} = -4 \rightarrow \frac{1}{16} = 2^{-4}$

CHECK Your Progress

1A. $\log_4 16 = 2$

1B. $\log_3 \frac{1}{27} = -3$

EXAMPLE 2 Exponential to Logarithmic Form

Write each equation in logarithmic form.

a. $10^3 = 1000$

$10^3 = 1000 \rightarrow \log_{10} 1000 = 3$

b. $9^{\frac{1}{2}} = 3$

$9^{\frac{1}{2}} = 3 \rightarrow \log_9 3 = \frac{1}{2}$

CHECK Your Progress

2A. $4^3 = 64$

2B. $125^{\frac{1}{3}} = 5$

You can use the definition of logarithm to find the value of a logarithmic expression.

EXAMPLE 3 Evaluate Logarithmic Expressions

Evaluate $\log_2 64$.

$\log_2 64 = y$	Let the logarithm equal y.
$64 = 2^y$	Definition of logarithm
$2^6 = 2^y$	$64 = 2^6$
$6 = y$	Property of Equality for Exponential Functions

So, $\log_2 64 = 6$.

CHECK Your Progress

Evaluate each expression.

3A. $\log_3 81$

3B. $\log_4 256$

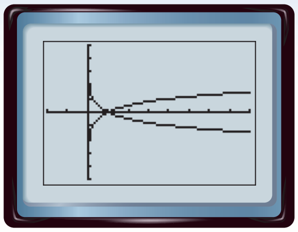
The function $y = \log_b x$, where $b > 0$ and $b \neq 1$, is called a **logarithmic function.** As shown in the graph on the previous page, this function is the inverse of the exponential function $y = b^x$ and has the following characteristics.

1. The function is continuous and one-to-one.
2. The domain is the set of all positive real numbers.
3. The y-axis is an asymptote of the graph.
4. The range is the set of all real numbers.
5. The graph contains the point (1, 0). That is, the x-intercept is 1.

GEOMETRY SOFTWARE LAB

The calculator screen shows the graphs of $y = \log_4 x$ and $y = \log_{\frac{1}{4}} x$.

KEYSTROKES: [Y=] [LOG] [X,T,θ,n] [ )] [÷] [LOG] 4 [ )] [ENTER]
[LOG] [X,T,θ,n] [ )] [÷] [LOG] 1 [÷] 4 [ )] [GRAPH]



THINK AND DISCUSS

1. How do the shapes of the graphs compare?

2. How do the asymptotes and the x-intercepts of the graphs compare?

3. Describe the relationship between the graphs.

4. Graph each pair of functions on the same screen. Then compare and contrast the graphs.

a.	$y = \log_4 x$	$y = \log_4 x + 2$
b.	$y = \log_4 x$	$y = \log_4 (x + 2)$
c.	$y = \log_4 x$	$y = 3 \log_4 x$

5. Describe the relationship between $y = \log_4 x$ and $y = -1(\log_4 x)$.

6. What are a reasonable domain and range for each function?

7. What is a reasonable viewing window in order to see the trends of both functions?

Study Tip

Look Back

To review **composition of functions**, see Lesson 7-5.

Since the exponential function $f(x) = b^x$ and the logarithmic function $g(x) = \log_b x$ are inverses of each other, their composites are the identity function. That is, $f[g(x)] = x$ and $g[f(x)] = x$.

$$f[g(x)] = x \qquad g[f(x)] = x$$
$$f(\log_b x) = x \qquad g(b^x) = x$$
$$b^{\log_b x} = x \qquad \log_b b^x = x$$

Thus, if their bases are the same, exponential and logarithmic functions "undo" each other. You can use this inverse property of exponents and logarithms to simplify expressions and solve equations. For example, $\log_6 6^8 = 8$ and $3^{\log_3 (4x - 1)} = 4x - 1$.

Solve Logarithmic Equations and Inequalities A **logarithmic equation** is an equation that contains one or more logarithms. You can use the definition of a logarithm to help you solve logarithmic equations.

EXAMPLE 4 Solve a Logarithmic Equation

Solve $\log_4 n = \frac{5}{2}$.

$\log_4 n = \frac{5}{2}$ — Original equation

$n = 4^{\frac{5}{2}}$ — Definition of logarithm

$n = (2^2)^{\frac{5}{2}}$ — $4 = 2^2$

$n = 2^5$ or 32 — Power of a Power

CHECK Your Progress

Solve each equation.

4A. $\log_9 x = \frac{3}{2}$ **4B.** $\log_{16} x = \frac{5}{2}$

A **logarithmic inequality** is an inequality that involves logarithms. In the case of inequalities, the following property is helpful.

KEY CONCEPT — *Logarithmic to Exponential Inequality*

Symbols If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.
If $b > 1$, $x > 0$, and $\log_b x < y$, then $0 < x < b^y$.

Examples $\log_2 x > 3$, $x > 2^3$ $\log_3 x < 5$, $0 < x < 3^5$

Study Tip

Special Values

If $b > 0$ and $b \neq 1$, then the following statements are true.

- $\log_b b = 1$ because $b^1 = b$.
- $\log_b 1 = 0$ because $b^0 = 1$.

EXAMPLE 5 Solve a Logarithmic Inequality

Solve $\log_5 x < 2$. Check your solution.

$\log_5 x < 2$ — Original inequality

$0 < x < 5^2$ — Logarithmic to exponential inequality

$0 < x < 25$ — Simplify.

The solution set is $\{x \mid 0 < x < 25\}$.

CHECK Try 5 to see if it satisfies the inequality.

$\log_5 x < 2$ — Original inequality

$\log_5 5 \overset{?}{<} 2$ — Substitute 5 for x.

$1 < 2$ ✓ — $\log_5 5 = 1$ because $5^1 = 5$.

CHECK Your Progress

Solve each inequality. Check your solution.

5A. $\log_4 x > 3$ **5B.** $\log_2 x < 4$

Use the following property to solve logarithmic equations that have logarithms with the same base on each side.

KEY CONCEPT *Property of Equality for Logarithmic Functions*

Symbols If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_7 x = \log_7 3$, then $x = 3$.

Personal Tutor

EXAMPLE 6 Solve Equations with Logarithms on Each Side

Solve $\log_5 (p^2 - 2) = \log_5 p$. Check your solution.

$\log_5 (p^2 - 2) = \log_5 p$	Original equation
$p^2 - 2 = p$	Property of Equality for Logarithmic Functions
$p^2 - p - 2 = 0$	Subtract p from each side.
$(p - 2)(p + 1) = 0$	Factor.
$p - 2 = 0$ or $p + 1 = 0$	Zero Product Property
$p = 2$ $\quad$ $p = -1$	Solve each equation.

CHECK Substitute each value into the original equation.

Check $p = 2$.

$\log_5 (2^2 - 2) \stackrel{?}{=} \log_5 2$ Substitute 2 for p.

$\log_5 2 = \log_5 2$ ✓ Simplify.

Check $p = -1$.

$\log_5 [(-1)^2 - 2] \stackrel{?}{=} \log_5 (-1)$ Substitute -1 for p.

Since $\log_5 (-1)$ is undefined, -1 is an *extraneous* solution and must be eliminated. Thus, the solution is 2.

Study Tip

Extraneous Solutions

The domain of a logarithmic function does not include negative values. For this reason, be sure to check for extraneous solutions of logarithmic equations.

CHECK Your Progress

Solve each equation. Check your solution.

6A. $\log_3 (x^2 - 15) = \log_3 2x$ $\qquad$ **6B.** $\log_{14} (m^2 - 30) = \log_{14} m$

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Use the following property to solve logarithmic inequalities that have the same base on each side. Exclude values from your solution set that would result in taking the logarithm of a number less than or equal to zero in the original inequality.

KEY CONCEPT *Property of Inequality for Logarithmic Functions*

Symbols If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

Example If $\log_2 x > \log_2 9$, then $x > 9$.

This property also holds for $\leq$ and $\geq$.

Study Tip

Look Back

To review **compound inequalities**, see Lesson 1-6.

EXAMPLE 7 Solve Inequalities with Logarithms on Each Side

Solve $\log_{10}(3x - 4) < \log_{10}(x + 6)$. Check your solution.

$\log_{10}(3x - 4) < \log_{10}(x + 6)$	Original inequality
$3x - 4 < x + 6$	Property of Inequality for Logarithmic Functions
$2x < 10$	Addition and Subtraction Properties of Inequalities
$x < 5$	Divide each side by 2.

We must exclude from this solution all values of x such that $3x - 4 \le 0$ or $x + 6 \le 0$.

Thus, $x > \frac{4}{3}$, $x > -6$, and $x < 5$. This compound inequality simplifies to $\frac{4}{3} < x < 5$. The solution set is $\left\{x \,\middle|\, \frac{4}{3} < x < 5\right\}$.

CHECK Your Progress

7. Solve $\log_5(2x + 1) \le \log_5(x + 4)$. Check your solution.

CHECK Your Understanding

Example 1 (p. 510) **Write each equation in logarithmic form.**

1. $5^4 = 625$ **2.** $7^{-2} = \frac{1}{49}$ **3.** $3^5 = 243$

Example 2 (p. 510) **Write each equation in exponential form.**

4. $\log_3 81 = 4$ **5.** $\log_{36} 6 = \frac{1}{2}$ **6.** $\log_{125} 5 = \frac{1}{3}$

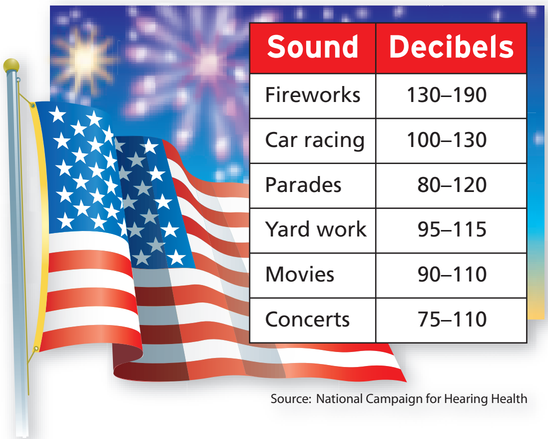
Example 3 (p. 510) **Evaluate each expression.**

7. $\log_4 256$ **8.** $\log_2 \frac{1}{8}$ **9.** $\log_6 216$

Example 4 (p. 512) **Solve each equation. Check your solutions.**

10. $\log_9 x = \frac{3}{2}$ **11.** $\log_{\frac{1}{10}} x = -3$ **12.** $\log_b 9 = 2$

SOUND For Exercises 13–15, use the following information.
An equation for loudness L, in decibels, is $L = 10 \log_{10} R$, where R is the relative intensity of the sound.



Sound	Decibels
Fireworks	130–190
Car racing	100–130
Parades	80–120
Yard work	95–115
Movies	90–110
Concerts	75–110

Source: National Campaign for Hearing Health

13. Solve $130 = 10 \log_{10} R$ to find the relative intensity of a fireworks display with a loudness of 130 decibels.

14. Solve $75 = 10 \log_{10} R$ to find the relative intensity of a concert with a loudness of 75 decibels.

15. How many times more intense is the fireworks display than the concert? In other words, find the ratio of their intensities.

Example 5 (p. 512)

Solve each inequality. Check your solutions.

16. $\log_4 x < 2$ **17.** $\log_3 (2x - 1) \leq 2$ **18.** $\log_{16} x \geq \frac{1}{4}$

Example 6 (p. 513)

Solve each equation. Check your solutions.

19. $\log_5 (3x - 1) = \log_5 (2x^2)$ **20.** $\log_{10} (x^2 - 10x) = \log_{10} (-21)$

Example 7 (p. 514)

Solve each inequality. Check your solutions.

21. $\log_2 (3x - 5) > \log_2 (x + 7)$ **22.** $\log_5 (5x - 7) \leq \log_5 (2x + 5)$

Exercises

HOMEWORK HELP

For Exercises	See Examples
23–28	1
29–34	2
35–43	3
44–51	4
52–55	5
56, 57	6
58, 59	7

Write each equation in exponential form.

23. $\log_5 125 = 3$ **24.** $\log_{13} 169 = 2$ **25.** $\log_4 \frac{1}{4} = -1$

26. $\log_{100} \frac{1}{10} = -\frac{1}{2}$ **27.** $\log_8 4 = \frac{2}{3}$ **28.** $\log_{\frac{1}{5}} 25 = -2$

Write each equation in logarithmic form.

29. $8^3 = 512$ **30.** $3^3 = 27$ **31.** $5^{-3} = \frac{1}{125}$

32. $\left(\frac{1}{3}\right)^{-2} = 9$ **33.** $100^{\frac{1}{2}} = 10$ **34.** $2401^{\frac{1}{4}} = 7$

Evaluate each expression.

35. $\log_2 16$ **36.** $\log_{12} 144$ **37.** $\log_{16} 4$

38. $\log_9 243$ **39.** $\log_2 \frac{1}{32}$ **40.** $\log_3 \frac{1}{81}$

41. $\log_{10} 0.001$ **42.** $\log_4 16^x$ **43.** $\log_3 27^x$

Solve each equation. Check your solutions.

44. $\log_9 x = 2$ **45.** $\log_{25} n = \frac{3}{2}$ **46.** $\log_{\frac{1}{7}} x = -1$

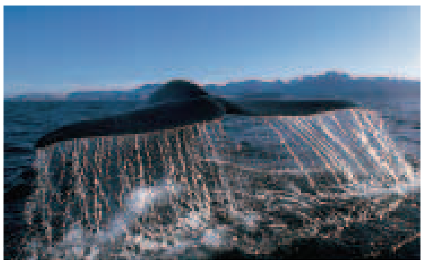
47. $\log_{10} (x^2 + 1) = 1$ **48.** $\log_b 64 = 3$ **49.** $\log_b 121 = 2$

WORLD RECORDS **For Exercises 50 and 51, use the information given for Exercises 13–15 to find the relative intensity of each sound.**

Source: *The Guinness Book of Records*

50. The loudest animal sounds are the low-frequency pulses made by blue whales when they communicate. These pulses have been measured up to 188 decibels.

51. The loudest insect is the African cicada that produces a calling song that measures 106.7 decibels at a distance of 50 centimeters.



EXTRA PRACTICE

See pages 910, 934.

Math Online

Self-Check Quiz at algebra2.com

Real-World Link

The Loma Prieta earthquake measured 7.1 on the Richter scale and interrupted the 1989 World Series in San Francisco.

Source: U.S. Geological Survey

Solve each equation or inequality. Check your solutions.

52. $\log_2 c > 8$

53. $\log_{64} y \le \frac{1}{2}$

54. $\log_{\frac{1}{3}} p < 0$

55. $\log_2 (3x - 8) \ge 6$

56. $\log_6 (2x - 3) = \log_6 (x + 2)$

57. $\log_7 (x^2 + 36) = \log_7 100$

58. $\log_2 (4y - 10) \ge \log_2 (y - 1)$

59. $\log_{10} (a^2 - 6) > \log_{10} a$

Show that each statement is true.

60. $\log_5 25 = 2 \log_5 5$

61. $\log_{16} 2 \cdot \log_2 16 = 1$

62. $\log_7 [\log_3 (\log_2 8)] = 0$

63. Sketch the graphs of $y = \log_{\frac{1}{2}} x$ and $y = \left(\frac{1}{2}\right)^x$ on the same axes. Then describe the relationship between the graphs.

64. Sketch the graphs of $y = \log_3 x$, $y = \log_3 (x + 2)$, $y = \log_3 x - 3$. Then describe the relationship between the graphs.

EARTHQUAKES **For Exercises 65 and 66, use the following information.**
The magnitude of an earthquake is measured on a logarithmic scale called the Richter scale. The magnitude M is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion.

65. How many times as great is the amplitude caused by an earthquake with a Richter scale rating of 7 as an aftershock with a Richter scale rating of 4?

66. How many times as great was the motion caused by the 1906 San Francisco earthquake that measured 8.3 on the Richter scale as that caused by the 2001 Bhuj, India, earthquake that measured 6.9?

67. **NOISE ORDINANCE** A proposed city ordinance will make it illegal to create sound in a residential area that exceeds 72 decibels during the day and 55 decibels during the night. How many times as intense is the noise level allowed during the day than at night? (*Hint:* See information on page 514.)

Graphing Calculator

FAMILY OF GRAPHS **For Exercises 68 and 69, use the following information.**
Consider the functions $y = \log_2 x + 3$, $y = \log_2 x - 4$, $y = \log_2 (x - 1)$, and $y = \log_2 (x + 2)$.

68. Use a graphing calculator to sketch the graphs on the same screen. Describe this family of graphs in terms of its parent graph $y = \log_2 x$.

69. What are a reasonable domain and range for each function?

H.O.T. Problems

70. **OPEN ENDED** Give an example of an exponential equation and its related logarithmic equation.

71. **Which One Doesn't Belong?** Find the expression that does not belong. Explain.

$\log_4 16$	$\log_2 16$	$\log_2 4$	$\log_3 9$

72. **FIND THE ERROR** Paul and Clemente are solving $\log_3 x = 9$. Who is correct? Explain your reasoning.

Paul

$\log_3 x = 9$

$3^x = 9$

$3^x = 3^2$

$x = 2$

Clemente

$\log_3 x = 9$

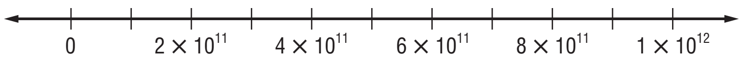
$x = 3^9$

$x = 19{,}683$

David Weintraub/Photo Researchers

73. **CHALLENGE** Using the definition of a logarithmic function where $y = \log_b x$, explain why the base b cannot equal 1.

74. ***Writing in Math*** Use the information about sound on page 509 to explain how a logarithmic scale can be used to measure sound. Include the relative intensities of a pin drop, a whisper, normal conversation, kitchen noise, and a jet engine written in scientific notation. Also include a plot of each of these relative intensities on the scale below and an explanation as to why the logarithmic scale might be preferred over the scale below.



STANDARDIZED TEST PRACTICE

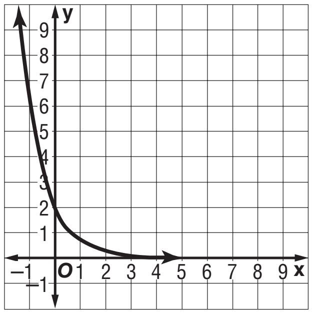
75. **ACT/SAT** What is the equation of the function?

A $y = 2(3)^x$

B $y = 2\left(\frac{1}{3}\right)^x$

C $y = 3\left(\frac{1}{2}\right)^x$

D $y = 3(2)^x$



76. **REVIEW** What is the solution to the equation $3^x = 11$?

F $x = 2$

G $x = \log_{10} 2$

H $x = \log_{10} 11 + \log_{10} 3$

J $x = \dfrac{\log_{10} 11}{\log_{10} 3}$

Spiral Review

Simplify each expression. (Lesson 9-1)

77. $x^{\sqrt{6}} \cdot x^{\sqrt{6}}$

78. $\left(b^{\sqrt{6}}\right)^{\sqrt{24}}$

Solve each equation. Check your solutions. (Lesson 8-6)

79. $\dfrac{2x+1}{x} - \dfrac{x+1}{x-4} = \dfrac{-20}{x^2-4x}$

80. $\dfrac{2a-5}{a-9} - \dfrac{a-3}{3a+2} = \dfrac{5}{3a^2-25a-18}$

Solve each equation by using the method of your choice. Find exact solutions. (Lesson 5-6)

81. $9y^2 = 49$

82. $2p^2 = 5p + 6$

83. **BANKING** Donna Bowers has \$8000 she wants to save in the bank. A 12-month certificate of deposit (CD) earns 4% annual interest, while a regular savings account earns 2% annual interest. Ms. Bowers doesn't want to tie up all her money in a CD, but she has decided she wants to earn \$240 in interest for the year. How much money should she put in to each type of account? (Lesson 4-4)

GET READY for the Next Lesson

Simplify. Assume that no variable equals zero. (Lesson 6-1)

84. $x^4 \cdot x^6$

85. $(2a^2b)^3$

86. $\dfrac{a^4n^7}{a^3n}$

87. $\left(\dfrac{b^7}{a^4}\right)^0$

EXTEND 9-2

Graphing Calculator Lab

Modeling Data Using Exponential Functions

We are often confronted with data for which we need to find an equation that best fits the information. We can find exponential and logarithmic functions of best fit using a TI-83/84 Plus graphing calculator.

ACTIVITY

The population per square mile in the United States has changed dramatically over a period of years. The table shows the number of people per square mile for several years.

U.S. Population Density

Year	People per square mile	Year	People per square mile
1790	4.5	1900	21.5
1800	6.1	1910	26.0
1810	4.3	1920	29.9
1820	5.5	1930	34.7
1830	7.4	1940	37.2
1840	9.8	1950	42.6
1850	7.9	1960	50.6
1860	10.6	1970	57.5
1870	10.9	1980	64.0
1880	14.2	1990	70.3
1890	17.8	2000	80.0

Source: Northeast-Midwest Institute

a. Use a graphing calculator to enter the data and draw a scatter plot that shows how the number of people per square mile is related to the year.

Step 1 Enter the year into L1 and the people per square mile into L2.

KEYSTROKES: *See pages 92 and 93 to review how to enter lists.*

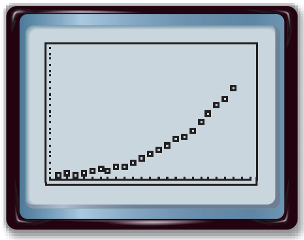
Be sure to clear the Y= list. Use the ▶ key to move the cursor from L1 to L2.

Step 2 Draw the scatter plot.

KEYSTROKES: *See pages 92 and 93 to review how to graph a scatter plot.*

Make sure that **Plot 1** is on, the scatter plot is chosen, **Xlist** is L1, and **Ylist** is L2. Use the viewing window [1780, 2020] with a scale factor of 10 by [0, 115] with a scale factor of 5.

We see from the graph that the equation that best fits the data is a curve. Based on the shape of the curve, try an exponential model.



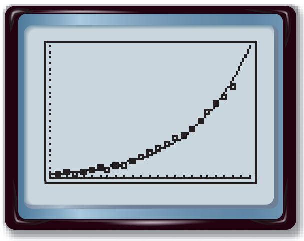
[1780, 2020] scl: 10 by [0, 115] scl: 5

Step 3 To determine the exponential equation that best fits the data, use the exponential regression feature of the calculator.

KEYSTROKES: STAT ▶ 0 2nd [L1] , 2nd [L2] ENTER

The calculator also reports an r value of 0.991887235. Recall that this number is a correlation coefficient that indicates how well the equation fits the data. A perfect fit would be $r = 1$. Therefore, we can conclude that this equation is a pretty good fit for the data.

To check this equation visually, overlap the graph of the equation with the scatter plot.



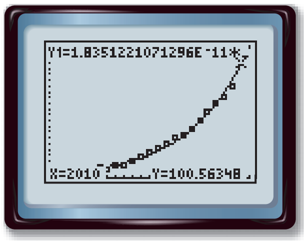
[1780, 2020] scl: 10 by [0, 115] scl: 5

KEYSTROKES: Y= VARS 5 ▶ ▶ 1 GRAPH

The *residual* is the difference between actual and predicted data. The predicted population per square mile in 2000 using this model was 86.9. (To calculate, press 2nd [CALC] 1 2000 ENTER.) So, the residual for 2000 was 80.0 − 86.9, or −6.9.

b. If this trend continues, what will be the population per square mile in 2010?

To determine the population per square mile in 2010, from the graphics screen, find the value of y when $x = 2010$.



[1780, 2020] scl: 10 by [0, 115] scl: 5

KEYSTROKES: 2nd [CALC] 1 2010 ENTER

The calculator returns a value of approximately 100.6. If this trend continues, in 2010, there will be approximately 100.6 people per square mile.

EXERCISES

Jewel received $30 from her aunt and uncle for her seventh birthday. Her father deposited it into a bank account for her. Both Jewel and her father forgot about the money and made no further deposits or withdrawals. The table shows the account balance for several years.

Elapsed Time (years)	Balance
0	$30.00
5	$41.10
10	$56.31
15	$77.16
20	$105.71
25	$144.83
30	$198.43

1. Use a graphing calculator to draw a scatter plot for the data.

2. Calculate and graph the curve of best fit that shows how the elapsed time is related to the balance. Use ExpReg for this exercise.

3. Write the equation of best fit.

4. Write a sentence that describes the fit of the graph to the data.

5. Based on the graph, estimate the balance in 41 years. Check this using the CALC value.

6. Do you think there are any other types of equations that would be good models for these data? Why or why not?

9-3 Properties of Logarithms

Main Ideas

- Simplify and evaluate expressions using the properties of logarithms.
- Solve logarithmic equations using the properties of logarithms.

GET READY for the Lesson

In Lesson 6-1, you learned that the product of powers is the sum of their exponents.

$$9 \cdot 81 = 3^2 \cdot 3^4 \text{ or } 3^{2+4}$$

In Lesson 9-2, you learned that logarithms *are* exponents, so you might expect that a similar property applies to logarithms. Let's consider a specific case. Does $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$? Investigate by simplifying the expression on each side of the equation.

$\log_3 (9 \cdot 81)$	$= \log_3 (3^2 \cdot 3^4)$	Replace 9 with 3^2 and 81 with 3^4.
	$= \log_3 3^{(2+4)}$	Product of Powers
	$= 2 + 4$ or 6	Inverse Property of Exponents and Logarithms

$\log_3 9 + \log_3 81$	$= \log_3 3^2 + \log_3 3^4$	Replace 9 with 3^2 and 81 with 3^4.
	$= 2 + 4$ or 6	Inverse Property of Exponents and Logarithms

Both expressions are equal to 6. So, $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$.

Properties of Logarithms Since logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. The Product Property of Logarithms can be derived from the Product of Powers Property of Exponents.

KEY CONCEPT *Product Property of Logarithms*

Words The logarithm of a product is the sum of the logarithms of its factors.

Symbols For all positive numbers m, n, and b, where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.

Example $\log_3 (4)(7) = \log_3 4 + \log_3 7$

To show that this property is true, let $b^x = m$ and $b^y = n$. Then, using the definition of logarithm, $x = \log_b m$ and $y = \log_b n$.

$b^x b^y = mn$	Substitution
$b^{x+y} = mn$	Product of Powers
$\log_b b^{x+y} = \log_b mn$	Property of Equality for Logarithmic Functions
$x + y = \log_b mn$	Inverse Property of Exponents and Logarithms
$\log_b m + \log_b n = \log_b mn$	Replace x with $\log_b m$ and y with $\log_b n$.

You can use the Product Property of Logarithms to approximate logarithmic expressions.

EXAMPLE Use the Product Property

1 Use $\log_2 3 \approx 1.5850$ to approximate the value of $\log_2 48$.

$\log_2 48 = \log_2 (2^4 \cdot 3)$ — Replace 48 with 16 • 3 or $2^4 \cdot 3$.

$= \log_2 2^4 + \log_2 3$ — Product Property

$= 4 + \log_2 3$ — Inverse Property of Exponents and Logarithms

$\approx 4 + 1.5850$ or 5.5850 — Replace $\log_2 3$ with 1.5850.

Thus, $\log_2 48$ is approximately 5.5850.

Study Tip

Answer Check

You can check this answer by evaluating $2^{5.5850}$ on a calculator. The calculator should give a result of about 48, since $\log_2 48 \approx 5.5850$ means $2^{5.5850} \approx 48$.

CHECK Your Progress

1. Use $\log_4 2 = 0.5$ to approximate the value of $\log_4 32$.

Recall that the quotient of powers is found by subtracting exponents. The property for the logarithm of a quotient is similar.

KEY CONCEPT — *Quotient Property of Logarithms*

Words The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.

Symbols For all positive numbers m, n, and b, where $b \neq 1$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.

You will prove this property in Exercise 51.

EXAMPLE Use the Quotient Property

2 Use $\log_3 5 \approx 1.4650$ and $\log_3 20 \approx 2.7268$ to approximate $\log_3 4$.

$\log_3 4 = \log_3 \frac{20}{5}$ — Replace 4 with the quotient $\frac{20}{5}$.

$= \log_3 20 - \log_3 5$ — Quotient Property

$\approx 2.7268 - 1.4650$ or 1.2618 — $\log_3 20 \approx 2.7268$ and $\log_3 5 \approx 1.4650$

Thus, $\log_3 4$ is approximately 1.2618.

CHECK Use the definition of logarithm and a calculator.

3 [^] 1.2618 [ENTER] 3.999738507

Since $3^{1.2618} \approx 4$, the answer checks. ✓

CHECK Your Progress

2. Use $\log_5 7 \approx 1.2091$ and $\log_5 21 \approx 1.8917$ to approximate $\log_5 3$.

Real-World EXAMPLE

Personal Tutor

Real-World Career

Sound Technician

Sound technicians produce movie sound tracks in motion picture production studios, control the sound of live events such as concerts, or record music in a recording studio.

Math Online

For more information, go to algebra2.com.

3 SOUND The loudness L of a sound is measured in decibels and is given by $L = 10 \log_{10} R$, where R is the sound's relative intensity. Suppose one person talks with a relative intensity of 10^6 or 60 decibels. Would the sound of ten people each talking at that same intensity be ten times as loud, or 600 decibels? Explain your reasoning.

Let L_1 be the loudness of one person talking. $\rightarrow$ $L_1 = 10 \log_{10} 10^6$

Let L_2 be the loudness of ten people talking. $\rightarrow$ $L_2 = 10 \log_{10} (10 \cdot 10^6)$

Then the increase in loudness is $L_2 - L_1$.

$L_2 - L_1 = 10 \log_{10} (10 \cdot 10^6) - 10 \log_{10} 10^6$ Substitute for L_1 and L_2.

$= 10(\log_{10} 10 + \log_{10} 10^6) - 10 \log_{10} 10^6$ Product Property

$= 10 \log_{10} 10 + 10 \log_{10} 10^6 - 10 \log_{10} 10^6$ Distributive Property

$= 10 \log_{10} 10$ Subtract.

$= 10(1)$ or 10 Inverse Property of Exponents and Logarithms

The sound of ten people talking is perceived by the human ear to be only about 10 decibels louder than the sound of one person talking, or 70 decibels.

CHECK Your Progress

3. How much louder would 100 people talking at the same intensity be than just one person?

Online **Personal Tutor at** algebra2.com

Recall that the power of a power is found by multiplying exponents. The property for the logarithm of a power is similar.

KEY CONCEPT *Power Property of Logarithms*

Words The logarithm of a power is the product of the logarithm and the exponent.

Symbols For any real number p and positive numbers m and b, where $b \neq 1$, $\log_b m^p = p \log_b m$.

You will prove this property in Exercise 45.

EXAMPLE Power Property of Logarithms

4 Given $\log_4 6 \approx 1.2925$, approximate the value of $\log_4 36$.

$\log_4 36 = \log_4 6^2$ Replace 36 with 6^2.

$= 2 \log_4 6$ Power Property

$\approx 2(1.2925)$ or 2.585 Replace $\log_4 6$ with 1.2925.

CHECK Your Progress

4. Given $\log_3 7 \approx 1.7712$, approximate the value of $\log_3 49$.

Solve Logarithmic Equations You can use the properties of logarithms to solve equations involving logarithms.

EXAMPLE 5 Solve Equations Using Properties of Logarithms

Solve each equation.

a. $3 \log_5 x - \log_5 4 = \log_5 16$

$3 \log_5 x - \log_5 4 = \log_5 16$	Original equation
$\log_5 x^3 - \log_5 4 = \log_5 16$	Power Property
$\log_5 \frac{x^3}{4} = \log_5 16$	Quotient Property
$\frac{x^3}{4} = 16$	Property of Equality for Logarithmic Functions
$x^3 = 64$	Multiply each side by 4.
$x = 4$	Take the cube root of each side.

The solution is 4.

b. $\log_4 x + \log_4 (x - 6) = 2$

$\log_4 x + \log_4 (x - 6) = 2$	Original equation
$\log_4 x(x - 6) = 2$	Product Property
$x(x - 6) = 4^2$	Definition of logarithm
$x^2 - 6x - 16 = 0$	Subtract 16 from each side.
$(x - 8)(x + 2) = 0$	Factor.
$x - 8 = 0$ or $x + 2 = 0$	Zero Product Property
$x = 8$ $\quad$ $x = -2$	Solve each equation.

CHECK Substitute each value into the original equation.

$$\log_4 8 + \log_4 (8 - 6) \stackrel{?}{=} 2$$
$$\log_4 8 + \log_4 2 \stackrel{?}{=} 2$$
$$\log_4 (8 \cdot 2) \stackrel{?}{=} 2$$
$$\log_4 16 \stackrel{?}{=} 2$$
$$2 = 2 \checkmark$$

$$\log_4 (-2) + \log_4 (-2 - 6) \stackrel{?}{=} 2$$
$$\log_4 (-2) + \log_4 (-8) \stackrel{?}{=} 2$$

Since $\log_4 (-2)$ and $\log_4 (-8)$ are undefined, -2 is an extraneous solution and must be eliminated.

The only solution is 8.

Study Tip

Checking Solutions

It is wise to check all solutions to see if they are valid since the domain of a logarithmic function is not the complete set of real numbers.

CHECK Your Progress

5A. $2 \log_7 x = \log_7 27 + \log_7 3$ **5B.** $\log_6 x + \log_6 (x + 5) = 2$

CHECK Your Understanding

Examples 1, 2 (p. 521)

Use $\log_3 2 \approx 0.6309$ and $\log_3 7 \approx 1.7712$ to approximate the value of each expression.

1. $\log_3 18$ **2.** $\log_3 14$ **3.** $\log_3 \frac{7}{2}$ **4.** $\log_3 \frac{2}{3}$

Example 3
(p. 522)

5. **MOUNTAIN CLIMBING** As elevation increases, the atmospheric air pressure decreases. The formula for pressure based on elevation is $a = 15{,}500\ (5 - \log_{10} P)$, where a is the altitude in meters and P is the pressure in pascals (1 psi ≈ 6900 pascals). What is the air pressure at the summit in pascals for each mountain listed in the table at the right?

Mountain	Country	Height (m)
Everest	Nepal/Tibet	8850
Trisuli	India	7074
Bonete	Argentina/Chile	6872
McKinley	United States	6194
Logan	Canada	5959

Source: infoplease.com

Example 4
(p. 522)

Given $\log_2 7 \approx 2.8074$ and $\log_5 8 \approx 1.2920$ to approximate the value of each expression.

6. $\log_2 49$
7. $\log_5 64$

Example 5
(p. 523)

Solve each equation. Check your solutions.

8. $\log_3 42 - \log_3 n = \log_3 7$
9. $\log_2(3x) + \log_2 5 = \log_2 30$
10. $2 \log_5 x = \log_5 9$
11. $\log_{10} a + \log_{10} (a + 21) = 2$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–14	1
15–17	2
18–20	3
21–24	4
25–30	5

Use $\log_5 2 \approx 0.4307$ and $\log_5 3 \approx 0.6826$ to approximate the value of each expression.

12. $\log_5 50$
13. $\log_5 30$
14. $\log_5 20$
15. $\log_5 \frac{2}{3}$
16. $\log_5 \frac{3}{2}$
17. $\log_5 \frac{4}{3}$
18. $\log_5 9$
19. $\log_5 8$
20. $\log_5 16$

21. **EARTHQUAKES** The great Alaskan earthquake, in 1964, was about 100 times as intense as the Loma Prieta earthquake in San Francisco, in 1989. Find the difference in the Richter scale magnitudes of the earthquakes.

PROBABILITY For Exercises 22–24, use the following information.
In the 1930s, Dr. Frank Benford demonstrated a way to determine whether a set of numbers have been randomly chosen or the numbers have been manually chosen. If the sets of numbers were not randomly chosen, then the Benford formula, $P = \log_{10}\left(1 + \frac{1}{d}\right)$, predicts the probability of a digit d being the first digit of the set. For example, there is a 4.6% probability that the first digit is 9.

22. Rewrite the formula to solve for the digit if given the probability.
23. Find the digit that has a 9.7% probability of being selected.
24. Find the probability that the first digit is 1 ($\log_{10} 2 \approx 0.30103$).

Solve each equation. Check your solutions.

25. $\log_3 5 + \log_3 x = \log_3 10$
26. $\log_4 a + \log_4 9 = \log_4 27$
27. $\log_{10} 16 - \log_{10} (2t) = \log_{10} 2$
28. $\log_7 24 - \log_7 (y + 5) = \log_7 8$
29. $\log_2 n = \frac{1}{4} \log_2 16 + \frac{1}{2} \log_2 49$
30. $2 \log_{10} 6 - \frac{1}{3} \log_{10} 27 = \log_{10} x$

Solve for n.

31. $\log_a (4n) - 2 \log_a x = \log_a x$

32. $\log_b 8 + 3 \log_b n = 3 \log_b (x - 1)$

Solve each equation. Check your solutions.

33. $\log_{10} z + \log_{10} (z + 3) = 1$

34. $\log_6 (a^2 + 2) + \log_6 2 = 2$

35. $\log_2 (12b - 21) - \log_2 (b^2 - 3) = 2$

36. $\log_2 (y + 2) - \log_2 (y - 2) = 1$

37. $\log_3 0.1 + 2 \log_3 x = \log_3 2 + \log_3 5$

38. $\log_5 64 - \log_5 \frac{8}{3} + \log_5 2 = \log_5 (4p)$

Real-World Link

The Greek astronomer Hipparchus made the first known catalog of stars. He listed the brightness of each star on a scale of 1 to 6, the brightest being 1. With no telescope, he could only see stars as dim as the 6th magnitude.

Source: NASA

SOUND For Exercises 39–41, use the formula for the loudness of sound in Example 3 on page 546. Use $\log_{10} 2 \approx 0.3010$ and $\log_{10} 3 \approx 0.4771$.

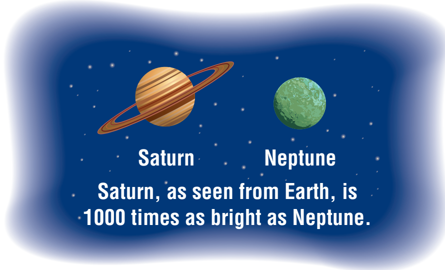
39. A certain sound has a relative intensity of R. By how many decibels does the sound increase when the intensity is doubled?

40. A certain sound has a relative intensity of R. By how many decibels does the sound decrease when the intensity is halved?

41. A stadium containing 10,000 cheering people can produce a crowd noise of about 90 decibels. If everyone cheers with the same relative intensity, how much noise, in decibels, is a crowd of 30,000 people capable of producing? Explain your reasoning.

STAR LIGHT For Exercises 42–44, use the following information.

The brightness, or apparent magnitude, m of a star or planet is given by $m = 6 - 2.5 \log_{10} \frac{L}{L_0}$, where L is the amount of light L coming to Earth from the star or planet and L_0 is the amount of light from a sixth magnitude star.



42. Find the difference in the magnitudes of Sirius and the crescent moon.

43. Find the difference in the magnitudes of Saturn and Neptune.

44. RESEARCH Use the Internet or other reference to find the magnitude of the dimmest stars that we can now see with ground-based telescopes.

EXTRA PRACTICE

See pages 910, 934.

Math Online

Self-Check Quiz at algebra2.com

H.O.T. Problems

45. REASONING Use the properties of exponents to prove the Power Property of Logarithms.

46. REASONING Use the properties of Logarithms to prove that $\log_a \frac{1}{x} = -\log_a x$.

47. CHALLENGE Simplify $\log_{\sqrt{a}} (a^2)$ to find an exact numerical value.

48. CHALLENGE Simplify $x^{3 \log_x 2 - \log_x 5}$ to find an exact numerical value.

CHALLENGE **Tell whether each statement is *true* or *false*. If true, show that it is true. If false, give a counterexample.**

49. For all positive numbers m, n, and b, where $b \neq 1$, $\log_b (m + n) = \log_b m + \log_b n$.

50. For all positive numbers m, n, x, and b, where $b \neq 1$, $n \log_b x + m \log_b x = (n + m) \log_b x$.

51. **REASONING** Use the properties of exponents to prove the Quotient Property of Logarithms.

52. ***Writing in Math*** Use the information given regarding exponents and logarithms on page 520 to explain how the properties of exponents and logarithms are related. Include examples like the one shown at the beginning of the lesson illustrating the Quotient Property and Power Property of Logarithms, and an explanation of the similarity between one property of exponents and its related property of logarithms in your answer.

STANDARDIZED TEST PRACTICE

53. **ACT/SAT** To what is $2 \log_5 12 - \log_5 8 - 2 \log_5 3$ equal?

A $\log_5 2$

B $\log_5 3$

C $\log_5 0.5$

D 1

54. **REVIEW** In a movie theater, 2 boys and 3 girls are seated randomly together. What is the probability that the 2 boys are seated next to each other?

F $\frac{1}{5}$ G $\frac{2}{5}$ H $\frac{1}{2}$ J $\frac{2}{3}$

Spiral Review

Evaluate each expression. (Lesson 9-2)

55. $\log_3 81$ **56.** $\log_9 \frac{1}{729}$ **57.** $\log_7 7^{2x}$

Solve each equation or inequality. Check your solutions. (Lesson 9-1)

58. $3^{5n+3} = 3^{33}$ **59.** $7^a = 49^{-4}$ **60.** $3^{d+4} > 9^d$

61. **PHYSICS** If a stone is dropped from a cliff, the equation $t = \frac{1}{4}\sqrt{d}$ represents the time t in seconds that it takes for the stone to reach the ground. If d represents the distance in feet that the stone falls, find how long it would take for a stone to hit the ground after falling from a 150-foot cliff. (Lesson 7-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation or inequality. Check your solutions.** (Lesson 9-2)

62. $\log_3 x = \log_3 (2x - 1)$ **63.** $\log_{10} 2^x = \log_{10} 32$

64. $\log_2 3x > \log_2 5$ **65.** $\log_5 (4x + 3) < \log_5 11$

CHAPTER 9

Mid-Chapter Quiz

Lessons 9-1 through 9-3

RABBIT POPULATION For Exercises 1 and 2, use the following information. (Lesson 9-1)

Rabbits reproduce at a tremendous rate and their population increases exponentially in the absence of natural enemies. Suppose there were originally 65,000 rabbits in a region and two years later there are 2,500,000.

1. Write an exponential function that could be used to model the rabbit population y in that region. Write the function in terms of x, the number of years since the original year.
2. Assume that the rabbit population continued to grow at that rate. Estimate the rabbit population in that region seven years later.
3. Determine whether $5(1.2)^x$ represents exponential *growth* or *decay*. Explain. (Lesson 9-1)
4. **SAVINGS** Suppose you deposit $500 in an account paying 4.5% interest compounded semiannually. Find the dollar value of the account rounded to the nearest penny after 10 years. (Lesson 9-1)

Evaluate each expression. (Lesson 9-2)

5. $\log_8 16$
6. $\log_4 4^{15}$
7. **MULTIPLE CHOICE** What is the value of n if $\log_3 3^{4n-1} = 11$? (Lesson 9-2)

   A 3

   B 4

   C 6

   D 12

Solve each equation or inequality. Check your solution. (Lessons 9-1 through 9-3)

8. $3^{4x} = 3^{3-x}$
9. $3^{2n} \leq \frac{1}{9}$
10. $3^{5x} \cdot 81^{1-x} = 9^{x-3}$
11. $49^x = 7^{x^2-15}$
12. $\log_2 (x + 6) > 5$
13. $\log_5 (4x - 1) = \log_5 (3x + 2)$
14. **MULTIPLE CHOICE** Find the value of x for $\log_2 (9x + 5) = 2 + \log_2 (x^2 - 1)$. (Lesson 9-3)

   F -0.4 H 1

   G 0   J 3

HEALTH For Exercises 15–17, use the following information. (Lesson 9-3)

The pH of a person's blood is given by the function $pH = 6.1 + \log_{10} B - \log_{10} C$, where B is the concentration of bicarbonate, which is a base, in the blood, and C is the concentration of carbonic acid in the blood.

Substance	pH
Lemon juice	2.3
Milk	6.4
Baking soda	8.4
Ammonia	11.9
Drain cleaner	14.0

15. Use the Quotient Property of Logarithms to simplify the formula for blood pH.
16. Most people have a blood pH of 7.4. What is the approximate ratio of bicarbonate to carbonic acid for blood with this pH?
17. If a person's ratio of bicarbonate to carbonic acid is 17.5:2.25, determine which substance has a pH closest to this person's blood.

ENERGY For Exercises 18–20, use the following information. (Lesson 9-3)

The energy E (in kilocalories per gram molecule) needed to transport a substance from the outside to the inside of a living cell is given by $E = 1.4(\log_{10} C_2 - \log_{10} C_1)$, where C_1 is the concentration of the substance outside the cell and C_2 is the concentration inside the cell.

18. Express the value of E as one logarithm.
19. Suppose the concentration of a substance inside the cell is twice the concentration outside the cell. How much energy is needed to transport the substance on the outside of the cell to the inside? (Use $\log_{10} 2 \approx 0.3010$.)
20. Suppose the concentration of a substance inside the cell is four times the concentration outside the cell. How much energy is needed to transport the substance from the outside of the cell to the inside?

9-4 Common Logarithms

Main Ideas

- Solve exponential equations and inequalities using common logarithms.
- Evaluate logarithmic expressions using the Change of Base Formula.

New Vocabulary

common logarithm

Change of Base Formula

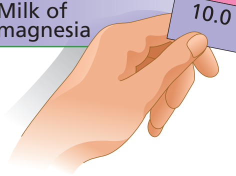
GET READY for the Lesson

The pH level of a substance measures its acidity. A low pH indicates an acid solution while a high pH indicates a basic solution. The pH levels of some common substances are shown.

The pH level of a substance is given by pH $= -\log_{10}[H^+]$, where H^+ is the substance's hydrogen ion concentration in moles per liter. Another way of writing this formula is pH $= -\log[H^+]$.

Acidity of Common Substances

Substance	pH Level
Battery acid	1.0
Sauerkraut	3.5
Tomatoes	4.2
Black coffee	5.0
Milk	6.4
Distilled water	7.0
Eggs	7.8
Milk of magnesia	10.0



Common Logarithms You have seen that the base 10 logarithm function, $y = \log_{10} x$, is used in many applications. Base 10 logarithms are called **common logarithms**. Common logarithms are usually written without the subscript 10.

$$\log_{10} x = \log x,\ x > 0$$

Most scientific calculators have a $\boxed{\text{LOG}}$ key for evaluating common logarithms.

EXAMPLE 1 Find Common Logarithms

Use a calculator to evaluate each expression to four decimal places.

a. **log 3**

KEYSTROKES: $\boxed{\text{LOG}}$ 3 $\boxed{\text{ENTER}}$ .4771212547

log 3 is about 0.4771.

b. **log 0.2**

KEYSTROKES: $\boxed{\text{LOG}}$ 0.2 $\boxed{\text{ENTER}}$ −.6989700043

log 0.2 is about −0.6990.

CHECK Your Progress

1A. log 5 **1B.** log 0.5

Study Tip

Technology

Nongraphing scientific calculators often require entering the number followed by the function, for example, 3 $\boxed{\text{LOG}}$.

Sometimes an application of logarithms requires that you use the inverse of logarithms, or exponentiation.

$$10^{\log x} = x$$

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Real-World EXAMPLE — Solve Logarithmic Equations

2 **EARTHQUAKES** **The amount of energy E, in ergs, that an earthquake releases is related to its Richter scale magnitude M by the equation $\log E = 11.8 + 1.5M$. The Chilean earthquake of 1960 measured 8.5 on the Richter scale. How much energy was released?**

$\log E = 11.8 + 1.5M$	Write the formula.
$\log E = 11.8 + 1.5(8.5)$	Replace M with 8.5.
$\log E = 24.55$	Simplify.
$10^{\log E} = 10^{24.55}$	Write each side using exponents and base 10.
$E = 10^{24.55}$	Inverse Property of Exponents and Logarithms
$E \approx 3.55 \times 10^{24}$	Use a calculator.

The amount of energy released by this earthquake was about 3.55×10^{24} ergs.

CHECK Your Progress

2. Use the equation above to find the energy released by the 2004 Sumatran earthquake, which measured 9.0 on the Richter scale and led to a tsunami.

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If both sides of an exponential equation cannot easily be written as powers of the same base, you can solve by taking the logarithm of each side.

EXAMPLE — Solve Exponential Equations Using Logarithms

3 **Solve $3^x = 11$.**

$3^x = 11$	Original equation
$\log 3^x = \log 11$	Property of Equality for Logarithmic Functions
$x \log 3 = \log 11$	Power Property of Logarithms
$x = \dfrac{\log 11}{\log 3}$	Divide each side by log 3.
$x \approx \dfrac{1.0414}{0.4771}$	Use a calculator.
$x \approx 2.1828$	

The solution is approximately 2.1828.

CHECK You can check this answer using a calculator or by using estimation. Since $3^2 = 9$ and $3^3 = 27$, the value of x is between 2 and 3. In addition, the value of x should be closer to 2 than 3, since 11 is closer to 9 than 27. Thus, 2.1828 is a reasonable solution. ✓

> **Study Tip**
>
> **Using Logarithms**
> When you use the Property of Equality for Logarithmic Functions, this is sometimes referred to as *taking the logarithm of each side.*

CHECK Your Progress

Solve each equation.

3A. $4^x = 15$ **3B.** $6^x = 42$

EXAMPLE 4 Solve Exponential Inequalities Using Logarithms

4 Solve $5^{3y} < 8^{y-1}$.

$$5^{3y} < 8^{y-1} \quad \text{Original inequality}$$

$$\log 5^{3y} < \log 8^{y-1} \quad \text{Property of Inequality for Logarithmic Functions}$$

$$3y \log 5 < (y-1) \log 8 \quad \text{Power Property of Logarithms}$$

$$3y \log 5 < y \log 8 - \log 8 \quad \text{Distributive Property}$$

$$3y \log 5 - y \log 8 < -\log 8 \quad \text{Subtract } y \log 8 \text{ from each side.}$$

$$y(3 \log 5 - \log 8) < -\log 8 \quad \text{Distributive Property}$$

$$y < \frac{-\log 8}{3 \log 5 - \log 8} \quad \text{Divide each side by } 3 \log 5 - \log 8.$$

$$y < -0.7564 \quad \text{Use a calculator.}$$

The solution set is $\{y \mid y < -0.7564\}$.

CHECK Test $y = -1$.

$$5^{3y} < 8^{y-1} \quad \text{Original inequality}$$

$$5^{3(-1)} < 8^{(-1)-1} \quad \text{Replace } y \text{ with } -1.$$

$$5^{-3} < 8^{-2} \quad \text{Simplify.}$$

$$\frac{1}{125} < \frac{1}{64} \checkmark \quad \text{Negative Exponent Property}$$

CHECK Your Progress Solve each inequality.

4A. $3^{2x} \geq 6^{x+1}$ **4B.** $4^y < 5^{2y+1}$

Study Tip

Solving Inequalities

Remember that the direction of an inequality must be switched if both sides are multiplied or divided by a negative number. Since $3 \log 5 - \log 8 > 0$, the inequality does not change.

Change of Base Formula The **Change of Base Formula** allows you to write equivalent logarithmic expressions that have different bases.

KEY CONCEPT — *Change of Base Formula*

Symbols For all positive numbers, a, b and n, where $a \neq 1$ and $b \neq 1$,

$$\log_a n = \frac{\log_b n}{\log_b a}. \quad \begin{array}{l} \leftarrow \text{log base } b \text{ of original number} \\ \leftarrow \text{log base } b \text{ of old base} \end{array}$$

Example $\log_5 12 = \dfrac{\log_{10} 12}{\log_{10} 5}$

To prove this formula, let $\log_a n = x$.

$$a^x = n \quad \text{Definition of logarithm}$$

$$\log_b a^x = \log_b n \quad \text{Property of Equality for Logarithms}$$

$$x \log_b a = \log_b n \quad \text{Power Property of Logarithms}$$

$$x = \frac{\log_b n}{\log_b a} \quad \text{Divide each side by } \log_b a.$$

$$\log_a n = \frac{\log_b n}{\log_b a} \quad \text{Replace } x \text{ with } \log_a n.$$

The Change of Base Formula makes it possible to evaluate a logarithmic expression of any base by translating the expression into one that involves common logarithms.

EXAMPLE **Change of Base Formula**

5 **Express $\log_4 25$ in terms of common logarithms. Then approximate its value to four decimal places.**

$$\log_4 25 = \frac{\log_{10} 25}{\log_{10} 4} \quad \text{Change of Base Formula}$$

$$\approx 2.3219 \quad \text{Use a calculator.}$$

The value of $\log_4 25$ is approximately 2.3219.

CHECK Your Progress

5. Express $\log_6 8$ in terms of common logarithms. Then approximate its value to four decimal places.

CHECK Your Understanding

Example 1 (p. 528)

Use a calculator to evaluate each expression to four decimal places.

1. log 4 **2.** log 23 **3.** log 0.5

Example 2 (p. 529)

4. NUTRITION For health reasons, Sandra's doctor has told her to avoid foods that have a pH of less than 4.5. What is the hydrogen ion concentration of foods Sandra is allowed to eat? Use the information at the beginning of the lesson.

Example 3 (p. 529)

Solve each equation. Round to four decimal places.

5. $9^x = 45$ **6.** $3.1^{a-3} = 9.42$

7. $11^{x^2} = 25.4$ **8.** $7^{t-2} = 5^t$

Example 4 (p. 530)

Solve each inequality. Round to four decimal places.

9. $4^{5n} > 30$ **10.** $4^{p-1} \leq 3^p$

Example 5 (p. 531)

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

11. $\log_7 5$ **12.** $\log_3 42$ **13.** $\log_2 9$

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–19	1
20, 21	2
22–27	3
28–33	4
34–39	5

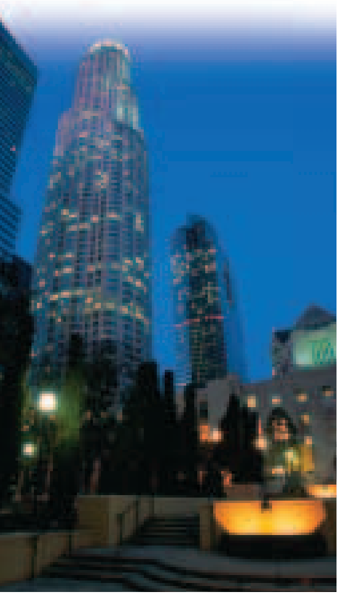
Use a calculator to evaluate each expression to four decimal places.

14. log 5 **15.** log 12 **16.** log 7.2

17. log 2.3 **18.** log 0.8 **19.** log 0.03

20. POLLUTION The acidity of water determines the toxic effects of runoff into streams from industrial or agricultural areas. A pH range of 6.0 to 9.0 appears to provide protection for freshwater fish. What is this range in terms of the water's hydrogen ion concentration?

21. BUILDING DESIGN The 1971 Sylmar earthquake in Los Angeles had a Richter scale magnitude of 6.3. Suppose an architect has designed a building strong enough to withstand an earthquake 50 times as intense as the Sylmar quake. Find the magnitude of the strongest quake this building can withstand.



Real-World Link There are an estimated 500,000 detectable earthquakes in the world each year. Of these earthquakes, 100,000 can be felt and 100 cause damage.

Source: earthquake.usgs.gov

Solve each equation or inequality. Round to four decimal places.

22. $5^x = 52$ **23.** $4^{3p} = 10$ **24.** $3^{n+2} = 14.5$

25. $9^{z-4} = 6.28$ **26.** $8.2^{n-3} = 42.5$ **27.** $2.1^{t-5} = 9.32$

28. $6^x \geq 42$ **29.** $8^{2a} < 124$ **30.** $4^{3x} \leq 72$

31. $8^{2n} > 52^{4n+3}$ **32.** $7^{p+2} \leq 13^{5-p}$ **33.** $3^{y+2} \geq 8^{3y}$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

34. $\log_2 13$ **35.** $\log_5 20$ **36.** $\log_7 3$

37. $\log_3 8$ **38.** $\log_4 (1.6)^2$ **39.** $\log_6 \sqrt{5}$

ACIDITY For Exercises 40–43, use the information at the beginning of the lesson to find each pH given the concentration of hydrogen ions.

40. ammonia: $[H^+] = 1 \times 10^{-11}$ mole per liter

41. vinegar: $[H^+] = 6.3 \times 10^{-3}$ mole per liter

42. lemon juice: $[H^+] = 7.9 \times 10^{-3}$ mole per liter

43. orange juice: $[H^+] = 3.16 \times 10^{-4}$ mole per liter

Solve each equation. Round to four decimal places.

44. $20^{x^2} = 70$ **45.** $2^{x^2-3} = 15$ **46.** $2^{2x+3} = 3^{3x}$

47. $16^{d-4} = 3^{3-d}$ **48.** $5^{5y-2} = 2^{2y+1}$ **49.** $8^{2x-5} = 5^{x+1}$

50. $2^n = \sqrt{3^{n-2}}$ **51.** $4^x = \sqrt{5^{x+2}}$ **52.** $3^y = \sqrt{2^{y-1}}$

MUSIC For Exercises 53 and 54, use the following information.
The musical cent is a unit in a logarithmic scale of relative pitch or intervals. One octave is equal to 1200 cents. The formula to determine the difference in cents between two notes with frequencies a and b is $n = 1200\left(\log_2 \frac{a}{b}\right)$.

53. Find the interval in cents when the frequency changes from 443 Hertz (Hz) to 415 Hz.

54. If the interval is 55 cents and the beginning frequency is 225 Hz, find the final frequency.

MONEY For Exercises 55 and 56, use the following information.
If you deposit P dollars into a bank account paying an annual interest rate r (expressed as a decimal), with n interest payments each year, the amount A you would have after t years is $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Marta places \$100 in a savings account earning 2% annual interest, compounded quarterly.

55. If Marta adds no more money to the account, how long will it take the money in the account to reach \$125?

56. How long will it take for Marta's money to double?

EXTRA PRACTICE
See pages 910, 934.
Math Online
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H.O.T. Problems

57. CHALLENGE Solve $\log_{\sqrt{a}} 3 = \log_a x$ for x and explain each step.

58. Write $\frac{\log_5 9}{\log_5 3}$ as a single logarithm.

59. CHALLENGE

a. Find the values of $\log_2 8$ and $\log_8 2$.

b. Find the values of $\log_9 27$ and $\log_{27} 9$.

c. Make and prove a conjecture about the relationship between $\log_a b$ and $\log_b a$.

60. ***Writing in Math*** Use the information about acidity of common substances on page 528 to explain why a logarithmic scale is used to measure acidity. Include the hydrogen ion concentration of three substances listed in the table, and an explanation as to why it is important to be able to distinguish between a hydrogen ion concentration of 0.00001 mole per liter and 0.0001 mole per liter in your answer.

STANDARDIZED TEST PRACTICE

61. ACT/SAT If $2^4 = 3^x$, then what is the approximate value of x?

A 0.63

B 2.34

C 2.52

D 4

62. REVIEW Which equation is equivalent to $\log_4 \frac{1}{16} = x$?

F $\frac{1^4}{16} = x^4$

G $\left(\frac{1}{16}\right)^4 = x$

H $4^x = \frac{1}{16}$

J $4^{\frac{1}{16}} = x$

Spiral Review

Use $\log_7 2 \approx 0.3562$ and $\log_7 3 \approx 0.5646$ to approximate the value of each expression. (Lesson 9-3)

63. $\log_7 16$

64. $\log_7 27$

65. $\log_7 36$

Solve each equation or inequality. Check your solutions. (Lesson 9-2)

66. $\log_4 r = 3$

67. $\log_8 z \leq -2$

68. $\log_3 (4x - 5) = 5$

69. Use synthetic substitution to find $f(-2)$ for $f(x) = x^3 + 6x - 2$. (Lesson 6-7)

70. MONEY Viviana has two dollars worth of nickels, dimes, and quarters. She has 18 total coins, and the number of nickels equals 25 minus twice the number of dimes. How many nickels, dimes, and quarters does she have? (Lesson 3-5)

GET READY for the Next Lesson

PREREQUISITE SKILL Write an equivalent exponential equation. (Lesson 9-2)

71. $\log_2 3 = x$

72. $\log_3 x = 2$

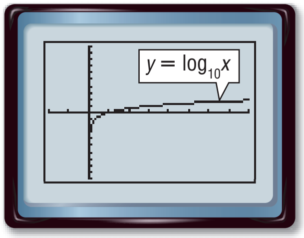
73. $\log_5 125 = 3$

EXTEND 9-4

Graphing Calculator Lab

Solving Logarithmic Equations and Inequalities

You have solved logarithmic equations algebraically. You can also solve logarithmic equations by graphing or by using a table. The calculator has $y = \log_{10} x$ as a built-in function. Enter [Y=] [LOG] [X,T,θ,n] [GRAPH] to view this graph. To graph logarithmic functions with bases other than 10, you must use the Change of Base Formula, $\log_a n = \dfrac{\log_b n}{\log_b a}$.



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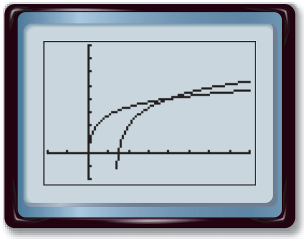
ACTIVITY 1 **Solve $\log_2 (6x - 8) = \log_3 (20x + 1)$.**

Step 1 **Graph each side of the equation.**

Graph each side of the equation as a separate function. Enter $\log_2 (6x - 8)$ as **Y1** and $\log_3 (20x + 1)$ as **Y2**. Then graph the two equations.

KEYSTROKES: [Y=] [LOG] 6 [X,T,θ,n] [−] 8 [)] [÷] [LOG] 2 [)] [ENTER] [LOG] 20 [X,T,θ,n] [+] 1 [)] [÷] [LOG] 3 [)] [GRAPH]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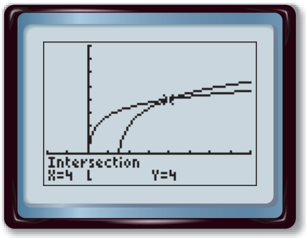
Step 2 **Use the intersect feature.**

Use the **intersect** feature on the **CALC** menu to approximate the ordered pair of the point at which the curves cross.

KEYSTROKES: *See page 121 to review how to use the intersect feature.*



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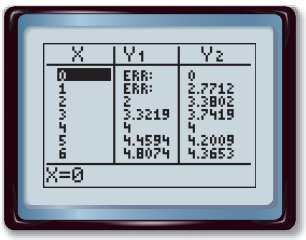
The calculator screen shows that the x-coordinate of the point at which the curves cross is 4. Therefore, the solution of the equation is 4.

Step 3 **Use the TABLE feature.**

KEYSTROKES: *See page 508.*



Examine the table to find the x-value for which the y-values for the graphs are equal. At $x = 4$, both functions have a y-value of 4. Thus, the solution of the equation is 4.

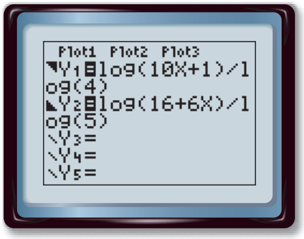
You can use a similar procedure to solve logarithmic inequalities using a graphing calculator.

ACTIVITY 2 **Solve $\log_4 (10x + 1) < \log_5 (16 + 6x)$.**

Step 1 **Enter the inequalities.**

Rewrite the problem as a system of inequalities.

The first inequality is $\log_4 (10x + 1) < y$, which can be written as $y > \log_4 (10x + 1)$. Since this inequality includes the *greater than* symbol, shade above the curve. First enter the boundary and then use the arrow and [ENTER] keys to choose the shade above icon, ◥.



The second inequality is $y < \log_5 (16 + 6x)$. Shade below the curve since this inequality contains *less than*.

KEYSTROKES: [Y=] [◀] [◀] [ENTER] [ENTER] [▶] [▶] [LOG] 10 [X,T,θ,n] [+] 1 [)] [÷] [LOG] 4 [)] [ENTER] [◀] [◀] [ENTER] [ENTER] [ENTER] [▶] [▶] [LOG] 16 [+] 6 [X,T,θ,n] [)] [÷] [LOG] 5 [)]

Step 2 **Graph the system.**

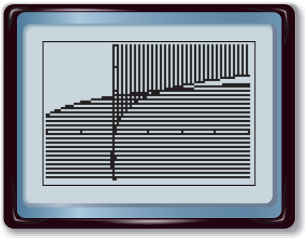
KEYSTROKES: [GRAPH]

The left boundary of the solution set is where the first inequality is undefined. It is undefined for $10x + 1 \leq 0$.

$$10x + 1 \leq 0$$

$$10x \leq -1$$

$$x \leq -\frac{1}{10}$$



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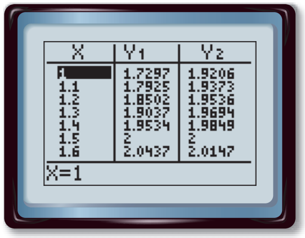
Use the calculator's **intersect** feature to find the right boundary. You can conclude that the solution set is $\{x \mid -0.1 < x < 1.5\}$.

Step 3 **Use the TABLE feature to check your solution.**

Start the table at −0.1 and show x-values in increments of 0.1. Scroll through the table.

KEYSTROKES: [2nd] [TBLSET] −0.1 [ENTER] .5 [ENTER] [2nd] [TABLE]



The table confirms the solution of the inequality is $\{x \mid -0.1 < x < 1.5\}$.

EXERCISES

Solve each equation or inequality. Check your solution.

1. $\log_2 (3x + 2) = \log_3 (12x + 3)$

2. $\log_6 (7x + 1) = \log_4 (4x - 4)$

3. $\log_2 3x = \log_3 (2x + 2)$

4. $\log_{10} (1 - x) = \log_5 (2x + 5)$

5. $\log_4 (9x + 1) > \log_3 (18x - 1)$

6. $\log_3 (3x - 5) \geq \log_3 (x + 7)$

7. $\log_5 (2x + 1) < \log_4 (3x - 2)$

8. $\log_2 2x \leq \log_4 (x + 3)$

9-5 Base *e* and Natural Logarithms

Main Ideas

- Evaluate expressions involving the natural base and natural logarithms.
- Solve exponential equations and inequalities using natural logarithms.

New Vocabulary

natural base, *e*

natural base exponential function

natural logarithm

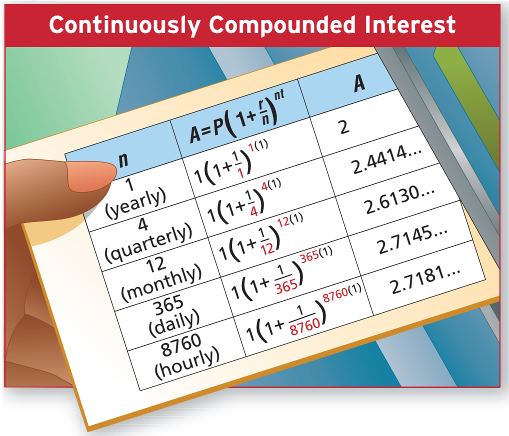
natural logarithmic function

GET READY for the Lesson

Suppose a bank compounds interest on accounts *continuously*, that is, with no waiting time between interest payments.

To develop an equation to determine continuously compounded interest, examine what happens to the value A of an account for increasingly larger numbers of compounding periods n. Use a principal P of \$1, an interest rate r of 100% or 1, and time t of 1 year.

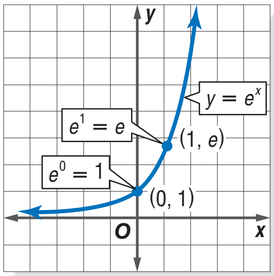
Continuously Compounded Interest



n	$A = P\left(1+\frac{r}{n}\right)^{nt}$	A
1 (yearly)	$1\left(1+\frac{1}{1}\right)^{1(1)}$	2
4 (quarterly)	$1\left(1+\frac{1}{4}\right)^{4(1)}$	2.4414...
12 (monthly)	$1\left(1+\frac{1}{12}\right)^{12(1)}$	2.6130...
365 (daily)	$1\left(1+\frac{1}{365}\right)^{365(1)}$	2.7145...
8760 (hourly)	$1\left(1+\frac{1}{8760}\right)^{8760(1)}$	2.7181...

Base *e* and Natural Logarithms In the table above, as n increases, the expression $1\left(1+\frac{1}{n}\right)^{n(1)}$ or $\left(1+\frac{1}{n}\right)^{n}$ approaches the irrational number 2.71828.... This number is referred to as the **natural base, *e*.**

An exponential function with base e is called a **natural base exponential function**. The graph of $y = e^x$ is shown at the right. Natural base exponential functions are used extensively in science to model quantities that grow and decay continuously.



Most calculators have an e^x function for evaluating natural base expressions.

Study Tip

Simplifying Expressions with e

You can simplify expressions involving e in the same manner in which you simplify expressions involving π.

Examples:

- $\pi^2 \cdot \pi^3 = \pi^5$
- $e^2 \cdot e^3 = e^5$

EXAMPLE Evaluate Natural Base Expressions

1 **Use a calculator to evaluate each expression to four decimal places.**

a. e^2 **KEYSTROKES:** [2nd] [e^x] 2 [ENTER] 7.389056099

$e^2 \approx 7.3891$

b. $e^{-1.3}$ **KEYSTROKES:** [2nd] [e^x] -1.3 [ENTER] .272531793

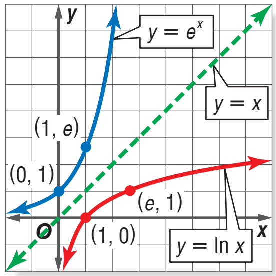
$e^{-1.3} \approx 0.2725$

CHECK Your Progress

1A. e^5

1B. $e^{-2.2}$

The logarithm with base e is called the **natural logarithm**, sometimes denoted by $\log_e x$, but more often abbreviated $\ln x$. The **natural logarithmic function**, $y = \ln x$, is the inverse of the natural base exponential function, $y = e^x$. The graph of these two functions shows that $\ln 1 = 0$ and $\ln e = 1$.



Study Tip

Calculator Keystrokes

On graphing calculators, you press the LN key before the number. On other calculators, usually you must type the number before pressing the LN key.

Most calculators have an LN key for evaluating natural logarithms.

EXAMPLE 2 Evaluate Natural Logarithmic Expressions

Use a calculator to evaluate each expression to four decimal places.

a. $\ln 4$ **KEYSTROKES:** LN 4 ENTER 1.386294361

$\ln 4 \approx 1.3863$

b. $\ln 0.05$ **KEYSTROKES:** LN 0.05 ENTER –2.995732274

$\ln 0.05 \approx -2.9957$

CHECK Your Progress

2A. $\ln 7$ **2B.** $\ln 0.25$

You can write an equivalent base e exponential equation for a natural logarithmic equation and vice versa by using the fact that $\ln x = \log_e x$.

EXAMPLE 3 Write Equivalent Expressions

Write an equivalent exponential or logarithmic equation.

a. $e^x = 5$

$e^x = 5 \longrightarrow \log_e 5 = x$

$\ln 5 = x$

b. $\ln x \approx 0.6931$

$\ln x \approx 0.6931 \longrightarrow \log_e x \approx 0.6931$

$x \approx e^{0.6931}$

CHECK Your Progress

3A. $e^x = 6$ **3B.** $\ln x \approx 0.5352$

Since the natural base function and the natural logarithmic function are inverses, these two functions can be used to "undo" each other.

$$e^{\ln x} = x \qquad \ln e^x = x$$

For example, $e^{\ln 7} = 7$ and $\ln e^{4x + 3} = 4x + 3$.

Equations and Inequalities with *e* and ln Equations and inequalities involving base e are easier to solve using natural logarithms than using common logarithms. All of the properties of logarithms that you have learned apply to natural logarithms as well.

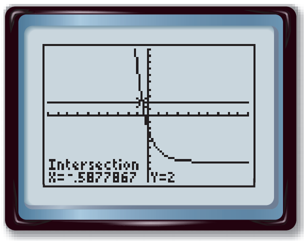
EXAMPLE 4 Solve Base *e* Equations

Solve $5e^{-x} - 7 = 2$. Round to the nearest ten-thousandth.

$5e^{-x} - 7 = 2$	Original equation
$5e^{-x} = 9$	Add 7 to each side.
$e^{-x} = \frac{9}{5}$	Divide each side by 5.
$\ln e^{-x} = \ln \frac{9}{5}$	Property of Equality for Logarithms
$-x = \ln \frac{9}{5}$	Inverse Property of Exponents and Logarithms
$x = -\ln \frac{9}{5}$	Divide each side by -1.
$x \approx -0.5878$	Use a calculator.

The solution is about -0.5878.

CHECK You can check this value by substituting -0.5878 into the original equation and evaluating, or by finding the intersection of the graphs of $y = 5e^{-x} - 7$ and $y = 2$.



CHECK Your Progress

Solve each equation. Round to the nearest ten-thousandth.

4A. $3e^x + 2 = 4$ **4B.** $4e^{-x} - 9 = -2$

Study Tip

Continuously Compounded Interest

Although no banks actually pay interest compounded continuously, the equation $A = Pe^{rt}$ is so accurate in computing the amount of money for quarterly compounding, or daily compounding, that it is often used for this purpose.

When interest is compounded continuously, the amount A in an account after t years is found using the formula $A = Pe^{rt}$, where P is the amount of principal and r is the annual interest rate.

Real-World EXAMPLE 5 Solve Base *e* Inequalities

Personal Tutor

SAVINGS Suppose you deposit $1000 in an account paying 2.5% annual interest, compounded continuously.

a. What is the balance after 10 years?

$A = Pe^{rt}$	Continuous compounding formula
$= 1000e^{(0.025)(10)}$	Replace P with 1000, r with 0.025, and t with 10.
$= 1000e^{0.25}$	Simplify.
≈ 1284.03	Use a calculator.

The balance after 10 years would be $1284.03.

CHECK If the account was earning simple interest, the formula for the interest, would be $I = prt$. In that case, the interest would be $I = (1000)(0.025)(10)$ or $250. Continuously compounded interest should be greater than simple interest at the same rate. Thus, the solution $1284.03 is reasonable.

b. **How long will it take for the balance in your account to reach at least $1500?**

Words	The balance is at least $1500.
Variable	Let A represent the amount in the account.
Inequality	$A \geq 1500$

$$\ln e^{(0.025)t} \geq 1500 \quad \text{Replace } A \text{ with } 1000e^{(0.025)t}.$$
$$\ln e^{(0.025)t} \geq 1.5 \quad \text{Divide each side by 1000.}$$
$$\ln e^{(0.025)t} \geq \ln 1.5 \quad \text{Property of Equality for Logarithms}$$
$$0.025t \geq \ln 1.5 \quad \text{Inverse Property of Exponents and Logarithms}$$
$$t \geq \frac{\ln 1.5}{0.025} \quad \text{Divide each side by 0.025.}$$
$$t \geq 16.22 \quad \text{Use a calculator.}$$

It will take at least 16.22 years for the balance to reach $1500.

CHECK Your Progress

Suppose you deposit $5000 in an account paying 3% annual interest, compounded continuously.

5A. What is the balance after 5 years?

5B. How long will it take for the balance in your account to reach at least $7000?

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Study Tip

Equations with ln

As with other logarithmic equations, remember to check for extraneous solutions.

EXAMPLE Solve Natural Log Equations and Inequalities

6 **Solve each equation or inequality. Round to the nearest ten-thousandth.**

a. $\ln 5x = 4$

$$\ln 5x = 4 \quad \text{Original equation}$$
$$e^{\ln 5x} = e^4 \quad \text{Write each side using exponents and base } e.$$
$$5x = e^4 \quad \text{Inverse Property of Exponents and Logarithms}$$
$$x = \frac{e^4}{5} \quad \text{Divide each side by 5.}$$
$$x \approx 10.9196 \quad \text{Use a calculator. Check using substitution or graphing.}$$

b. $\ln (x - 1) > -2$

$$\ln (x - 1) > -2 \quad \text{Original inequality}$$
$$e^{\ln (x - 1)} > e^{-2} \quad \text{Write each side using exponents and base } e.$$
$$x - 1 > e^{-2} \quad \text{Inverse Property of Exponents and Logarithms}$$
$$x > e^{-2} + 1 \quad \text{Add 1 to each side.}$$
$$x > 1.1353 \quad \text{Use a calculator. Check using substitution.}$$

CHECK Your Progress

6A. $\ln 3x = 7$

6B. $\ln (3x + 2) < 5$

CHECK Your Understanding

Examples 1, 2 (pp. 536, 537)

Use a calculator to evaluate each expression to four decimal places.

1. e^6 **2.** $e^{-3.4}$ **3.** $e^{0.35}$

4. $\ln 1.2$ **5.** $\ln 0.1$ **6.** $\ln 3.25$

Example 3 (p. 537)

Write an equivalent exponential or logarithmic equation.

7. $e^x = 4$ **8.** $\ln 1 = 0$

Example 4 (p. 538)

Solve each equation. Round to the nearest ten-thousandth.

9. $2e^x - 5 = 1$ **10.** $3 + e^{-2x} = 8$

Example 5 (pp. 538–539)

ALTITUDE For Exercises 11 and 12, use the following information.
The altimeter in an airplane gives the altitude or height h (in feet) of a plane above sea level by measuring the outside air pressure P (in kilopascals). The height and air pressure are related by the model $P = 101.3\,e^{-\frac{h}{26{,}200}}$.

11. Find a formula for the height in terms of the outside air pressure.

12. Use the formula you found in Exercise 11 to approximate the height of a plane above sea level when the outside air pressure is 57 kilopascals.

Example 6 (p. 539)

Solve each equation or inequality. Round to the nearest ten-thousandth.

13. $e^x > 30$ **14.** $\ln x < 6$

15. $2 \ln 3x + 1 = 5$ **16.** $\ln x^2 = 9$

Exercises

HOMEWORK HELP

For Exercises	See Examples
17–20	1
21–24	2
25–32	3
33–40	4
41–46	5
47–54	6

Use a calculator to evaluate each expression to four decimal places.

17. e^4 **18.** e^5 **19.** $e^{-1.2}$ **20.** $e^{0.5}$

21. $\ln 3$ **22.** $\ln 10$ **23.** $\ln 5.42$ **24.** $\ln 0.03$

Write an equivalent exponential or logarithmic equation.

25. $e^{-x} = 5$ **26.** $e^2 = 6x$ **27.** $\ln e = 1$ **28.** $\ln 5.2 = x$

29. $e^{x+1} = 9$ **30.** $e^{-1} = x^2$ **31.** $\ln \frac{7}{3} = 2x$ **32.** $\ln e^x = 3$

Solve each equation. Round to the nearest ten-thousandth.

33. $3e^x + 1 = 5$ **34.** $2e^x - 1 = 0$ **35.** $-3e^{4x} + 11 = 2$ **36.** $8 + 3e^{3x} = 26$

37. $2e^x - 3 = -1$ **38.** $-2e^x + 3 = 0$ **39.** $-2 + 3e^{3x} = 7$ **40.** $1 - \frac{1}{3}e^{5x} = -5$

POPULATION For Exercises 41 and 42, use the following information.
In 2005, the world's population was about 6.5 billion. If the world's population continues to grow at a constant rate, the future population P, in billions, can be predicted by $P = 6.5e^{0.02t}$, where t is the time in years since 2005.

41. According to this model, what will the world's population be in 2015?

42. Some experts have estimated that the world's food supply can support a population of at most 18 billion. According to this model, for how many more years will the food supply be able to support the trend in world population growth?

Real-World Link

To determine the doubling time on an account paying an interest rate r that is compounded annually, investors use the "Rule of 72." Thus, the amount of time needed for the money in an account paying 6% interest compounded annually to double is $\frac{72}{6}$ or 12 years.

Source: datachimp.com

MONEY **For Exercises 43–46, use the formula for continuously compounded interest found in Example 5.**

43. If you deposit \$100 in an account paying 3.5% interest compounded continuously, how long will it take for your money to double?

44. Suppose you deposit A dollars in an account paying an interest rate of r, compounded continuously. Write an equation giving the time t needed for your money to double, or the *doubling time*.

45. Explain why the equation you found in Exercise 44 might be referred to as the "Rule of 70."

46. **MAKE A CONJECTURE** State a rule that could be used to approximate the amount of time t needed to triple the amount of money in a savings account paying r percent interest compounded continuously.

Solve each equation or inequality. Round to the nearest ten-thousandth.

47. $\ln 2x = 4$ **48.** $\ln 3x = 5$ **49.** $\ln (x + 1) = 1$ **50.** $\ln (x - 7) = 2$

51. $e^x < 4.5$ **52.** $e^x > 1.6$ **53.** $e^{5x} \geq 25$ **54.** $e^{-2x} \leq 7$

E-MAIL **For Exercises 55 and 56, use the following information.**
The number of people N who will receive a forwarded e-mail can be approximated by $N = \frac{P}{1 + (P - S)e^{-0.35t}}$, where P is the total number of people online, S is the number of people who start the e-mail, and t is the time in minutes. Suppose four people want to send an e-mail to all those who are online at that time.

55. If there are 156,000 people online, how many people will have received the e-mail after 25 minutes?

56. How much time will pass before half of the people will receive the e-mail?

EXTRA PRACTICE
See pages 911, 934.

Math Online
Self-Check Quiz at algebra2.com

Solve each equation. Round to the nearest ten-thousandth.

57. $\ln x + \ln 3x = 12$ **58.** $\ln 4x + \ln x = 9$

59. $\ln (x^2 + 12) = \ln x + \ln 8$ **60.** $\ln x + \ln (x + 4) = \ln 5$

H.O.T. Problems

61. **OPEN ENDED** Give an example of an exponential equation that requires using natural logarithms instead of common logarithms to solve.

62. **FIND THE ERROR** Colby and Elsu are solving $\ln 4x = 5$. Who is correct? Explain your reasoning.

Colby

$\ln 4x = 5$

$10^{\ln 4x} = 10^5$

$4x = 100{,}000$

$x = 25{,}000$

Elsu

$\ln 4x = 5$

$e^{\ln 4x} = e^5$

$4x = e^5$

$x = \frac{e^5}{4}$

x 37.1033

63. **CHALLENGE** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For all positive numbers x and y, $\frac{\log x}{\log y} = \frac{\ln x}{\ln y}$.

64. *Writing in Math* Use the information about banking on page 536 to explain how the natural base e is used in banking. Include an explanation of how to calculate the value of an account whose interest is compounded continuously, and an explanation of how to use natural logarithms to find the time at which the account will have a specified value in your answer.

STANDARDIZED TEST PRACTICE

65. ACT/SAT A recent study showed that the number of Australian homes with a computer doubles every 8 months. Assuming that the number is increasing continuously, at approximately what monthly rate must the number of Australian computer owners be increasing for this to be true?

A 68%

B 8.66%

C 0.0866%

D 0.002%

66. REVIEW Which is the first *incorrect* step in simplifying $\log_3 \frac{3}{48}$?

Step 1: $\log_3 \frac{3}{48} = \log_3 3 - \log_3 48$

Step 2: $= 1 - 16$

Step 3: $= -15$

F Step 1

G Step 2

H Step 3

J Each step is correct.

Spiral Review

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 9-4)

67. $\log_4 68$ **68.** $\log_6 0.047$ **69.** $\log_{50} 23$

Solve each equation. Check your solutions. (Lesson 9-3)

70. $\log_3 (a + 3) + \log_3 (a - 3) = \log_3 16$ **71.** $\log_{11} 2 + 2 \log_{11} x = \log_{11} 32$

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation. (Lesson 8-4)

72. $mn = 4$ **73.** $\frac{a}{b} = c$ **74.** $y = -7x$

75. BASKETBALL Alexis has never scored a 3-point field goal, but she has scored a total of 59 points so far this season. She has made a total of 42 shots including free throws and 2-point field goals. How many free throws and 2-point field goals has Alexis scored? (Lesson 3-2)

GET READY for the Next Lesson

PREREQUISITE SKILL Solve each equation. Round to the nearest hundredth. (Lesson 9-1)

76. $2^x = 10$ **77.** $5^x = 12$ **78.** $6^x = 13$

79. $2(1 + 0.1)^x = 50$ **80.** $10(1 + 0.25)^x = 200$ **81.** $400(1 - 0.2)^x = 50$

READING MATH

Double Meanings

In mathematics, many words have specific definitions. However, when these words are used in everyday language, they frequently have a different meaning. Study each pair of sentences. How does the meaning of the word in boldface differ?

A. The number of boards that can be cut from a **log** depends on the size of the log.
B. The **log** of a number with base b represents the exponent to which b must be raised to produce that number.

A. Tinted paints are produced by adding small amounts of color to a **base** of white paint.
B. In the expression $\log_b x$, b is referred to as the **base** of the logarithm.

A. When a plant dies, it will **decay**, changing in form and substance, until it appears that the plant has disappeared.
B. If a quantity y satisfies a relationship of the form $y = ae^{-kt}$, the quantity y is described by an exponential **decay** model.

Read the following property and paragraph below. Which words are mathematical words? Which words are ordinary words? Which mathematical words have another meaning in everyday language?

KEY CONCEPT ***Product Property of Radicals***

For any nonnegative real numbers a and b and any integer n greater than 1, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Simplifying a square root means finding the square root of the greatest perfect square factor of the radicand. You can use the product property of radicals to simplify square roots.

Exercises

Write two sentences for each word. First, use the word in everyday language. Then use the word in a mathematical context.

1. index	**2.** negative	**3.** even	**4.** rational
5. irrational	**6.** like	**7.** rationalize	**8.** coordinates
9. real	**10.** degree	**11.** absolute	**12.** identity

9-6 Exponential Growth and Decay

Main Ideas

- Use logarithms to solve problems involving exponential decay.
- Use logarithms to solve problems involving exponential growth.

New Vocabulary

rate of decay
rate of growth

GET READY for the Lesson

Certain assets, like homes, can *appreciate* or increase in value over time. Others, like cars, *depreciate* or decrease in value with time. Suppose you buy a car for \$22,000 and the value of the car decreases by 16% each year. The table shows the value of the car each year for up to 5 years after it was purchased.

Years after Purchase	Value of Car (\$)
0	22,000.00
1	18,480.00
2	15,523.20
3	13,039.49
4	10,953.17
5	9200.66

Exponential Decay The depreciation of the value of a car is an example of exponential decay. When a quantity *decreases* by a fixed percent each year, or other period of time, the amount y of that quantity after t years is given by $y = a(1 - r)^t$, where a is the initial amount and r is the percent of decrease expressed as a decimal. The percent of decrease r is also referred to as the **rate of decay**.

EXAMPLE 1 Exponential Decay of the Form $y = a(1 - r)^t$

CAFFEINE A cup of coffee contains 130 milligrams of caffeine. If caffeine is eliminated from the body at a rate of 11% per hour, how long will it take for half of this caffeine to be eliminated?

Explore The problem gives the amount of caffeine consumed and the rate at which the caffeine is eliminated. It asks you to find the time it will take for half of the caffeine to be eliminated.

Plan Use the formula $y = a(1 - r)^t$. Let t be the number of hours since drinking the coffee. The amount remaining y is half of 130 or 65.

Solve

$y = a(1 - r)^t$	Exponential decay formula
$65 = 130(1 - 0.11)^t$	Replace y with 65, a with 130, and r with 11% or 0.11.
$0.5 = (0.89)^t$	Divide each side by 130.
$\log 0.5 = \log (0.89)^t$	Property of Equality for Logarithms
$\log 0.5 = t \log (0.89)$	Power Property for Logarithms
$\dfrac{\log 0.5}{\log 0.89} = t$	Divide each side by log 0.89.
$5.9480 \approx t$	Use a calculator.

It will take approximately 6 hours.

Study Tip

Rate of Change Remember to rewrite the rate of change as a decimal before using it in the formula.

Check Use the formula to find how much of the original 130 milligrams of caffeine would remain after 6 hours.

$$y = a(1 - r)^t \quad \text{Exponential decay formula}$$

$$= 130(1 - 0.11)^6 \quad \text{Replace } a \text{ with 130, } r \text{ with 0.11, and } t \text{ with 6.}$$

$$\approx 64.6 \quad \text{Use a calculator.}$$

Half of 130 is 65, so the answer seems reasonable. Half of the caffeine will be eliminated from the body in about 6 hours.

CHECK Your Progress

1. **SHOPPING** A store is offering a clearance sale on a certain type of digital camera. The original price for the camera was $198. The price decreases 10% each week until all of the cameras are sold. How many weeks will it take for the price of the cameras to drop below half of the original price?

Another model for exponential decay is given by $y = ae^{-kt}$, where k is a constant. This is the model preferred by scientists. Use this model to solve problems involving radioactive decay. Radioactive decay is the decrease in the intensity of a radioactive material over time. Being able to solve problems involving radioactive decay allows scientists to use carbon dating methods.

EXAMPLE Exponential Decay of the Form $y = ae^{-kt}$

Real-World Career

Paleontologist

Paleontologists study fossils found in geological formations. They use these fossils to trace the evolution of plant and animal life and the geologic history of Earth.

Math Online

For more information, go to algebra2.com.

2 PALEONTOLOGY The *half-life* of a radioactive substance is the time it takes for half of the atoms of the substance to disintegrate. All life on Earth contains Carbon-14, which decays continuously at a fixed rate. The half-life of Carbon-14 is 5760 years. That is, every 5760 years half of a mass of Carbon-14 decays away.

a. What is the value of k and the equation of decay for Carbon-14?

Let a be the initial amount of the substance. The amount y that remains after 5760 years is then represented by $\frac{1}{2}a$ or $0.5a$.

$$y = ae^{-kt} \quad \text{Exponential decay formula}$$

$$0.5a = ae^{-k(5760)} \quad \text{Replace } y \text{ with } 0.5a \text{ and } t \text{ with 5760.}$$

$$0.5 = e^{-5760k} \quad \text{Divide each side by } a.$$

$$\ln 0.5 = \ln e^{-5760k} \quad \text{Property of Equality for Logarithmic Functions}$$

$$\ln 0.5 = -5760k \quad \text{Inverse Property of Exponents and Logarithms}$$

$$\frac{\ln 0.5}{-5760} = k \quad \text{Divide each side by } -5760.$$

$$\frac{-0.6931472}{-5760} \approx k \quad \text{Use a calculator.}$$

$$0.00012 \approx k \quad \text{Simplify.}$$

The value of k for Carbon-14 is 0.00012. Thus, the equation for the decay of Carbon-14 is $y = ae^{-0.00012t}$, where t is given in years.

(continued on the next page)

CHECK Use the formula to find the amount of a sample remaining after 5760 years. Use an original amount of 1.

$y = ae^{-0.00012t}$	Original equation
$= 1e^{-0.00012(5760)}$	$a = 1$ and $t = 5760$
≈ 0.501	Use a calculator.

About half of the amount remains. The answer checks.

b. A paleontologist examining the bones of a woolly mammoth estimates that they contain only 3% as much Carbon-14 as they would have contained when the animal was alive. How long ago did the mammoth die?

Let a be the initial amount of Carbon-14 in the animal's body. Then the amount y that remains after t years is 3% of a or $0.03a$.

$y = ae^{-0.00012t}$	Formula for the decay of Carbon-14
$0.03a = ae^{-0.00012t}$	Replace y with $0.03a$.
$0.03 = e^{-0.00012t}$	Divide each side by a.
$\ln 0.03 = \ln e^{-0.00012t}$	Property of Equality for Logarithms
$\ln 0.03 = -0.00012t$	Inverse Property of Exponents and Logarithms
$\frac{\ln 0.03}{-0.00012} = t$	Divide each side by -0.00012.
$29{,}221 \approx t$	Use a calculator.

The mammoth lived about 29,000 years ago.

CHECK Your Progress

2. A specimen that originally contained 150 milligrams of Carbon-14 now contains 130 milligrams. How old is the fossil?

Exponential Growth When a quantity *increases* by a fixed percent each time period, the amount y of that quantity after t time periods is given by $y = a(1 + r)^t$, where a is the initial amount and r is the percent of increase expressed as a decimal. The percent of increase r is also referred to as the **rate of growth**.

Test-Taking Tip

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

1.5% = 0.015

STANDARDIZED TEST EXAMPLE 3

Personal Tutor

In 1910, the population of a city was 120,000. Since then, the population has increased by 1.5% per year. If the population continues to grow at this rate, what will the population be in 2010?

A 138,000 **B** 531,845 **C** 1,063,690 **D** 1.4×10^{11}

Read the Test Item

You need to find the population of the city 2010 − 1910, or 100, years later. Since the population is growing at a fixed percent each year, use the formula $y = a(1 + r)^t$.

Real-World Link

The Indian city of Varanasi is the world's oldest continuously inhabited city.

Source: tourismofindia.com

Solve the Test Item

$y = a(1 + r)^t$	Exponential growth formula
$= 120{,}000(1 + 0.015)^{100}$	Replace a with 120,000, r with 0.015, and t with 2010 — 1910, or 100.
$= 120{,}000(1.015)^{100}$	Simplify.
$\approx 531{,}845.48$	Use a calculator.

The answer is B.

CHECK Your Progress

3. Home values in Millersport increase about 4% per year. Mr. Thomas purchased his home eight years ago for \$122,000. What is the value of his home now?

F $\$1.36 \times 10^5$ **G** \$126,880 **H** \$166,965 **J** \$175,685

Online Personal Tutor at algebra2.com

Another model for exponential growth, preferred by scientists, is $y = ae^{kt}$, where k is a constant. Use this model to find the constant k.

EXAMPLE Exponential Growth of the Form $y = ae^{kt}$

4 **POPULATION As of 2005, China was the world's most populous country, with an estimated population of 1.31 billion people. The second most populous country was India, with 1.08 billion. The populations of India and China can be modeled by $I(t) = 1.08e^{0.0103t}$ and $C(t) = 1.31e^{0.0038t}$, respectively. According to these models, when will India's population be more than China's?**

You want to find t, the number of years, such that $I(t) > C(t)$.

$I(t) > C(t)$	
$1.08e^{0.0103t} > 1.31e^{0.0038t}$	Replace $I(t)$ with $1.08e^{0.0103t}$ and $C(t)$ with $1.31e^{0.0038t}$.
$\ln 1.08e^{0.0103t} > \ln 1.31e^{0.0038t}$	Property of Inequality for Logarithms
$\ln 1.08 + \ln e^{0.0103t} > \ln 1.31 + \ln e^{0.0038t}$	Product Property of Logarithms
$\ln 1.08 + 0.0103t > \ln 1.31 + 0.0038t$	Inverse Property of Exponents and Logarithms
$0.0065t > \ln 1.31 - \ln 1.08$	Subtract $0.0038t$ from each side.
$t > \dfrac{\ln 1.31 - \ln 1.08}{0.0065}$	Divide each side by 0.006.
$t > 29.70$	Use a calculator.

After 30 years, or in 2035, India will be the most populous country.

In Motion | In Motion for MAC

Concepts in Motion

Interactive Lab
algebra2.com

CHECK Your Progress

4. **BACTERIA** Two different types of bacteria in two different cultures reproduce exponentially. The first type can be modeled by $B_1(t) = 1200\ e^{0.1532t}$, and the second can be modeled by $B_2(t) = 3000\ e^{0.0466t}$, where t is the number of hours. According to these models, how many hours will it take for the amount of B_1 to exceed the amount of B_2?

Getty Images

CHECK Your Understanding

Example 1 (pp. 544–545)

1. **POLICE** Police use blood alcohol content (BAC) to measure the percent concentration of alcohol in a person's bloodstream. In most states, a BAC of 0.08 percent means a person is not allowed to drive. Each hour after drinking, a person's BAC may decrease by 15%. If a person has a BAC of 0.18, how many hours will he need to wait until he can legally drive?

Example 2 (pp. 545–546)

SPACE For Exercises 2–4, use the following information.
A radioisotope is used as a power source for a satellite. The power output P (in watts) is given by $P = 50\, e^{-\frac{t}{250}}$, where t is the time in days.

2. Is the formula for power output an example of exponential growth or decay? Explain your reasoning.
3. Find the power available after 100 days.
4. Ten watts of power are required to operate the equipment in the satellite. How long can the satellite continue to operate?

Example 3 (pp. 546–547)

5. **STANDARDIZED TEST PRACTICE** The weight of a bar of soap decreases by 2.5% each time it is used. If the bar weighs 95 grams when it is new, what is its weight to the nearest gram after 15 uses?

   A 57.5 g **B** 59.4 g **C** 65 g **D** 93 g

Example 4 (p. 547)

POPULATION GROWTH For Exercises 6 and 7, use the following information.
Fayette County, Kentucky, grew from a population of 260,512 in 2000 to a population of 268,080 in 2005.

6. Write an exponential growth equation of the form $y = ae^{kt}$ for Fayette County, where t is the number of years after 2000.
7. Use your equation to predict the population of Fayette County in 2015.

Exercises

HOMEWORK HELP

For Exercises	See Examples
8	1
9–11	2
12–14	3
15, 16	4

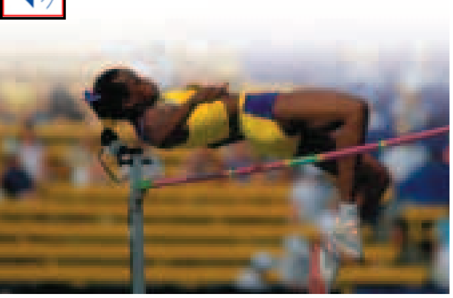
8. **COMPUTERS** Zeus Industries bought a computer for \$2500. If it depreciates at a rate of 20% per year, what will be its value in 2 years?

9. **HEALTH** A certain medication is eliminated from the bloodstream at a steady rate. It decays according to the equation $y = ae^{-0.1625t}$, where t is in hours. Find the half-life of this substance.

10. **PALENTOLOGY** A paleontologist finds a bone of a human. In the laboratory, she finds that the Carbon-14 found in the bone is $\frac{2}{3}$ of that found in living bone tissue. How old is this bone?

11. **ANTHROPOLOGY** An anthropologist studying the bones of a prehistoric person finds there is so little remaining Carbon-14 in the bones that instruments cannot measure it. This means that there is less than 0.5% of the amount of Carbon-14 the bones would have contained when the person was alive. How long ago did the person die?

12. **REAL ESTATE** The Martins bought a condominium for \$145,000. Assuming that the value of the condo will appreciate at most 5% a year, how much will the condo be worth in 5 years?



Real-World Link

The women's high jump competition first took place in the USA in 1895, but it did not become an Olympic event until 1926.

Source: www.princeton.edu

ECONOMICS **For Exercises 13 and 14, use the following information.**
The annual Gross Domestic Product (GDP) of a country is the value of all of the goods and services produced in the country during a year. During the period 2001–2004, the Gross Domestic Product of the United States grew about 2.8% per year, measured in 2004 dollars. In 2001, the GDP was $9891 billion.

13. Assuming this rate of growth continues, what will the GDP of the United States be in the year 2015?

14. In what year will the GDP reach $20 trillion?

BIOLOGY **For Exercises 15 and 16, use the following information.**
Bacteria usually reproduce by a process known as *binary fission*. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes.

15. Find the constant k for this type of bacteria under ideal conditions.

16. Write the equation for modeling the exponential growth of this bacterium.

17. **OLYMPICS** In 1928, when the high jump was first introduced as a women's sport at the Olympic Games, the winning women's jump was 62.5 inches, while the winning men's jump was 76.5 inches. Since then, the winning jump for women has increased by about 0.38% per year, while the winning jump for men has increased at a slower rate, 0.3%. If these rates continue, when will the women's winning high jump be higher than the men's?

18. **HOME OWNERSHIP** The Mendes family bought a new house 10 years ago for $120,000. The house is now worth $191,000. Assuming a steady rate of growth, what was the yearly rate of appreciation?

FOOD **For Exercises 19 and 20, use the table of suggested times for cooking potatoes in a microwave oven. Assume that the number of minutes is a function of some power of the number of potatoes.**

Number of 8 oz. Potatoes	Cooking Time (min)
2	10
4	15

Source: wholehealthmd.com

19. Write an equation in the form $t = an^b$, where t is the time in minutes, n is the number of potatoes, and a and b are constants. (*Hint:* Use a system of equations to find the constants.)

20. According to the formula, how long should you cook six 8-ounce potatoes in a microwave?

EXTRA PRACTICE
See pages 911, 934.

Math Online
Self-Check Quiz at
algebra2.com

H.O.T. Problems

21. **REASONING** Explain how to solve $y = (1 + r)^t$ for t.

22. **OPEN ENDED** Give an example of a quantity that grows or decays at a fixed rate. Write a real-world problem involving the rate and solve by using logarithms.

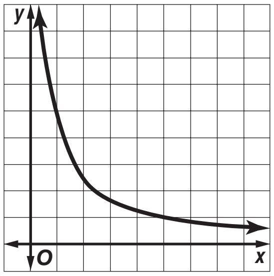
23. **CHALLENGE** The half-life of radium is 1620 years. When will a 20-gram sample of radium be completely gone? Explain your reasoning.

24. ***Writing in Math*** Use the information about car values on page 544 to explain how you can use exponential decay to determine the current value of a car. Include a description of how to find the percent decrease in the value of the car each year and a description of how to find the value of a car for any given year when the rate of depreciation is known.

Karl Weatherly/CORBIS

STANDARDIZED TEST PRACTICE

25. **ACT/SAT** The curve represents a portion of the graph of which function?



A $y = 50 - x$ **C** $y = e^{-x}$

B $y = \log x$ **D** $xy = 5$

26. **REVIEW** A radioactive element decays over time, according to the equation

$$y = x\left(\frac{1}{4}\right)^{\frac{t}{200}},$$

where x = the number of grams present initially and t = time in years. If 500 grams were present initially, how many grams will remain after 400 years?

F 12.5 grams **H** 62.5 grams

G 31.25 grams **J** 125 grams

Spiral Review

Write an equivalent exponential or logarithmic equation. (Lesson 9-5)

27. $e^3 = y$ **28.** $e^{4n-2} = 29$ **29.** $\ln 4 + 2\ln x = 8$

Solve each equation or inequality. Round to four decimal places. (Lesson 9-4)

30. $16^x = 70$ **31.** $2^{3p} > 1000$ **32.** $\log_b 81 = 2$

BUSINESS **For Exercises 33–35, use the following information.**
A small corporation decides that 8% of its profits would be divided among its six managers. There are two sales managers and four nonsales managers. Fifty percent would be split equally among all six managers. The other 50% would be split among the four nonsales managers. Let p represent the profits. (Lesson 8-2)

33. Write an expression to represent the share of the profits each nonsales manager will receive.

34. Simplify this expression.

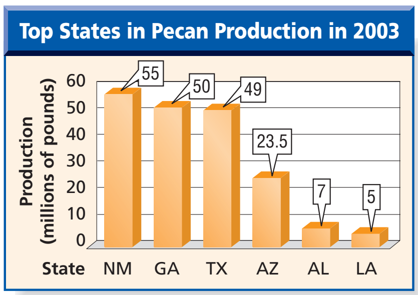
35. Write an expression in simplest form to represent the share of the profits each sales manager will receive.

AGRICULTURE **For Exercises 36–38, use the graph at the right.**
U.S. growers were forecasted to produce 264 million pounds of pecans in 2003. (Lesson 6-1)

36. Write the number of pounds of pecans forecasted by U.S. growers in 2003 in scientific notation.

37. Write the number of pounds of pecans produced by Georgia in 2003 in scientific notation.

38. What percent of the overall pecan production for 2003 can be attributed to Georgia?



Source: www.nass.usda.gov

EXTEND 9-6

Graphing Calculator Lab
Cooling

In this lab, you will explore the type of equation that models the change in the temperature of water as it cools under various conditions.

SET UP the Lab

- Collect a variety of containers, such as a foam cup, a ceramic coffee mug, and an insulated cup.
- Boil water or collect hot water from a tap.
- Choose a container to test and fill with hot water. Place the temperature probe in the cup.
- Connect the temperature probe to your data collection device.

ACTIVITY

Step 1 Program the device to collect 20 or more samples in 1-minute intervals.

Step 2 Wait a few seconds for the probe to warm to the temperature of the water.

Step 3 Press the button to begin collecting data.

ANALYZE THE RESULTS

1. When the data collection is complete, graph the data in a scatter plot. Use time as the independent variable and temperature as the dependent variable. Write a sentence that describes the points on the graph.
2. Use the STAT menu to find an equation to model the data you collected. Try linear, quadratic, and exponential models. Which model appears to fit the data best? Explain.
3. Would you expect the temperature of the water to drop below the temperature of the room? Explain your reasoning.
4. Use the data collection device to find the temperature of the air in the room. Graph the function $y = t$, where t is the temperature of the room along with the scatter plot and the model equation. Describe the relationship among the graphs. What is the meaning of the relationship in the context of the experiment?

MAKE A CONJECTURE

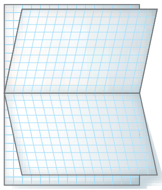
5. Do you think the results of the experiment would change if you used an insulated container? Repeat the experiment to verify your conjecture.
6. How might the results of the experiment change if you added ice to the water? Repeat the experiment to verify your conjecture.

CHAPTER 9 Study Guide and Review

Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer — GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Exponential Functions (Lesson 9-1)

- An exponential function is in the form $y = ab^x$, where $a \neq 0$, $b > 0$ and $b \neq 1$.
- Property of Equality for Exponential Functions: If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.
- Property of Inequality for Exponential Functions: If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Logarithms and Logarithmic Functions (Lessons 9-2 through 9-4)

- Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.
- The logarithm of a product is the sum of the logarithms of its factors.
- The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- The logarithm of a power is the product of the logarithm and the exponent.
- The Change of Base Formula: $\log_a n = \dfrac{\log_b n}{\log_b a}$

Natural Logarithms (Lesson 9-5)

- Since the natural base function and the natural logarithmic function are inverses, these two can be used to "undo" each other.

Exponential Growth and Decay (Lesson 9-6)

- Exponential decay: $y = a(1 - r)^t$ or $y = ae^{-kt}$
- Exponential growth: $y = a(1 + r)^t$ or $y = ae^{kt}$

Key Vocabulary

common logarithm (p. 528)
exponential decay (p. 500)
exponential equation (p. 501)
exponential function (p. 499)
exponential growth (p. 500)
exponential inequality (p. 502)
logarithm (p. 510)
logarithmic equation (p. 512)
logarithmic function (p. 511)
logarithmic inequality (p. 512)
natural base, e (p. 536)
natural base exponential function (p. 536)
natural logarithm (p. 537)
natural logarithmic function (p. 537)
rate of decay (p. 544)
rate of growth (p. 546)

Vocabulary Check

State whether each sentence is *true* or *false*. If *false*, replace the underlined word(s) to make a true statement.

1. In $x = b^y$, y is called the <u>logarithm.</u>
2. The change in the number of bacteria in a Petri dish over time is an example of <u>exponential decay.</u>
3. The <u>natural logarithm</u> is the inverse of the exponential function with base 10.
4. The irrational number 2.71828... is referred to as the <u>natural base, e.</u>
5. If a savings account yields 2% interest per year, then 2% is the <u>rate of growth</u>.
6. Radioactive half-life is used to describe the <u>exponential decay</u> of a sample.
7. The inverse of an exponential function is a <u>composite function</u>.
8. If $24^{2y+3} = 24^{y-4}$, then $2y + 3 = y - 4$ by the <u>Property of Equality for Exponential Functions.</u>
9. The <u>Power Property of Logarithms</u> shows that $\ln 9 < \ln 81$.

Math Online Vocabulary Review at algebra2.com

Lesson-by-Lesson Review

9-1 Exponential Functions (pp. 498–506)

Determine whether each function represents exponential *growth* or *decay*.

10. $y = 5(0.7)^x$ **11.** $y = \frac{1}{3}(4)^x$

Write an exponential function for the graph that passes through the given points.

12. (0, −2) and (3, −54)

13. (0, 7) and (1, 1.4)

Solve each equation or inequality. Check your solution.

14. $9^x = \frac{1}{81}$ **15.** $2^{6x} = 4^{5x+2}$

16. $49^{3p+1} = 7^{2p-5}$ **17.** $9^{x^2} \leq 27^{x^2-2}$

18. POPULATION The population of mice in a particular area is growing exponentially. On January 1, there were 50 mice, and by June 1, there were 200 mice. Write an exponential function of the form $y = ab^x$ that could be used to model the mouse population y of the area. Write the function in terms of x, the number of months since January.

Example 1 Write an exponential function for the graph that passes through (0, 2) and (1, 16).

$y = ab^x$	Exponential equation
$2 = ab^0$	Substitute (0, 2) into the exponential equation.
$2 = a$	Simplify.
$y = 2b^x$	Intermediate function
$16 = 2b^1$	Substitute (1, 16) into the intermediate function.
$8 = b$	Simplify.
$y = 2(8)^x$	

Example 2 Solve $64 = 2^{3n+1}$ for n.

$64 = 2^{3n+1}$	Original equation
$2^6 = 2^{3n+1}$	Rewrite 64 as 2^6 so each side has the same base.
$6 = 3n + 1$	Property of Equality for Exponential Functions
$\frac{5}{3} = n$	The solution is $\frac{5}{3}$.

9-2 Logarithms and Logarithmic Functions (pp. 509–517)

(continued on the next page)

Write each equation in logarithmic form.

19. $7^3 = 343$ **20.** $5^{-2} = \frac{1}{25}$

Write each equation in exponential form.

21. $\log_4 64 = 3$ **22.** $\log_8 2 = \frac{1}{3}$

Evaluate each expression.

23. $4^{\log_4 9}$ **24.** $\log_7 7^{-5}$

25. $\log_{81} 3$ **26.** $\log_{13} 169$

Example 3 Solve $\log_9 n > \frac{3}{2}$.

$\log_9 n > \frac{3}{2}$	Original inequality
$n > 9^{\frac{3}{2}}$	Logarithmic to exponential inequality
$n > (3^2)^{\frac{3}{2}}$	$9 = 3^2$
$n > 3^3$	Power of a Power
$n > 27$	Simplify.

Study Guide and Review

9-2 Logarithms and Logarithmic Functions (pp. 509–517)

Solve each equation or inequality.

27. $\log_4 x = \frac{1}{2}$

28. $\log_{81} 729 = x$

29. $\log_8 (x^2 + x) = \log_8 12$

30. $\log_8 (3y - 1) < \log_8 (y + 5)$

31. CHEMISTRY $\text{pH} = -\log(H^+)$, where H^+ is the hydrogen ion concentration of the substance. How many times as great is the acidity of orange juice with a pH of 3 as battery acid with a pH of 0?

Example 4 **Solve $\log_3 12 = \log_3 2x$.**

$\log_3 12 = \log_3 2x$	Original equation
$12 = 2x$	Property of Equality for Logarithmic Functions
$6 = x$	Divide each side by 2.

9-3 Properties of Logarithms (pp. 520–526)

Use $\log_9 7 \approx 0.8856$ and $\log_9 4 \approx 0.6309$ to approximate the value of each expression.

32. $\log_9 28$

33. $\log_9 49$

34. $\log_9 144$

35. $\log_9 63$

Solve each equation. Check your solutions.

36. $\log_5 7 + \frac{1}{2} \log_5 4 = \log_5 x$

37. $2\log_2 x - \log_2 (x + 3) = 2$

38. $\log_6 48 - \log_6 \frac{16}{5} + \log_6 5 = \log_6 5x$

39. SOUND Use the formula $L = 10 \log_{10} R$, where L is the loudness of a sound and R is the sound's relative intensity, to find out how much louder 10 alarm clocks would be than one alarm clock. Suppose the sound of one alarm clock is 80 decibels.

Example 5 **Use $\log_{12} 9 \approx 0.884$ and $\log_{12} 18 \approx 1.163$ to approximate the value of $\log_{12} 2$.**

$\log_{12} 2 = \log_{12} \frac{18}{9}$	Replace 2 with $\frac{18}{9}$.
$= \log_{12} 18 - \log_{12} 9$	Quotient Property
$\approx 1.163 - 0.884$ or 0.279	

Example 6

Solve $\log_3 4 + \log_3 x = 2 \log_3 6$.

$\log_3 4 + \log_3 x = 2 \log_3 6$	
$\log_3 4x = 2 \log_3 6$	Product Property of Logarithms
$\log_3 4x = \log_3 6^2$	Power Property of Logarithms
$4x = 36$	Property of Equality for Logarithmic Functions
$x = 9$	Divide each side by 4.

Mixed Problem Solving
For mixed problem-solving practice, see page 934.

9-4 Common Logarithms (pp. 528–533)

Solve each equation or inequality. Round to four decimal places.

40. $2^x = 53$

41. $2.3^{x^2} = 66.6$

42. $3^{4x-7} < 4^{2x+3}$

43. $6^{3y} = 8^{y-1}$

44. $12^{x-5} \geq 9.32$

45. $2.1^{x-5} = 9.32$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

46. $\log_4 11$

47. $\log_2 15$

48. MONEY Diane deposited \$500 into a bank account that pays an annual interest rate r of 3% compounded quarterly. Use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to find how long it will take for Diane's money to double.

Example 7 **Solve $5^x = 7$.**

$5^x = 7$	Original equation
$\log 5^x = \log 7$	Property of Equality for Logarithmic Functions
$x \log 5 = \log 7$	Power Property of Logarithms
$x = \dfrac{\log 7}{\log 5}$	Divide each side by log 5.
$x \approx \dfrac{0.8451}{0.6990}$ or 1.2090	Use a calculator.

9-5 Base *e* and Natural Logarithms (pp. 536–542)

Write an equivalent exponential or logarithmic equation.

49. $e^x = 6$

50. $\ln 7.4 = x$

Solve each equation or inequality.

51. $2e^x - 4 = 1$

52. $e^x > 3.2$

53. $-4e^{2x} + 15 = 7$

54. $\ln 3x \leq 5$

55. $\ln (x - 10) = 0.5$

56. $\ln x + \ln 4x = 10$

57. MONEY If you deposit \$1200 in an account paying 4.7% interest compounded continuously, how long will it take for your money to triple?

Example 8 **Solve $\ln (x + 4) > 5$.**

$\ln (x + 4) > 5$	Original inequality
$e^{\ln (x+4)} > e^5$	Write each side using exponents and base e.
$x + 4 > e^5$	Inverse Property of Exponents and Logarithms
$x > e^5 - 4$	Subtract 4 from each side.
$x > 144.4132$	Use a calculator.

9-6 Exponential Growth and Decay (pp. 544–550)

58. BUSINESS Able Industries bought a fax machine for \$250. It is expected to depreciate at a rate of 25% per year. What will be the value of the fax machine in 3 years?

59. BIOLOGY For a certain strain of bacteria, k is 0.872 when t is measured in days. Using the formula $y = ae^{kt}$, how long will it take 9 bacteria to increase to 738 bacteria?

60. CHEMISTRY Radium-226 has a half-life of 1800 years. Find the constant k in the decay formula for this compound.

61. POPULATION The population of a city 10 years ago was 45,600. Since then, the population has increased at a steady rate each year. If the population is currently 64,800, find the annual rate of growth for this city.

Example 9 **A certain culture of bacteria will grow from 500 to 4000 bacteria in 1.5 hours. Find the constant k for the growth formula. Use $y = ae^{kt}$.**

$y = ae^{kt}$	Exponential growth formula
$4000 = 500\, e^{k(1.5)}$	Replace y with 4000, a with 500, and t with 1.5.
$8 = e^{1.5k}$	Divide each side by 500.
$\ln 8 = \ln e^{1.5k}$	Property of Equality for Logarithmic Functions
$\ln 8 = 1.5k$	Inverse Property of Exponents and Logarithms
$\frac{\ln 8}{1.5} = k$	Divide each side by 1.5.
$1.3863 \approx k$	Use a calculator.

The constant k for this type of bacteria is about 1.3863.

CHAPTER 9 Practice Test

1. Write $3^7 = 2187$ in logarithmic form.
2. Write $\log_8 16 = \frac{4}{3}$ in exponential form.
3. Express $\log_3 5$ in terms of common logarithms. Then approximate its value to four decimal places.
4. Evaluate $\log_2 \frac{1}{32}$.

Use $\log_4 7 \approx 1.4037$ and $\log_4 3 \approx 0.7925$ to approximate the value of each expression.

5. $\log_4 21$
6. $\log_4 \frac{7}{12}$

Simplify each expression.

7. $\left(3^{\sqrt{8}}\right)^{\sqrt{2}}$
8. $81^{\sqrt{5}} \div 3^{\sqrt{5}}$

Solve each equation or inequality. Round to four decimal places if necessary.

9. $27^{2p+1} = 3^{4p-1}$
10. $\log_m 144 = -2$
11. $\log_3 3^{(4x-1)} = 15$
12. $4^{2x-3} = 9^{x+3}$
13. $2e^{3x} + 5 = 11$
14. $\log_2 x < 7$
15. $\log_9 (x + 4) + \log_9 (x - 4) = 1$
16. $\log_2 5 + \frac{1}{3} \log_2 27 = \log_2 x$

COINS For Exercises 17 and 18, use the following information.
You buy a commemorative coin for \$25. The value of the coin increases at a rate of 3.25% per year.

17. How much will the coin be worth in 15 years?
18. After how many years will the coin have doubled in value?

19. **MULTIPLE CHOICE** The population of a certain country can be modeled by the equation $P(t) = 40\,e^{0.02t}$, where P is the population in millions and t is the number of years since 1900. When will the population be 400 million?

 A 1946 **C** 2015

 B 1980 **D** 2045

STARS For Exercises 20–22, use the following information.
Some stars appear bright only because they are very close to us. Absolute magnitude M is a measure of how bright a star would appear if it were 10 parsecs, about 32 light years, away from Earth. A lower magnitude indicates a brighter star. Absolute magnitude is given by $M = m + 5 - 5 \log d$, where d is the star's distance from Earth measured in parsecs and m is its apparent magnitude.

Star	Apparent Magnitude	Distance (parsecs)
Sirius	−1.44	2.64
Vega	0.03	7.76

20. Sirius and Vega are two of the brightest stars. Which star appears brighter?
21. Find the absolute magnitudes of Sirius and Vega.
22. Which star is actually brighter? That is, which has a lower absolute magnitude?

23. **MULTIPLE CHOICE** Humans have about 1,400,000 hairs on their head and lose an average of 75 hairs each day. If a person's body were to *never* replace a hair, approximately how many years would it take for a person to have 1000 hairs left on their head? (Assume that a person can live significantly longer than the average life span.)

 F 85 years **H** 257 years

 G 113 years **J** 511 years

24. **DINOSAURS** A paleontologist finds that the Carbon-14 found in the bone is $\frac{1}{12}$ of that found in living bone tissue. Could this bone have belonged to a dinosaur? Explain your reasoning. (*Hint:* The dinosaurs lived from 220 million to 63 million years ago.)

25. **HEALTH** Radioactive iodine is used to determine the health of the thyroid gland. It decays according to the equation $y = ae^{-0.0856t}$, where t is in days. Find the half-life of this substance.

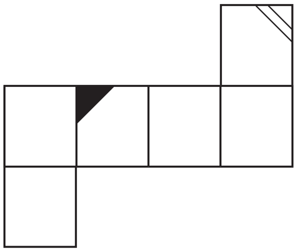
CHAPTER 9

Standardized Test Practice

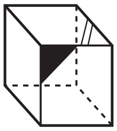
Cumulative, Chapters 1–9

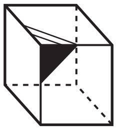
Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

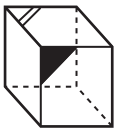
1. The net below shows the surface of a 3-dimensional figure.

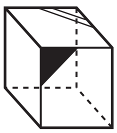


Which 3-dimensional figure does this net represent?

A 

B 

C 

D 

TEST-TAKING TIP

Question 1 If you don't know how to solve a problem, eliminate the answer choices you know are incorrect and then guess from the remaining choices. Even eliminating only one answer choice greatly increases your chance of guessing the correct answer.

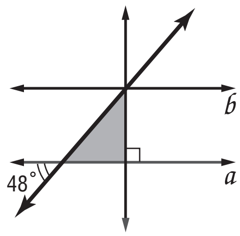
2. An equation can be used to find the total cost of a pizza with a certain diameter. Using the table below, find the equation that best represents y, the total cost, as a function of x, the diameter in inches.

Diameter, x (in.)	Total Cost, y
9	\$10.80
12	\$14.40
20	\$24.00

F $y = 1.2x$ **H** $y = 0.83x$

G $x = 1.2y$ **J** $x = 0.83y$

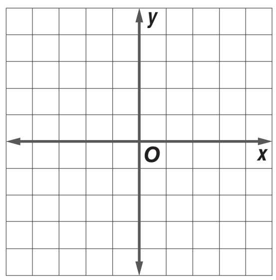
3. In the figure below, lines a and b are parallel. What are the measures of the angles in the shaded triangle below?



A 42, 48, 90 **C** 48, 52, 90

B 42, 90, 132 **D** 48, 90, 132

4. What are the slope and y-intercept of a line that contains the point $(-1, 4)$ and has the same x-intercept as $x + 2y = -3$?

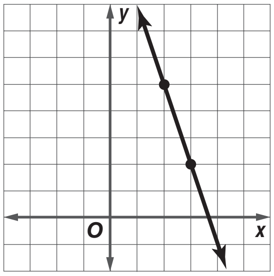


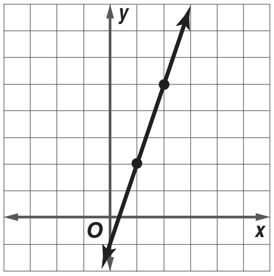
F $m = 2, b = 6$ **H** $m = \frac{1}{2}, b = -2$

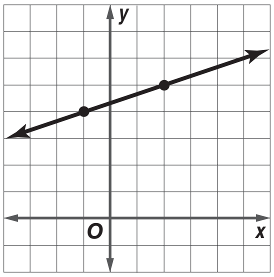
G $m = -7, b = -3$ **J** $m = -\frac{1}{7}, b = -3$

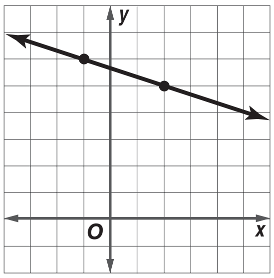
Preparing for Standardized Tests
For test-taking strategies and more practice, see pages 941–956.

5. Which graph best represents the line passing through the point (2, 5) and perpendicular to $y = 3x$?

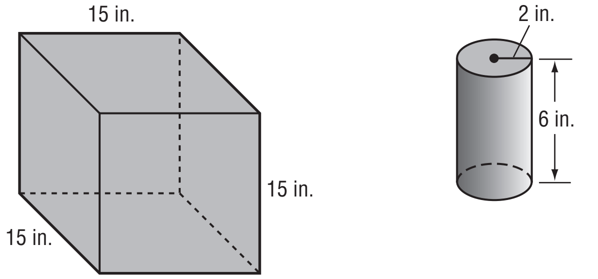
A 

B 

C 

D 

6. GRIDDABLE Matt has a square trough as shown below. He plans to fill it by emptying cylindrical cans of water with the dimensions as shown.



About how many cylindrical cans will it take to fill the trough?

7. Carson is making a circle graph showing the favorite movie types of customers at his store. The table summarizes the data. What central angle should Carson use for the section representing Comedy?

Type	Customers
Comedy	35
Romance	42
Horror	7
Drama	12
Other	4

F 35 **H** 126
G 63 **J** 150

Pre-AP

Record your answers on a sheet of paper. Show your work.

8. Sarah received $2500 for a graduation gift. She put it into a savings account in which the interest rate was 5.5% per year.

a. How much did she have in her savings account after 5 years?

b. After how many years will the amount in her savings account have doubled?

NEED EXTRA HELP?								
If You Missed Question...	1	2	3	4	5	6	7	8
Go to Lesson	1-3	2-4	3-1	2-4	2-4	6-8	10-3	10-6