UNIT 4
Discrete Mathematics

Focus

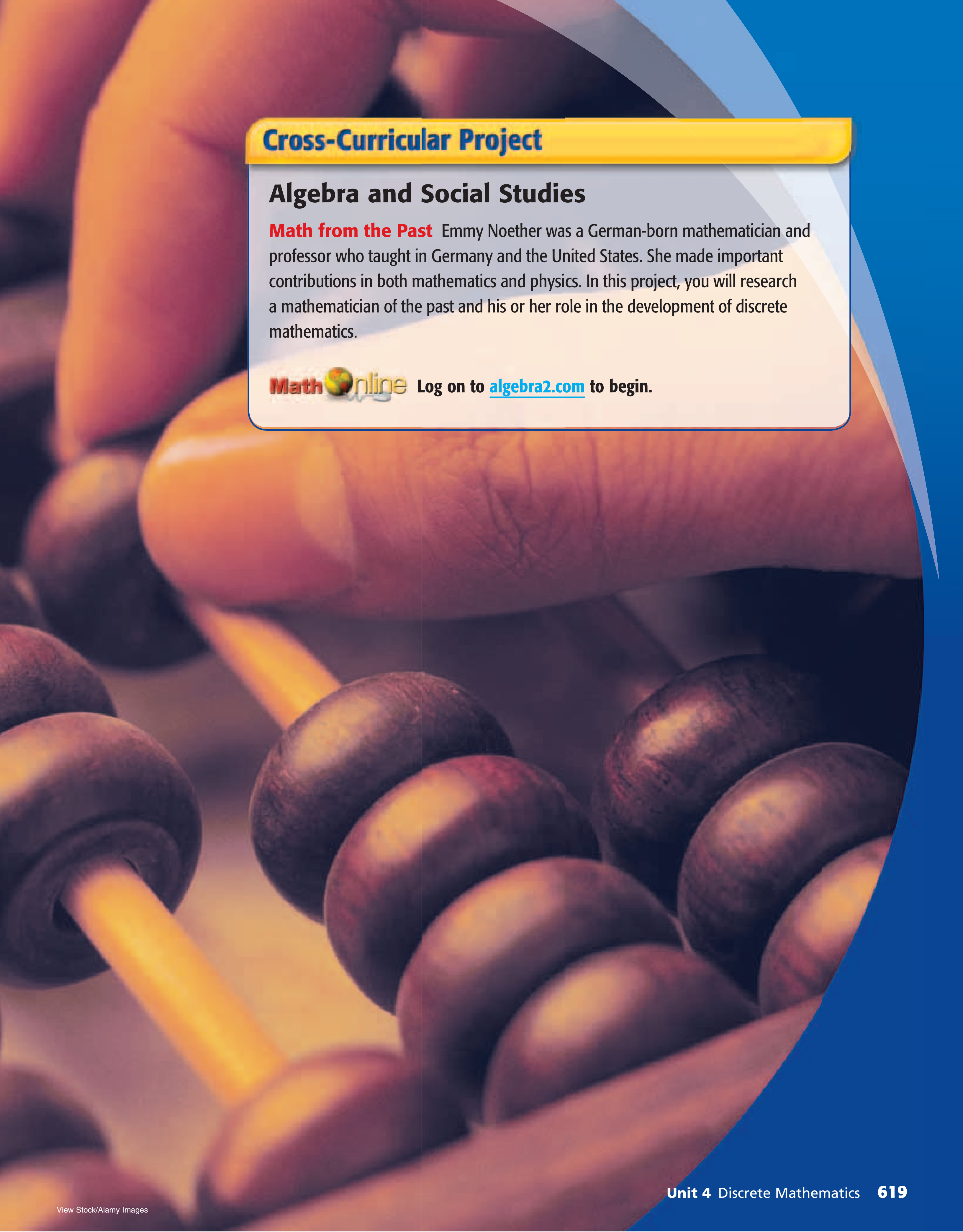
Use multiple representations, technology, applications and modeling, and numerical fluency in discrete problem-solving contexts.

CHAPTER 11
Sequences and Series

BIG Idea Use sequences and series as well as tools and technology to represent, analyze, and solve real-life problems.

CHAPTER 12
Probability and Statistics

BIG Idea Use probability and statistical models to describe everyday situations involving chance.



Cross-Curricular Project

Algebra and Social Studies

Math from the Past Emmy Noether was a German-born mathematician and professor who taught in Germany and the United States. She made important contributions in both mathematics and physics. In this project, you will research a mathematician of the past and his or her role in the development of discrete mathematics.

Math Online **Log on to algebra2.com to begin.**

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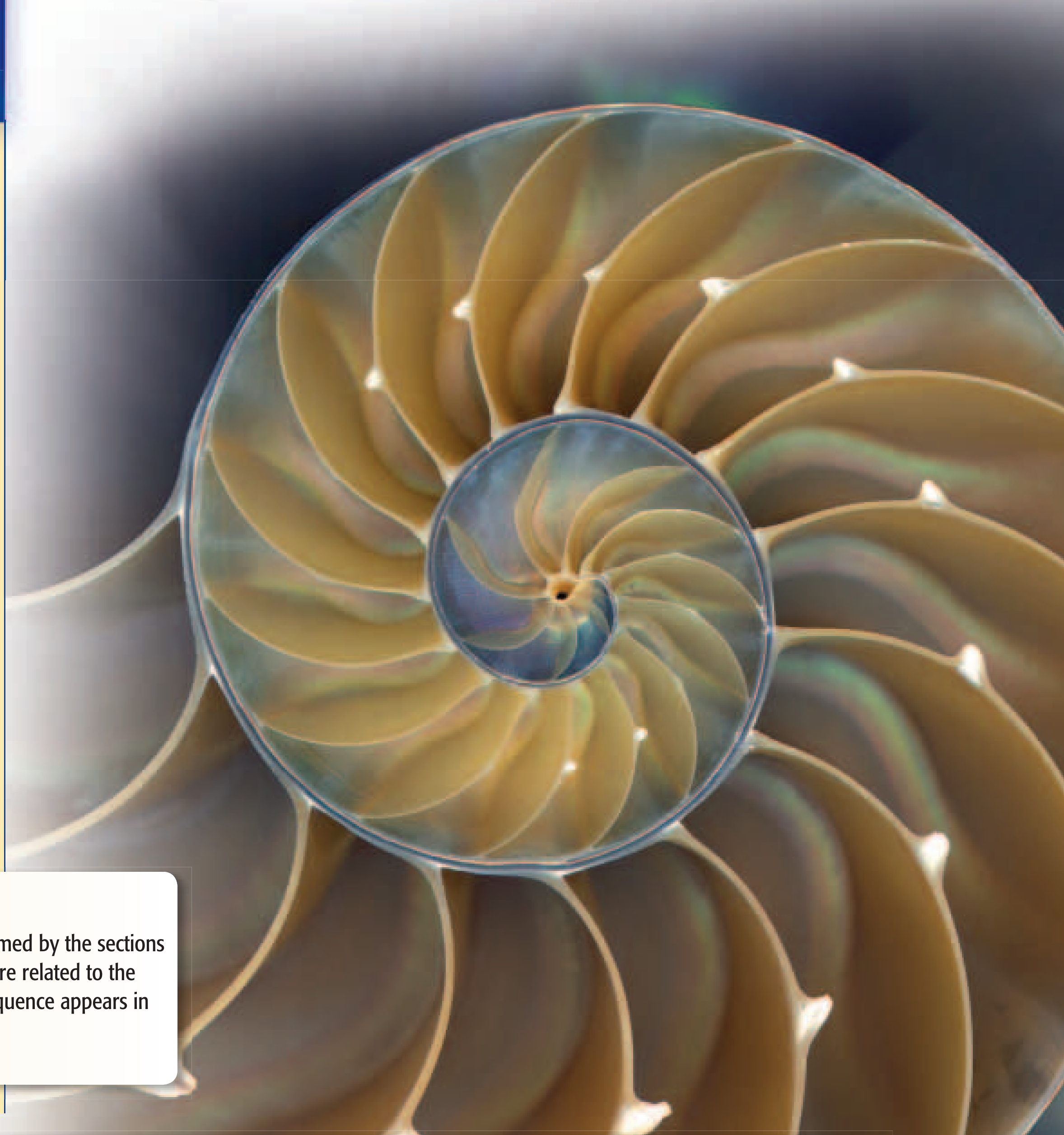
CHAPTER 11

Sequences and Series



BIG Ideas

- Use arithmetic and geometric sequences and series.
- Use special sequences and iterate functions.
- Expand powers by using the Binomial Theorem.
- Prove statements by using mathematical induction.

Key Vocabulary

arithmetic sequence (p. 622)
arithmetic series (p. 629)
geometric sequence (p. 636)
geometric series (p. 643)
inductive hypothesis (p. 670)
mathematical induction (p. 670)
recursive formula (p. 658)

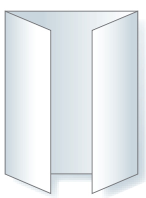
Real-World Link

Chambered Nautilus The spiral formed by the sections of the shell of a chambered nautilus are related to the Fibonacci sequence. The Fibonacci sequence appears in many objects naturally.

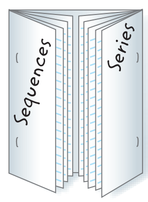
FOLDABLES Study Organizer

Sequences and Series Make this Foldable to help you organize your notes. Begin with one sheet of 11″ by 17″ paper and four sheets of notebook paper.

1. **Fold** the short sides of the 11″ by 17″ paper to meet in the middle.



2. **Fold** the notebook paper in half lengthwise. Insert two sheets of notebook paper under each tab and staple the edges. Take notes under the appropriate tabs.



GET READY for Chapter 11

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

Math Online Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Solve each equation. (Lesson 1-3)

1. $-40 = 10 + 5x$
2. $162 = 2x^4$
3. $12 - 3x = 27$
4. $3x^3 + 4 = -20$
5. **FAIR** Jeremy goes to a state fair with \$36. The entrance fee is \$12 and each ride costs \$4. How many rides can Jeremy go on? (Lesson 1-3)

Graph each function. (Lesson 2-1)

6. $\{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}$
7. $\{(1, -20), (2, -16), (3, -12), (4, -8), (5, -4)\}$
8. $\left\{(1, 64), (2, 16), (3, 4), (4, 1), \left(5, \frac{1}{4}\right)\right\}$
9. $\left\{(1, 2), (2, 3), \left(3, \frac{7}{2}\right), \left(4, \frac{15}{4}\right), \left(5, \frac{31}{8}\right)\right\}$
10. **HOBBIES** Arthur has a collection of 21 model cars. He decides to buy 2 more model cars every time he goes to the toy store. The function $C(t) = 21 + 2t$ counts the number of model cars $C(t)$ he has after t trips to the toy store. How many model cars will he have after he has been to the toy store 6 times? (Lesson 1-3)

Evaluate each expression for the given value(s) of the variable(s). (Lesson 1-1)

11. $x + (y - 1)z$ if $x = 3$, $y = 8$, and $z = 2$
12. $\frac{x}{2}(y + z)$ if $x = 10$, $y = 3$, and $z = 25$
13. $a \cdot b^{c-1}$ if $a = 2$, $b = \frac{1}{2}$, and $c = 7$
14. $\frac{a(1 - bc)^2}{1 - b}$ if $a = -2$, $b = 3$, and $c = 5$

QUICKReview

EXAMPLE 1

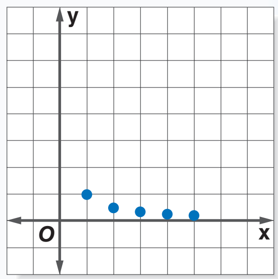
Solve the equation $14 = 2x^3 + 700$.

$-686 = 2x^3$	Subtract 700 from each side.
$-343 = x^3$	Divide each side by 2.
$\sqrt[3]{-343} = \sqrt[3]{x^3}$	Take the cube root of each side.
$-7 = x$	Simplify.

EXAMPLE 2

Graph the function $\left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$.

The domain of a function is the set of all possible x-values. So, the domain of this function is $\{1, 2, 3, 4, 5\}$. The range of a function is the set of all possible y-values. So, the range of this function is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$.



EXAMPLE 3

Evaluate the expression 2^{m+k+b} if $m = 4$, $k = -5$, and $b = 1$.

$2^{4+(-5)+1}$	Substitute.
$= 2^0$	Simplify.
$= 1$	Zero Exponent Rule

11-1 Arithmetic Sequences

Main Ideas

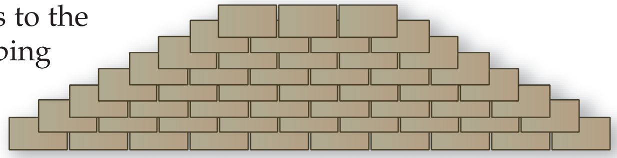
- Use arithmetic sequences.
- Find arithmetic means.

New Vocabulary

sequence
term
arithmetic sequence
common difference
arithmetic means

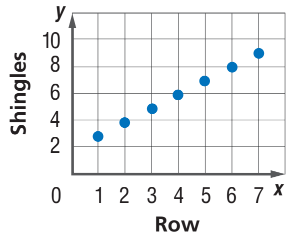
GET READY for the Lesson

A roofer is nailing shingles to the roof of a house in overlapping rows. There are three shingles in the top row. Since the roof widens from top to bottom, one more shingle is needed in each successive row.



Row	1	2	3	4	5	6	7
Shingles	3	4	5	6	7	8	9

Arithmetic Sequences The numbers 3, 4, 5, 6, …, representing the number of shingles in each row, are an example of a sequence of numbers. A **sequence** is a list of numbers in a particular order. Each number in a sequence is called a **term**. The first term is symbolized by a_1, the second term is symbolized by a_2, and so on.



A sequence can also be thought of as a discrete function whose domain is the set of positive integers over some interval.

Many sequences have patterns. For example, in the sequence above for the number of shingles, each term can be found by adding 1 to the previous term. A sequence of this type is called an arithmetic sequence. An **arithmetic sequence** is a sequence in which each term after the first is found by adding a constant, called the **common difference**, to the previous term.

Study Tip

Sequences

The numbers in a sequence may not be ordered. For example, the numbers 84, 102, 97, 72, 93, 84, 87, 92, … are a sequence that represents the number of games won by the Houston Astros each season beginning with 1997.

EXAMPLE Find the Next Terms

1 **Find the next four terms of the arithmetic sequence 55, 49, 43, … .**

Find the common difference d by subtracting two consecutive terms.

$49 - 55 = -6$ and $43 - 49 = -6$ So, $d = -6$.

Now add -6 to the third term of the sequence, and then continue adding -6 until the next four terms are found.

$$43 \xrightarrow{+(-6)} 37 \xrightarrow{+(-6)} 31 \xrightarrow{+(-6)} 25 \xrightarrow{+(-6)} 19$$

The next four terms of the sequence are 37, 31, 25, and 19.

CHECK Your Progress

1. Find the next four terms of the arithmetic sequence $-1.6, -0.7, 0.2, \ldots$.

It is possible to develop a formula for each term of an arithmetic sequence in terms of the first term a_1 and the common difference d. Consider the sequence in Example 1.

Sequence	numbers	55	49	43	37	...	
	symbols	a_1	a_2	a_3	a_4	...	a_n
Expressed in Terms of d and the First Term	numbers	$55 + 0(-6)$	$55 + 1(-6)$	$55 + 2(-6)$	$55 + 3(-6)$	...	$55 + (n-1)(-6)$
	symbols	$a_1 + 0 \cdot d$	$a_1 + 1 \cdot d$	$a_1 + 2 \cdot d$	$a_1 + 3 \cdot d$	...	$a_1 + (n-1)d$

The following formula generalizes this pattern for any arithmetic sequence.

KEY CONCEPT *nth Term of an Arithmetic Sequence*

The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by the following formula, where n is any positive integer.

$$a_n = a_1 + (n - 1)d$$

You can use the formula to find a term in a sequence given the first term and the common difference or given the first term and some successive terms.

Real-World Link

A hydraulic crane uses fluid to transmit forces from point to point. The brakes of a car use this same principle.

Source: howstuffworks.com

Real-World EXAMPLE Find a Particular Term

Personal Tutor

2 **CONSTRUCTION** **The table at the right shows typical costs for a construction company to rent a crane for one, two, three, or four months. If the sequence continues, how much would it cost to rent the crane for twelve months?**

Months	Cost ($)
1	75,000
2	90,000
3	105,000
4	120,000

Explore Since the difference between any two successive costs is $15,000, the costs form an arithmetic sequence with common difference 15,000.

Plan You can use the formula for the nth term of an arithmetic sequence with $a_1 = 75{,}000$ and $d = 15{,}000$ to find a_{12}, the cost for twelve months.

Solve

$a_n = a_1 + (n - 1)d$ Formula for nth term

$a_{12} = 75{,}000 + (12 - 1)15{,}000$ $n = 12$, $a_1 = 75{,}000$, $d = 15{,}000$

$a_{12} = 240{,}000$ Simplify.

It would cost $240,000 to rent the crane for twelve months.

Check You can find terms of the sequence by adding 15,000. a_5 through a_{12} are 135,000, 150,000, 165,000, 180,000, 195,000, 210,000, 225,000, and 240,000. Therefore, $240,000 is correct.

CHECK Your Progress

2. The construction company has a budget of $350,000 for crane rental. The job is expected to last 18 months. Will the company be able to afford the crane rental for the entire job? Explain.

Online Personal Tutor at algebra2.com

If you are given some of the terms of a sequence, you can use the formula for the nth term of a sequence to write an equation to help you find the nth term.

EXAMPLE Write an Equation for the nth Term

3 **Write an equation for the nth term of the arithmetic sequence 8, 17, 26, 35, ...**

In this sequence, $a_1 = 8$ and $d = 9$. Use the nth term formula to write an equation.

$a_n = a_1 + (n - 1)d$ — Formula for nth term

$a_n = 8 + (n - 1)9$ — $a_1 = 8$, $d = 9$

$a_n = 8 + 9n - 9$ — Distributive Property

$a_n = 9n - 1$ — Simplify.

An equation is $a_n = 9n - 1$.

Study Tip

Checking Solutions

You can check to see that the equation you wrote to describe a sequence is correct by finding the first few terms of the sequence.

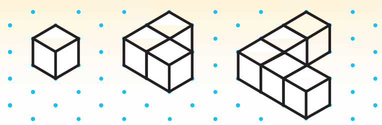
CHECK Your Progress

3. Write an equation for the nth term of the arithmetic sequence -1.5, -3.5, -5.5, ...

ALGEBRA LAB

Arithmetic Sequences

Study the figures below. The length of an edge of each cube is 1 centimeter.



MODEL AND ANALYZE

1. Based on the pattern, draw the fourth figure on a piece of isometric dot paper.
2. Find the volumes of the four figures.
3. Suppose the number of cubes in the pattern continues. Write an equation that gives the volume of Figure n.
4. What would the volume of the twelfth figure be?

Arithmetic Means Sometimes you are given two terms of a sequence, but they are not successive terms of that sequence. The terms between any two nonsuccessive terms of an arithmetic sequence are called **arithmetic means**. In the sequence below, 41, 52, and 63 are the three arithmetic means between 30 and 74.

$$19, 30, \underbrace{41, 52, 63}_{\text{three arithmetic means between 30 and 74}}, 74, 85, 96, \ldots$$

The formula for the nth term of a sequence can be used to find arithmetic means between given terms of a sequence.

EXAMPLE Find Arithmetic Means

4 **Find the four arithmetic means between 16 and 91.**

> **Study Tip**
>
> **Alternate Method**
>
> You may prefer this method. The four means will be $16 + d$, $16 + 2d$, $16 + 3d$, and $16 + 4d$. The common difference is $d = 91 - (16 + 4d)$ or $d = 15$.

You can use the nth term formula to find the common difference. In the sequence 16, __?__, __?__, __?__, __?__, 91, ..., a_1 is 16 and a_6 is 91.

$a_n = a_1 + (n - 1)d$	Formula for the nth term
$a_6 = 16 + (6 - 1)d$	$n = 6$, $a_1 = 16$
$91 = 16 + 5d$	$a_6 = 91$
$75 = 5d$	Subtract 16 from each side.
$15 = d$	Divide each side by 5.

Now use the value of d to find the four arithmetic means.

$$16 \xrightarrow{+15} 31 \xrightarrow{+15} 46 \xrightarrow{+15} 61 \xrightarrow{+15} 76$$

The arithmetic means are 31, 46, 61, and 76. **CHECK** $76 + 15 = 91$ ✓

CHECK Your Progress

4. Find the three arithmetic means between 15.6 and 60.4.

CHECK Your Understanding

Example 1 (p. 622)

Find the next four terms of each arithmetic sequence.

1. 12, 16, 20, ...

2. 3, 1, −1, ...

Find the first five terms of each arithmetic sequence described.

3. $a_1 = 5, d = 3$

4. $a_1 = 14, d = -2$

5. $a_1 = \frac{1}{2}, d = \frac{1}{4}$

6. $a_1 = 0.5, d = -0.2$

Example 2 (p. 623)

7. Find a_{13} for the arithmetic sequence −17, −12, −7, ...

Find the indicated term of each arithmetic sequence.

8. $a_1 = 3, d = -5, n = 24$

9. $a_1 = -5, d = 7, n = 13$

10. $a_1 = -4, d = \frac{1}{3}, n = 8$

11. $a_1 = 6.6, d = 1.05, n = 32$

12. **ENTERTAINMENT** A basketball team has a halftime promotion where a fan gets to shoot a 3-pointer to try to win a jackpot. The jackpot starts at \$5000 for the first game and increases \$500 each time there is no winner. Ellis has tickets to the fifteenth game of the season. How much will the jackpot be for that game if no one wins by then?

Example 3 (p. 624)

13. Write an equation for the nth term of the arithmetic sequence −26, −15, −4, 7, ...

14. Complete: 68 is the __?__th term of the arithmetic sequence −2, 3, 8, ...

Example 4 (p. 625)

15. Find the three arithmetic means between 44 and 92.

16. Find the three arithmetic means between 2.5 and 12.5.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
17–26	1
27–34	2
35–40	3
41–44	4

Find the next four terms of each arithmetic sequence.

17. 9, 16, 23, ...

18. 31, 24, 17, ...

19. −6, −2, 2, ...

20. −8, −5, −2, ...

Find the first five terms of each arithmetic sequence described.

21. $a_1 = 2, d = 13$

22. $a_1 = 41, d = 5$

23. $a_1 = 6, d = -4$

24. $a_1 = 12, d = -3$

25. Find a_8 if $a_n = 4 + 3n$.

26. If $a_n = 1 - 5n$, what is a_{10}?

Find the indicated term of each arithmetic sequence.

27. $a_1 = 3, d = 7, n = 14$

28. $a_1 = -4, d = -9, n = 20$

29. $a_1 = 35, d = 3, n = 101$

30. $a_1 = 20, d = 4, n = 81$

31. a_{12} for −17, −13, −9, ...

32. a_{12} for 8, 3, −2, ...

Real-World Link

Upon its completion in 1370, the Leaning Tower of Pisa leaned about 1.7 meters from vertical. Today, it leans about 5.2 meters from vertical.

Source: Associated Press

33. TOWER OF PISA To prove that objects of different weights fall at the same rate, Galileo dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. How many feet would an object fall in the sixth second?

34. GEOLOGY Geologists estimate that the continents of Europe and North America are drifting apart at a rate of an average of 12 miles every 1 million years, or about 0.75 inch per year. If the continents continue to drift apart at that rate, how many inches will they drift in 50 years? (*Hint*: $a_1 = 0.75$)

Complete the statement for each arithmetic sequence.

35. 170 is the __?__ term of −4, 2, 8, ...

36. 124 is the __?__ term of −2, 5, 12, ...

Write an equation for the *n*th term of each arithmetic sequence.

37. 7, 16, 25, 34, ...

38. 18, 11, 4, −3, ...

39. −3, −5, −7, −9, ...

40. −4, 1, 6, 11, ...

Find the arithmetic means in each sequence.

41. 55, __?__, __?__, __?__, 115

42. 10, __?__, __?__, −8

43. −8, __?__, __?__, __?__, __?__, 7

44. 3, __?__, __?__, __?__, __?__, __?__, 27

Find the next four terms of each arithmetic sequence.

45. $\frac{1}{3}, 1, \frac{5}{3}, \ldots$

46. $\frac{18}{5}, \frac{16}{5}, \frac{14}{5}, \ldots$

47. 6.7, 6.3, 5.9, ...

48. 1.3, 3.8, 6.3, ...

Find the first five terms of each arithmetic sequence described.

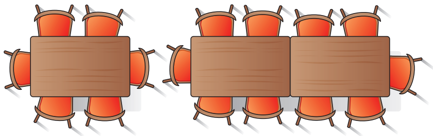
49. $a_1 = \frac{4}{3}, d = -\frac{1}{3}$

50. $a_1 = \frac{5}{8}, d = \frac{3}{8}$

51. **VACATION DAYS** Kono's employer gives him 1.5 vacation days for each month he works. If Kono has 11 days at the end of one year and takes no vacation time during the next year, how many days will he have at the end of that year?

52. **DRIVING** Olivia was driving her car at a speed of 65 miles per hour. To exit the highway, she began decelerating at a rate of 5 mph per second. How long did it take Olivia to come to a stop?

SEATING For Exercises 53–55, use the following information.
The rectangular tables in a reception hall are often placed end-to-end to form one long table. The diagrams below show the number of people who can sit at each of the table arrangements.



53. Make drawings to find the next three numbers as tables are added one at a time to the arrangement.
54. Write an equation representing the nth number in this pattern.
55. Is it possible to have seating for exactly 100 people with such an arrangement? Explain.

Find the indicated term of each arithmetic sequence.

56. $a_1 = 5, d = \frac{1}{3}, n = 12$

57. $a_1 = \frac{5}{2}, d = -\frac{3}{2}, n = 11$

58. a_{21} for 121, 118, 115, ...

59. a_{43} for 5, 9, 13, 17, ...

Use the given information to write an equation that represents the *n*th number in each arithmetic sequence.

60. The 15th term of the sequence is 66. The common difference is 4.
61. The 100th term of the sequence is 100. The common difference is 7.
62. The tenth term of the sequence is 84. The 21st term of the sequence is 161.
63. The 63rd term of the sequence is 237. The 90th term of the sequence is 75.

64. The 18th term of a sequence is 367. The 30th term of the sequence is 499. How many terms of this sequence are less than 1000?

EXTRA PRACTICE
See pages 914, 936.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

65. **OPEN ENDED** Write a real-life application that can be described by an arithmetic sequence with common difference −5.

66. **REASONING** Explain why the sequence 4, 5, 7, 10, 14, ... is not arithmetic.

67. **CHALLENGE** The numbers x, y, and z are the first three terms of an arithmetic sequence. Express z in terms of x and y.

68. ***Writing in Math*** Use the information on pages 622 and 623 to explain the relationship between n and a_n in the formula for the nth term of an arithmetic sequence. If n is the independent variable and a_n is the dependent variable, what kind of equation relates n and a_n? Explain what a_1 and d mean in the context of the graph.

STANDARDIZED TEST PRACTICE

69. ACT/SAT What is the first term in the arithmetic sequence?

$$\_\_\_\_,\ 8\frac{1}{3},\ 7,\ 5\frac{2}{3},\ 4\frac{1}{3},\ \ldots$$

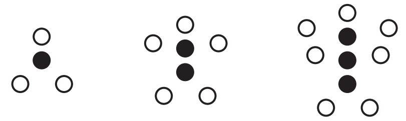
A 3

B $9\frac{2}{3}$

C $10\frac{1}{3}$

D 11

70. REVIEW The figures below show a pattern of filled circles and white circles that can be described by a relationship between 2 variables.



Which rule relates w, the number of white circles, to f, the number of dark circles?

F $w = 3f$

G $f = \frac{1}{2}w - 1$

H $w = 2f + 1$

J $f = \frac{1}{3}w$

Spiral Review

Find the exact solution(s) of each system of equations. (Lesson 10-7)

71. $x^2 + 2y^2 = 33$
$x^2 + y^2 - 19 = 2x$

72. $x^2 + 2y^2 = 33$
$x^2 - y^2 = 9$

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation. (Lesson 10-6)

73. $y^2 - 3x + 6y + 12 = 0$

74. $x^2 - 14x + 4 = 9y^2 - 36y$

75. If y varies directly as x and $y = 5$ when $x = 2$, find y when $x = 6$. (Lesson 8-4)

Simplify each expression. (Lesson 8-1)

76. $\frac{39a^3b^4}{13a^4b^3}$

77. $\frac{k+3}{5k\ell} \cdot \frac{10k\ell}{k+3}$

78. $\frac{5y - 15z}{42x^2} \div \frac{y - 3z}{14x}$

Find all the zeros of each function. (Lesson 6-8)

79. $f(x) = 8x^3 - 36x^2 + 22x + 21$

80. $g(x) = 12x^4 + 4x^3 - 3x^2 - x$

81. SAVINGS Mackenzie has \$57 in her bank account. She begins receiving a weekly allowance of \$15, of which she deposits 20% in her bank account. Write an equation that represents how much money is in Mackenzie's account after x weeks. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression for the given values of the variable.** (Lesson 1-1)

82. $3n - 1$; $n = 1, 2, 3, 4$

83. $6 - j$; $j = 1, 2, 3, 4$

84. $4m + 7$; $m = 1, 2, 3, 4, 5$

85. $4 - 2k$; $k = 3, 4, 5, 6, 7$

11-2 Arithmetic Series

Main Ideas

- Find sums of arithmetic series.
- Use sigma notation.

New Vocabulary

series
arithmetic series
sigma notation
index of summation

GET READY for the Lesson

Austin, Texas has a strong musical tradition. It is home to many indoor and outdoor music venues where new and established musicians perform regularly. Some of these venues are amphitheaters that generally get wider as the distance from the stage increases.

Suppose a section of an amphitheater can seat 18 people in the first row and each row can seat 4 more people than the previous row.

Arithmetic Series The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is 18 + 22 + 26 + 30 or 96. A **series** is an indicated sum of the terms of a sequence. Since 18, 22, 26, 30 is an arithmetic sequence, 18 + 22 + 26 + 30 is an **arithmetic series**.

Study Tip

Indicated Sum

The sum of a series is the result when the terms of the series are added. An *indicated sum* is the expression that illustrates the series, which includes the terms + or −.

S_n represents the sum of the first n terms of a series. For example, S_4 is the sum of the first four terms.

To develop a formula for the sum of any arithmetic series, consider the series below.

$$S_9 = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$$

Write S_9 in two different orders and add the two equations.

$$\begin{aligned}
S_9 &= 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60 \\
(+)\ S_9 &= 60 + 53 + 46 + 39 + 32 + 25 + 18 + 11 + 4 \\
\hline
2S_9 &= 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 \\
2S_9 &= 9(64) \\
S_9 &= \frac{9}{2}(64)
\end{aligned}$$

Note that the sum had 9 terms.

The sum of the first and last terms of the series is 64.

An arithmetic series S_n has n terms, and the sum of the first and last terms is $a_1 + a_n$. Thus, the formula $S_n = \frac{n}{2}(a_1 + a_n)$ represents the sum of any arithmetic series.

KEY CONCEPT — Sum of an Arithmetic Series

The sum S_n of the first n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \text{ or } S_n = \frac{n}{2}(a_1 + a_n).$$

EXAMPLE 1 Find the Sum of an Arithmetic Series

Find the sum of the first 100 positive integers.

The series is $1 + 2 + 3 + \ldots + 100$. Since you can see that $a_1 = 1$, $a_{100} = 100$, and $d = 1$, you can use either sum formula for this series.

Method 1		Method 2
$S_n = \frac{n}{2}(a_1 + a_n)$	Sum formula	$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$
$S_{100} = \frac{100}{2}(1 + 100)$	$n = 100$, $a_1 = 1$, $a_{100} = 100$, $d = 1$	$S_{100} = \frac{100}{2}[2(1) + (100 - 1)1]$
$S_{100} = 50(101)$	Simplify.	$S_{100} = 50(101)$
$S_{100} = 5050$	Multiply.	$S_{100} = 5050$

CHECK Your Progress

1. Find the sum of the first 50 positive even integers.

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Real-World EXAMPLE 2 Find the First Term

RADIO A radio station is giving away a total of \$124,000 in August. If they increase the amount given away each day by \$100, how much should they give away the first day?

You know the values of n, S_n, and d. Use the sum formula that contains d.

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	Sum formula
$S_{31} = \frac{31}{2}[2a_1 + (31 - 1)100]$	$n = 31$, $d = 100$
$124{,}000 = \frac{31}{2}(2a_1 + 3000)$	$S_{31} = 124{,}000$
$8000 = 2a_1 + 3000$	Multiply each side by $\frac{2}{31}$.
$5000 = 2a_1$	Subtract 3000 from each side.
$2500 = a_1$	Divide each side by 2.

The radio station should give away \$2500 the first day.

Real-World Link

99.0% of teens ages 12–17 listen to the radio at least once a week. 79.1% listen at least once a day.

Source: Radio Advertising Bureau

CHECK Your Progress

2. EXERCISE Aiden did pushups every day in March. He started on March 1st and increased the number of pushups done each day by one. He did a total of 1085 pushups for the month. How many pushups did Aiden do on March 1st?

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Sometimes it is necessary to use both a sum formula and the formula for the nth term to solve a problem.

EXAMPLE **Find the First Three Terms**

3 **Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$.**

Step 1 Since you know a_1, a_n, and S_n, use $S_n = \frac{n}{2}(a_1 + a_n)$ to find n.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$741 = \frac{n}{2}(9 + 105)$$

$$741 = 57n$$

$$13 = n$$

Step 2 Find d.

$$a_n = a_1 + (n - 1)d$$

$$105 = 9 + (13 - 1)d$$

$$96 = 12d$$

$$8 = d$$

Step 3 Use d to determine a_2 and a_3.

$a_2 = 9 + 8$ or 17 $\qquad$ $a_3 = 17 + 8$ or 25

The first three terms are 9, 17, and 25.

CHECK Your Progress

3. Find the first three terms of an arithmetic series in which $a_1 = -16$, $a_n = 33$, and $S_n = 68$.

Sigma Notation Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series $3 + 6 + 9 + 12 + \cdots + 30$ can be expressed as $\sum_{n=1}^{10} 3n$. This expression is read *the sum of 3n as n goes from 1 to 10.*

In Motion

Concepts in Motion

Animation

algebra2.com

last value of n

$$\sum_{n=1}^{10} 3n$$

formula for the terms of the series

first value of n

The variable, in this case n, is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of n are 1, 2, 3, and so on, through 10.

There are many ways to represent a given series. If changes are made to the first and last values of the variable and to the formula for the terms of the series, the same terms can be produced. For example, the following expressions produce the same terms.

$$\sum_{r=4}^{9} (r - 3) \qquad \sum_{s=2}^{7} (s - 1) \qquad \sum_{j=0}^{5} (j + 1)$$

EXAMPLE Evaluate a Sum in Sigma Notation

4 Evaluate $\sum_{j=5}^{8}(3j-4)$.

Method 1

Find the terms by replacing j with 5, 6, 7, and 8. Then add.

$$\begin{aligned}\sum_{j=5}^{8}(3j-4) &= [3(5)-4]+[3(6)-4]+[3(7)-4]+[3(8)-4]\\ &= 11+14+17+20\\ &= 62\end{aligned}$$

Method 2

Since the sum is an arithmetic series, use the formula $S_n = \frac{n}{2}(a_1 + a_n)$.

There are 4 terms, $a_1 = 3(5) - 4$ *or* 11, and $a_4 = 3(8) - 4$ or 20.

$$\begin{aligned}S_4 &= \frac{4}{2}(11+20)\\ &= 62\end{aligned}$$

CHECK Your Progress

4. Evaluate $\sum_{k=2}^{6}(2k+1)$.

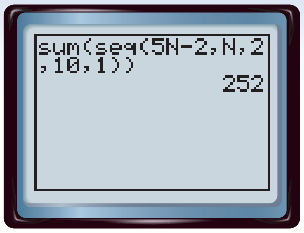
You can use the sum and sequence features on a graphing calculator to find the sum of a series.

GRAPHING CALCULATOR LAB

Sums of Series

The calculator screen shows the evaluation of $\sum_{N=2}^{10}(5N-2)$. The first four entries for seq(are

- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.



The last entry is always 1 for the types of series that we are considering.

THINK AND DISCUSS

1. Explain why you can use any letter for the index of summation.

2. Evaluate $\sum_{n=1}^{8}(2n-1)$ and $\sum_{j=5}^{12}(2j-9)$. Make a conjecture as to their relationship and explain why you think it is true.

Study Tip

Graphing Calculators

On the TI-83/84 Plus, sum(is located on the LIST MATH menu. The function seq(is located on the LIST OPS menu.

CHECK Your Understanding

Example 1 (p. 630)

Find the sum of each arithmetic series.

1. $5 + 11 + 17 + \cdots + 95$
2. $12 + 17 + 22 + \cdots + 102$
3. $38 + 35 + 32 + \cdots + 2$
4. $101 + 90 + 79 + \cdots + 2$

5. **TRAINING** To train for a race, Rosmaria runs 1.5 hours longer each week than she did the previous week. In the first week, Rosmaria ran 3 hours. How much time will Rosmaria spend running if she trains for 12 weeks?

Examples 1, 2 (p. 630)

Find S_n for each arithmetic series described.

6. $a_1 = 4, a_n = 100, n = 25$
7. $a_1 = 40, n = 20, d = -3$
8. $d = -4, n = 21, a_n = 52$
9. $d = 5, n = 16, a_n = 72$

Example 2 (p. 630)

Find a_1 for each arithmetic series described.

10. $d = 8, n = 19, S_{19} = 1786$
11. $d = -2, n = 12, S_{12} = 96$

Example 3 (p. 631)

Find the first three terms of each arithmetic series described.

12. $a_1 = 11, a_n = 110, S_n = 726$
13. $n = 8, a_n = 36, S_n = 120$

Example 4 (p. 632)

Find the sum of each arithmetic series.

14. $\sum_{n=1}^{7} (2n + 1)$
15. $\sum_{k=3}^{7} (3k + 4)$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
16–21, 34–37	1
22–25	1, 2
26–29	2
30–33	3
38–43	4

Find S_n for each arithmetic series described.

16. $a_1 = 7, a_n = 79, n = 8$
17. $a_1 = 58, a_n = -7, n = 26$
18. $a_1 = 7, d = -2, n = 9$
19. $a_1 = 3, d = -4, n = 8$
20. $a_1 = 5, d = \frac{1}{2}, n = 13$
21. $a_1 = 12, d = \frac{1}{3}, n = 13$
22. $d = -3, n = 21, a_n = -64$
23. $d = 7, n = 18, a_n = 72$

24. **TOYS** Jamila is making a wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?

25. **CONSTRUCTION** A construction company will be fined for each day it is late completing a bridge. The daily fine will be $4000 for the first day and will increase by $1000 each day. Based on its budget, the company can only afford $60,000 in total fines. What is the maximum number of days it can be late?

Find a_1 for each arithmetic series described.

26. $d = 3.5, n = 20, S_{20} = 1005$
27. $d = -4, n = 42, S_{42} = -3360$
28. $d = 0.5, n = 31, S_{31} = 573.5$
29. $d = -2, n = 18, S_{18} = 18$

EXTRA PRACTICE
See pages 914, 936.

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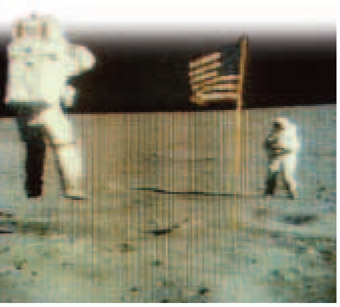
Find the first three terms of each arithmetic series described.

30. $a_1 = 17, a_n = 197, S_n = 2247$
31. $a_1 = -13, a_n = 427, S_n = 18{,}423$
32. $n = 31, a_n = 78, S_n = 1023$
33. $n = 19, a_n = 103, S_n = 1102$



Real-World Link

Six missions of the Apollo Program landed humans on the Moon. Apollo 11 was the first mission to do so.

Source: nssdc.gsfc.nasa.gov

Find the sum of each arithmetic series.

34. $6 + 13 + 20 + 27 + \cdots + 97$

35. $7 + 14 + 21 + 28 + \cdots + 98$

36. $34 + 30 + 26 + \cdots + 2$

37. $16 + 10 + 4 + \cdots + (-50)$

38. $\sum_{n=1}^{6} (2n + 11)$

39. $\sum_{n=1}^{5} (2 - 3n)$

40. $\sum_{k=7}^{11} (42 - 9k)$

41. $\sum_{t=19}^{23} (5t - 3)$

42. $\sum_{n=1}^{300} (7n - 3)$

43. $\sum_{k=1}^{150} (11 + 2k)$

Find S_n for each arithmetic series described.

44. $a_1 = 43, n = 19, a_n = 115$

45. $a_1 = 76, n = 21, a_n = 176$

46. $a_1 = 91, d = -4, a_n = 15$

47. $a_1 = -2, d = \frac{1}{3}, a_n = 9$

48. $d = \frac{1}{5}, n = 10, a_n = \frac{23}{10}$

49. $d = -\frac{1}{4}, n = 20, a_n = -\frac{53}{12}$

50. Find the sum of the first 1000 positive even integers.

51. What is the sum of the multiples of 3 between 3 and 999, inclusive?

52. AEROSPACE On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than it did the previous second. How far would an object fall in the first ten seconds after being dropped?

53. SALARY Mr. Vacarro's salary this year is $41,000. If he gets a raise of $2500 each year, how much will Mr. Vacarro earn in ten years?

Graphing Calculator

Use a graphing calculator to find the sum of each arithmetic series.

54. $\sum_{n=21}^{75} (2n + 5)$

55. $\sum_{n=10}^{50} (3n - 1)$

56. $\sum_{n=20}^{60} (4n + 3)$

57. $\sum_{n=17}^{90} (1.5n + 13)$

58. $\sum_{n=22}^{64} (-n + 70)$

59. $\sum_{n=26}^{50} (-2n + 100)$

H.O.T. Problems

60. OPEN ENDED Write an arithmetic series for which $S_5 = 10$.

CHALLENGE State whether each statement is *true* or *false*. Explain your reasoning.

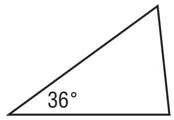
61. Doubling each term in an arithmetic series will double the sum.

62. Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum.

63. ***Writing in Math*** Use the information on page 629 to explain how arithmetic series apply to amphitheaters. Explain what the sequence and the series that can be formed from the given numbers represent, and show two ways to find the seating capacity of the amphitheater if it has ten rows of seats.

STANDARDIZED TEST PRACTICE

64. ACT/SAT The measures of the angles of a triangle form an arithmetic sequence. If the measure of the smallest angle is 36°, what is the measure of the largest angle?



A 75° **B** 84° **C** 90° **D** 97°

65. REVIEW How many 5-inch cubes can be placed completely inside a box that is 10 inches long, 15 inches wide, and 5 inches tall?

F 5 **H** 20

G 6 **J** 15

Spiral Review

Find the indicated term of each arithmetic sequence. (Lesson 11-1)

66. $a_1 = 46, d = 5, n = 14$ **67.** $a_1 = 12, d = -7, n = 22$

Solve each system of inequalities by graphing. (Lesson 10-7)

68. $9x^2 + y^2 < 81$
$x^2 + y^2 \geq 16$

69. $(y - 3)^2 \geq x + 2$
$x^2 \leq y + 4$

Write an equivalent logarithmic equation. (Lesson 9-2)

70. $5^x = 45$ **71.** $7^3 = x$ **72.** $b^y = x$

73. PAINTING Two employees of a painting company paint houses together. One painter can paint a house alone in 3 days, and the other painter can paint the same size house alone in 4 days. How long will it take them to paint one house if they work together? (Lesson 8-6)

Simplify. (Lesson 7-5)

74. $5\sqrt{3} - 4\sqrt{3}$ **75.** $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$ **76.** $(\sqrt{10} - \sqrt{6})(\sqrt{5} + \sqrt{3})$

Solve each equation by completing the square. (Lesson 5-5)

77. $x^2 + 9x + 20.25 = 0$ **78.** $9x^2 + 96x + 256 = 0$ **79.** $x^2 - 3x - 20 = 0$

Use a graphing calculator to find the value of each determinant. (Lesson 4-5)

80. $\begin{vmatrix} 1.3 & 7.2 \\ 6.1 & 5.4 \end{vmatrix}$ **81.** $\begin{vmatrix} 6.1 & 4.8 \\ 9.7 & 3.5 \end{vmatrix}$ **82.** $\begin{vmatrix} 8 & 6 & -5 \\ 10 & -7 & 3 \\ 9 & 14 & -6 \end{vmatrix}$

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

83. $a + 4b = 6$
$3a + 2b = -2$

84. $10x - y = 13$
$3x - 4y = 15$

85. $3c - 7d = -1$
$2c - 6d = -6$

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate the expression $a \cdot b^{n-1}$ for the given values of a, b, and n.** (Lesson 1-1)

86. $a = 1, b = 2, n = 5$ **87.** $a = 2, b = -3, n = 4$ **88.** $a = 18, b = \frac{1}{3}, n = 6$

11-3 Geometric Sequences

Main Ideas

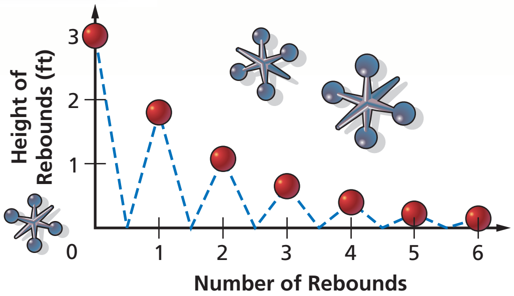
- Use geometric sequences.
- Find geometric means.

New Vocabulary

geometric sequence
common ratio
geometric means

GET READY for the Lesson

When you drop a ball, it never rebounds to the height from which you dropped it. Suppose a ball is dropped from a height of three feet, and each time it falls, it rebounds to 60% of the height from which it fell. The heights of the ball's rebounds form a sequence.



Geometric Sequences The height of the first rebound of the ball is 3(0.6) or 1.8 feet. The height of the second rebound is 1.8(0.6) or 1.08 feet. The height of the third rebound is 1.08(0.6) or 0.648 feet. The sequence of heights is an example of a **geometric sequence**. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a nonzero constant r called the **common ratio**.

As with an arithmetic sequence, you can label the terms of a geometric sequence as a_1, a_2, a_3, and so on, $a_1 \neq 0$. The nth term is a_n and the previous term is a_{n-1}. So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

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STANDARDIZED TEST EXAMPLE — Find the Next Term

1 What is the missing term in the geometric sequence: 8, 20, 50, 125, _____ ?

A 75 **B** 200 **C** 250 **D** 312.5

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the common ratio is 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: $125(2.5) = 312.5$. The answer is D.

Test-Taking Tip

Since the terms of this sequence are increasing, the missing term must be greater than 125. You can immediately eliminate 75 as a possible answer.

CHECK Your Progress

1. What is the missing term in the geometric sequence: −120, 60, −30, 15, _______ ?

F −7.5 **G** 0 **H** 7.5 **J** 10

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You have seen that each term of a geometric sequence after the first term can be expressed in terms of r and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of r and the first term a_1. Study the patterns in the table for the sequence 2, 6, 18, 54, … .

Sequence	**numbers**	2	6	18	54	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of *r* and the Previous Term	**numbers**	2	2(3)	6(3)	18(3)	…	
	symbols	a_1	$a_1 \cdot r$	$a_2 \cdot r$	$a_3 \cdot r$	…	$a_{n-1} \cdot r$
Expressed in Terms of *r* and the First Term	**numbers**	2	2(3)	2(9)	2(27)	…	
		$2(3^0)$	$2(3^1)$	$2(3^2)$	$2(3^3)$	…	
	symbols	$a_1 \cdot r^0$	$a_1 \cdot r^1$	$a_1 \cdot r^2$	$a_1 \cdot r^3$	…	$a_1 \cdot r^{n-1}$

The three entries in the last column all describe the nth term of a geometric sequence. This leads to the following formula.

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KEY CONCEPT — ***nth Term of a Geometric Sequence***

The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by the following formula, where n is any positive integer.

$$a_n = a_1 \cdot r^{n-1}$$

Study Tip

Finding a Term
For small values of r and n, it may be easier to multiply by r successively to find a given term than to use the formula.

EXAMPLE 2 Find a Term Given the First Term and the Ratio

Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_8 = (-3) \cdot (-2)^{8-1}$	$n = 8$, $a_1 = -3$, $r = -2$
$a_8 = (-3) \cdot (-128)$	$(-2)^7 = -128$
$a_8 = 384$	Multiply.

CHECK Your Progress

2. Find the sixth term of a geometric sequence for which $a_1 = -\frac{1}{9}$ and $r = 3$.

EXAMPLE 3 Write an Equation for the *n*th Term

Write an equation for the nth term of the geometric sequence 3, 12, 48, 192, … .

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_n = 3 \cdot 4^{n-1}$	$a_1 = 3$, $r = 4$

CHECK Your Progress

3. Write an equation for the nth term of the geometric sequence 18, −3, $\frac{1}{2}$, $-\frac{1}{12}$, … .

You can also use the formula for the nth term if you know the common ratio and one term of a geometric sequence, but not the first term.

EXAMPLE Find a Term Given One Term and the Ratio

4 **Find the tenth term of a geometric sequence for which $a_4 = 108$ and $r = 3$.**

Step 1 Find the value of a_1.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_4 = a_1 \cdot 3^{4-1}$	$n = 4, r = 3$
$108 = 27a_1$	$a_4 = 108$
$4 = a_1$	Divide each side by 27.

Step 2 Find a_{10}.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_{10} = 4 \cdot 3^{10-1}$	$n = 10, a_1 = 4, r = 3$
$a_{10} = 78{,}732$	Use a calculator.

The tenth term is 78,732.

CHECK Your Progress

4. Find the eighth term of a geometric sequence for which $a_3 = 16$ and $r = 4$.

Geometric Means In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called *arithmetic means*. Similarly, the missing terms(s) between two nonsuccessive terms of a geometric sequence are called **geometric means**. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, … . You can use the common ratio to find the geometric means in a sequence.

EXAMPLE Find Geometric Means

5 **Find three geometric means between 2.25 and 576.**

Use the nth term formula to find the value of r. In the sequence 2.25, _?_, _?_, _?_, 576, a_1 is 2.25 and a_5 is 576.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_5 = 2.25 \cdot r^{5-1}$	$n = 5, a_1 = 2.25$
$576 = 2.25r^4$	$a_5 = 576$
$256 = r^4$	Divide each side by 2.25.
$\pm 4 = r$	Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find three geometric means.

$r = 4$	$r = -4$
$a_2 = 2.25(4)$ or 9	$a_2 = 2.25(-4)$ or -9
$a_3 = 9(4)$ or 36	$a_3 = -9(-4)$ or 36
$a_4 = 36(4)$ or 144	$a_4 = 36(-4)$ or -144

The geometric means are 9, 36, and 144, or -9, 36, and -144.

CHECK Your Progress

5. Find two geometric means between 4 and 13.5.

Study Tip

Alternate Method

You may prefer this method. The three means will be $2.25r$, $2.25r^2$, and $2.25r^3$. Then the common ratio is $r = \frac{576}{2.25r^3}$ or $r^4 = \frac{576}{2.25}$. Thus, $r = 4$.

CHECK Your Understanding

Example 1 (p. 636)

1. Find the next two terms of the geometric sequence 20, 30, 45, … .

2. Find the first five terms of the geometric sequence for which $a_1 = -2$ and $r = 3$.

3. **STANDARDIZED TEST PRACTICE** What is the missing term in the geometric sequence: $-\frac{1}{4}, \frac{1}{2}, -1, 2,$ ___ ?

A -4 B $-3\frac{1}{2}$ C $3\frac{1}{2}$ D 4

Example 2 (p. 637)

4. Find a_9 for the geometric sequence 60, 30, 15, … .

5. Find a_8 for the geometric sequence $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$.

Find the indicated term of each geometric sequence.

6. $a_1 = 7, r = 2, n = 4$

7. $a_1 = 3, r = \frac{1}{3}, n = 5$

Example 3 (p. 637)

8. Write an equation for the nth term of the geometric sequence 4, 8, 16, … .

9. Write an equation for the nth term of the geometric sequence 15, 5, $\frac{5}{3}$, … .

Example 4 (p. 638)

Find the indicated term of each geometric sequence.

10. $a_3 = 24, r = \frac{1}{2}, n = 7$

11. $a_3 = 32, r = -0.5, n = 6$

Example 5 (p. 638)

12. Find two geometric means between 1 and 27.

13. Find two geometric means between 2 and 54.

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–19	1
20–27	2
28, 29	3
30–33	4
34–37	5

Find the next two terms of each geometric sequence.

14. 405, 135, 45, …

15. 81, 108, 144, …

16. 16, 24, 36, …

17. 162, 108, 72, …

Find the first five terms of each geometric sequence described.

18. $a_1 = 2, r = -3$

19. $a_1 = 1, r = 4$

20. Find a_7 if $a_1 = 12$ and $r = \frac{1}{2}$.

21. Find a_6 if $a_1 = \frac{1}{3}$ and $r = 6$.

22. **INTEREST** An investment pays interest so that each year the value of the investment increases by 10%. How much is an initial investment of $1000 worth after 5 years?

23. **SALARIES** Geraldo's current salary is $40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases?

Find the indicated term of each geometric sequence.

24. $a_1 = \frac{1}{3}, r = 3, n = 8$

25. $a_1 = \frac{1}{64}, r = 4, n = 9$

26. a_9 for $a_1 = \frac{1}{5}, 1, 5, \ldots$

27. a_7 for $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$

28. $a_4 = 16, r = 0.5, n = 8$

29. $a_6 = 3, r = 2, n = 12$

Real-World Link

The largest ever ice construction was an ice palace built for a carnival in St. Paul, Minnesota, in 1992. It contained 10.8 million pounds of ice.

Source: *The Guinness Book of Records*

Write an equation for the nth term of each geometric sequence.

30. 36, 12, 4, …

31. 64, 16, 4, …

32. $-2, 10, -50, \ldots$

33. $4, -12, 36, \ldots$

Find the geometric means in each sequence.

34. 9, ___?___, ___?___, ___?___, 144

35. 4, ___?___, ___?___, ___?___, 324

36. 32, ___?___, ___?___, ___?___, ___?___, 1

37. 3, ___?___, ___?___, ___?___, ___?___, 96

Find the next two terms of each geometric sequence.

38. $\frac{5}{2}, \frac{5}{3}, \frac{10}{9}, \ldots$

39. $\frac{1}{3}, \frac{5}{6}, \frac{25}{12}, \ldots$

40. $1.25, -1.5, 1.8, \ldots$

41. $1.4, -3.5, 8.75, \ldots$

Find the first five terms of each geometric sequence described.

42. $a_1 = 243, r = \frac{1}{3}$

43. $a_1 = 576, r = -\frac{1}{2}$

44. ART A one-ton ice sculpture is melting so that it loses one-eighth of its weight per hour. How much of the sculpture will be left after five hours? Write your answer in pounds.

MEDICINE For Exercises 45 and 46, use the following information.

Iodine-131 is a radioactive element used to study the thyroid gland.

45. RESEARCH Use the Internet or other resource to find the *half-life* of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element.

46. How much of an 80-milligram sample of Iodine-131 would be left after 32 days?

EXTRA PRACTICE

See pages 914, 936.

Math Online

Self-Check Quiz at algebra2.com

Find the indicated term of each geometric sequence.

47. $a_1 = 16{,}807, r = \frac{3}{7}, n = 6$

48. $a_1 = 4096, r = \frac{1}{4}, n = 8$

49. a_8 for $4, -12, 36, \ldots$

50. a_6 for 540, 90, 15, …

51. $a_4 = 50, r = 2, n = 8$

52. $a_4 = 1, r = 3, n = 10$

H.O.T. Problems

53. OPEN ENDED Write a geometric sequence with a common ratio of $\frac{2}{3}$.

54. FIND THE ERROR Marika and Lori are finding the seventh term of the geometric sequence 9, 3, 1, … . Who is correct? Explain your reasoning.

Marika

$r = \frac{3}{9}$ or $\frac{1}{3}$

$a_7 = 9\left(\frac{1}{3}\right)^{7-1}$

$= \frac{1}{81}$

Lori

$r = \frac{9}{3}$ or 3

$a_7 = 9 \cdot 3^{7-1}$

$= 6561$

55. Which One Doesn't Belong? Identify the sequence that does not belong with the other three. Explain your reasoning.

1, 4, 16, …	3, 9, 27, …	9, 16, 25, …	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Bill Horseman/Stock Boston

CHALLENGE **Determine whether each statement is *true* or *false*. If true, explain. If false, provide a counterexample.**

56. Every sequence is either arithmetic or geometric.

57. There is no sequence that is both arithmetic and geometric.

58. ***Writing in Math*** Use the information on pages 636 and 637 to explain the relationship between n and a_n in the formula for the nth term of a geometric sequence. If n is the independent variable and a_n is the dependent variable, what kind of equation relates n and a_n? Explain what r represents in the context of the relationship.

STANDARDIZED TEST PRACTICE

59. **ACT/SAT** The first four terms of a geometric sequence are shown in the table. What is the tenth term in the sequence?

a_1	144
a_2	72
a_3	36
a_4	18

A 0

B $\frac{9}{64}$

C $\frac{9}{32}$

D $\frac{9}{16}$

60. **REVIEW** The table shows the cost of jelly beans depending on the amount purchased. Which conclusion can be made based on the table?

Cost of Jelly Beans	
Number of Pounds	Cost
5	\$14.95
20	\$57.80
50	\$139.50
100	\$269.00

F The cost of 10 pounds of jelly beans would be more than \$30.

G The cost of 200 pounds of jelly beans would be less than \$540.

H The cost of jelly beans is always more than \$2.70 per pound.

J The cost of jelly beans is always less than \$2.97 per pound.

Spiral Review

Find S_n for each arithmetic series described. (Lesson 11-2)

61. $a_1 = 11, a_n = 44, n = 23$

62. $a_1 = -5, d = 3, n = 14$

Find the arithmetic means in each sequence. (Lesson 11-1)

63. 15, __?__ , __?__ , 27

64. −8, __?__ , __?__ , __?__ , −24

65. **GEOMETRY** Find the perimeter of a triangle with vertices at (2, 4), (−1, 3) and (1, −3). (Lesson 10-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.** (Lesson 1-1)

66. $\dfrac{1 - 2^7}{1 - 2}$

67. $\dfrac{1 - \left(\frac{1}{2}\right)^6}{1 - \frac{1}{2}}$

68. $\dfrac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)}$

EXTEND 11-3

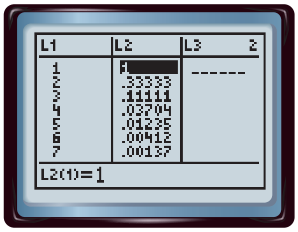
Graphing Calculator Lab
Limits

You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as n increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

ACTIVITY 1 Find the limit of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \ldots$.

Step 1 **Enter the sequence.**

- The formula for this sequence is $a_n = \left(\frac{1}{3}\right)^{n-1}$.
- Position the cursor on L1 in the STAT EDIT Edit … screen and enter the formula seq(N,N,1,10,1). This generates the values 1, 2, …, 10 of the index N.
- Position the cursor on L2 and enter the formula seq((1/3)^(N-1),N,1,10,1). This generates the first ten terms of the sequence.

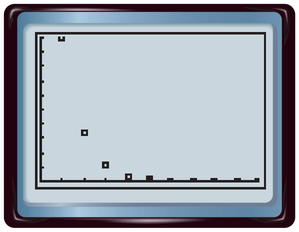


KEYSTROKES: *Review sequences in the Graphing Calculator Lab on page 632.*

Notice that as n increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \geq 8$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 **Graph the sequence.**

- Use a STAT PLOT to graph the sequence. Use L1 as the Xlist and L2 as the Ylist.



KEYSTROKES: *Review STAT PLOTs on page 92.*

The graph also shows that, as n increases, the terms approach 0. In fact, for $n \geq 6$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.

Exercises

Use a graphing calculator to find the limit, if it exists, of each sequence.

1. $a_n = \left(\frac{1}{2}\right)^n$

2. $a_n = \left(-\frac{1}{2}\right)^n$

3. $a_n = 4^n$

4. $a_n = \frac{1}{n^2}$

5. $a_n = \frac{2^n}{2^n + 1}$

6. $a_n = \frac{n^2}{n + 1}$

Other Calculator Keystrokes at algebra2.com

11-4 Geometric Series

Main Ideas

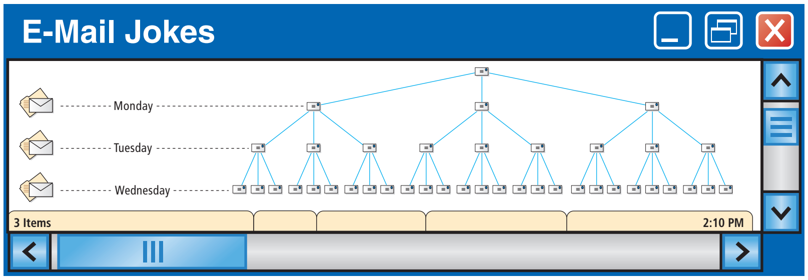
- Find sums of geometric series.
- Find specific terms of geometric series.

New Vocabulary

geometric series

GET READY for the Lesson

Suppose you e-mail a joke to three friends on Monday. Each of those friends sends the joke on to three of their friends on Tuesday. Each person who receives the joke on Tuesday sends it to three more people on Wednesday, and so on.



Geometric Series Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which $a_1 = 1$ and $r = 3$. The indicated sum of the numbers in the sequence, $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, is called a **geometric series**.

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above. Multiply each term in the series by the common ratio and subtract the result from the original series.

$$\begin{array}{rl} S_8 = & 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 \\ (-)\ 3S_8 = & \phantom{1 +{}} 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561 \\ \hline (1 - 3)S_8 = & 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 - 6561 \end{array}$$

first term in series

$$S_8 = \frac{1 - 6561}{1 - 3} \text{ or } 3280$$

last term in series multiplied by common ratio; in this case, a_9

common ratio

The expression for S_8 can be written as $S_8 = \dfrac{a_1 - a_1r^8}{1 - r}$. A rational expression like this can be used to find the sum of any geometric series.

Study Tip

Terms of Geometric Sequences

Remember that a_9 can also be written as a_1r^8.

KEY CONCEPT — Sum of a Geometric Series

The sum S_n of the first n terms of a geometric series is given by

$$S_n = \frac{a_1 - a_1r^n}{1 - r} \text{ or } S_n = \frac{a_1(1 - r^n)}{1 - r}, \text{ where } r \neq 1.$$

You cannot use the formula for the sum with a geometric series for which $r = 1$ because division by 0 would result. In a geometric series with $r = 1$, the terms are constant. For example, $4 + 4 + 4 + \cdots + 4$ is such a series. In general, the sum of n terms of a geometric series with $r = 1$ is $n \cdot a_1$.

Real-World Link

The development of vaccines for many diseases has helped to prevent infection. Vaccinations are commonly given to children.

Real-World EXAMPLE — Find the Sum of the First *n* Terms

1 **HEALTH** **Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic, and each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness?**

This is a geometric series with $a_1 = 5$, $r = 4$, and $n = 10$.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ — Sum formula

$S_{15} = \frac{5(1 - 4^{10})}{1 - 4}$ — $n = 10, a_1 = 5, r = 4$

$S_{15} = 1{,}747{,}625$ — Use a calculator.

After ten weeks, 1,747,625 people have been affected by the illness.

CHECK Your Progress

1. **GAMES** Maria arranges some rows of dominoes so that after she knocks over the first one, each domino knocks over two more dominoes when it falls. If there are ten rows, how many dominoes does Maria use?

You can use sigma notation to represent geometric series.

Personal Tutor

EXAMPLE — Evaluate a Sum Written in Sigma Notation

2 **Evaluate** $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.

Method 1 Since the sum is a geometric series, you can use the formula.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$

$S_6 = \frac{5(1 - 2^6)}{1 - 2}$ — $n = 6, a_1 = 5, r = 2$

$S_6 = \frac{5(-63)}{-1}$ — $2^6 = 64$

$S_6 = 315$ — Simplify.

David Kelly Crow/PhotoEdit

Method 2 Find the terms by replacing n with 1, 2, 3, 4, 5, and 6. Then add.

$$\sum_{n=1}^{6} 5 \cdot 2^{n-1}$$

$$= 5(2^{1-1}) + 5(2^{2-1}) + 5(2^{3-1}) + 5(2^{4-1}) + 5(2^{5-1}) + 5(2^{6-1})$$ Write out the sum.

$$= 5(1) + 5(2) + 5(4) + 5(8) + 5(16) + 5(32)$$ Simplify.

$$= 5 + 10 + 20 + 40 + 80 + 160$$ Multiply.

$$= 315$$ Add.

CHECK Your Progress

2. Evaluate $\sum_{n=1}^{5} -4 \cdot 3^{n-1}$.

Online Personal Tutor at algebra2.com

How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? You can use the formula for the nth term of a geometric sequence or series, $a_n = a_1 \cdot r^{n-1}$, to find an expression involving r^n.

$$a_n = a_1 \cdot r^{n-1}$$ Formula for nth term

$$a_n \cdot r = a_1 \cdot r^{n-1} \cdot r$$ Multiply each side by r.

$$a_n \cdot r = a_1 \cdot r^n$$ $r^{n-1} \cdot r^1 = r^{n-1+1}$ or r^n

Now substitute $a_n \cdot r$ for $a_1 \cdot r^n$ in the formula for the sum of geometric series. The result is $S_n = \frac{a_1 - a_n r}{1 - r}$.

EXAMPLE 3 Use the Alternate Formula for a Sum

Find the sum of a geometric series for which $a_1 = 15{,}625$, $a_n = -5$, and $r = -\frac{1}{5}$.

Since you do not know the value of n, use the formula derived above.

$$S_n = \frac{a_1 - a_n r}{1 - r}$$ Alternate sum formula

$$= \frac{15{,}625 - (-5)\left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)}$$ $a_1 = 15{,}625$; $a_n = -5$; $r = -\frac{1}{5}$

$$= \frac{15{,}624}{\frac{6}{5}} \text{ or } 13{,}020$$ Simplify.

CHECK Your Progress

3. Find the sum of a geometric series for which $a_1 = 1000$, $a_n = 125$, and $r = \frac{1}{2}$.

Specific Terms You can use the formula for the sum of a geometric series to help find a particular term of the series.

EXAMPLE 4 Find the First Term of a Series

Find a_1 in a geometric series for which $S_8 = 39{,}360$ and $r = 3$.

$S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ Sum formula

$39{,}360 = \dfrac{a_1(1 - 3^8)}{1 - 3}$ $S_8 = 39{,}360$; $r = 3$; $n = 8$

$39{,}360 = \dfrac{-6560a_1}{-2}$ Subtract.

$39{,}360 = 3280a_1$ Divide.

$12 = a_1$ Divide each side by 3280.

CHECK Your Progress

4. Find a_1 in a geometric series for which $S_7 = 258$ and $r = -2$.

CHECK Your Understanding

Example 1 (p. 644)

Find S_n for each geometric series described.

1. $a_1 = 5, r = 2, n = 14$

2. $a_1 = 243, r = -\frac{2}{3}, n = 5$

Find the sum of each geometric series.

3. $54 + 36 + 24 + 16 + \cdots$ to 6 terms

4. $3 - 6 + 12 - \cdots$ to 7 terms

5. **WEATHER** Heavy rain caused a river to rise. The river rose three inches the first day, and each day it rose twice as much as the previous day. How much did the river rise in five days?

Example 2 (pp. 644–645)

Find the sum of each geometric series.

6. $\sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1}$

7. $\sum_{n=1}^{7} 81\left(\frac{1}{3}\right)^{n-1}$

8. $\sum_{n=1}^{12} \frac{1}{6}(-2)^n$

9. $\sum_{n=1}^{8} \frac{1}{3} \cdot 5^{n-1}$

10. $\sum_{n=1}^{6} 100\left(\frac{1}{2}\right)^{n-1}$

11. $\sum_{n=1}^{9} \frac{1}{27}(-3)^{n-1}$

Example 3 (p. 645)

Find S_n for each geometric series described.

12. $a_1 = 12, a_5 = 972, r = -3$

13. $a_1 = 3, a_n = 46{,}875, r = -5$

14. $a_1 = 5, a_n = 81{,}920, r = 4$

15. $a_1 = -8, a_6 = -256, r = 2$

Example 4 (p. 646)

Find the indicated term for each geometric series described.

16. $S_n = \frac{381}{64}, r = \frac{1}{2}, n = 7; a_1$

17. $S_n = 33, a_n = 48, r = -2; a_1$

18. $S_n = 443, r = \frac{1}{3}, n = 6; a_1$

19. $S_n = -242, a_n = -162, r = 3; a_1$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
20–25, 28–31	1
26, 27	3
32, 33	2
34–37	4

Find S_n for each geometric series described.

20. $a_1 = 2, a_6 = 486, r = 3$

21. $a_1 = 3, a_8 = 384, r = 2$

22. $a_1 = 4, r = -3, n = 5$

23. $a_1 = 5, r = 3, n = 12$

24. $a_1 = 2401, r = -\frac{1}{7}, n = 5$

25. $a_1 = 625, r = \frac{3}{5}, n = 5$

26. $a_1 = 1296, a_n = 1, r = -\frac{1}{6}$

27. $a_1 = 343, a_n = -1, r = -\frac{1}{7}$

28. GENEALOGY In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?

29. LEGENDS There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have \$1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth?

Day	Payment
1	1¢
2	2¢
3	4¢
4	8¢
⋮	⋮
30	?
Total	?

Real-World Link

Some of the best-known legends involving a king are the Arthurian legends. According to the legends, King Arthur reigned over Britain before the Saxon conquest. Camelot was the most famous castle in the medieval legends of King Arthur.

Find the sum of each geometric series.

30. $4096 - 512 + 64 - \cdots$ to 5 terms

31. $7 + 21 + 63 + \cdots$ to 10 terms

32. $\sum_{n=1}^{9} 5 \cdot 2^{n-1}$

33. $\sum_{n=1}^{6} 2(-3)^{n-1}$

Find the indicated term for each geometric series described.

34. $S_n = 165, a_n = 48, r = -\frac{2}{3}; a_1$

35. $S_n = 688, a_n = 16, r = -\frac{1}{2}; a_1$

36. $S_n = -364, r = -3, n = 6; a_1$

37. $S_n = 1530, r = 2, n = 8; a_1$

Find S_n for each geometric series described.

38. $a_1 = 162, r = \frac{1}{3}, n = 6$

39. $a_1 = 80, r = -\frac{1}{2}, n = 7$

40. $a_1 = 625, r = 0.4, n = 8$

41. $a_1 = 4, r = 0.5, n = 8$

42. $a_2 = -36, a_5 = 972, n = 7$

43. $a_3 = -36, a_6 = -972, n = 10$

44. $a_1 = 4, a_n = 236{,}196, r = 3$

45. $a_1 = 125, a_n = \frac{1}{125}, r = \frac{1}{5}$

Find the sum of each geometric series.

46. $\frac{1}{16} + \frac{1}{4} + 1 + \cdots$ to 7 terms

47. $\frac{1}{9} - \frac{1}{3} + 1 - \cdots$ to 6 terms

48. $\sum_{n=1}^{8} 64\left(\frac{3}{4}\right)^{n-1}$

49. $\sum_{n=1}^{20} 3 \cdot 2^{n-1}$

50. $\sum_{n=1}^{16} 4 \cdot 3^{n-1}$

51. $\sum_{n=1}^{7} 144\left(-\frac{1}{2}\right)^{n-1}$

EXTRA PRACTICE
See pages 915, 936.
Math Online
Self-Check Quiz at
algebra2.com

Find the indicated term for each geometric series described.

52. $S_n = 315, r = 0.5, n = 6; a_2$

53. $S_n = 249.92, r = 0.2, n = 5, a_3$

54. WATER TREATMENT A certain water filtration system can remove 80% of the contaminants each time a sample of water is passed through it. If the same water is passed through the system three times, what percent of the original contaminants will be removed from the water sample?

Graphing Calculator

Use a graphing calculator to find the sum of each geometric series.

55. $\sum_{n=1}^{20} 3(-2)^{n-1}$

56. $\sum_{n=1}^{15} 2\left(\frac{1}{2}\right)^{n-1}$

57. $\sum_{n=1}^{10} 5(0.2)^{n-1}$

58. $\sum_{n=1}^{13} 6\left(\frac{1}{3}\right)^{n-1}$

H.O.T. Problems

59. OPEN ENDED Write a geometric series for which $r = \frac{1}{2}$ and $n = 4$.

60. REASONING Explain how to write the series $2 + 12 + 72 + 432 + 2592$ using sigma notation.

61. CHALLENGE If a_1 and r are integers, explain why the value of $\frac{a_1 - a_1 r^n}{1 - r}$ must also be an integer.

62. REASONING Explain, using geometric series, why the polynomial $1 + x + x^2 + x^3$ can be written as $\frac{x^4 - 1}{x - 1}$, assuming $x \neq 1$.

63. *Writing in Math* Use the information on page 643 to explain how e-mailing a joke is related to a geometric series. Include an explanation of how the situation could be changed to make it better to use a formula than to add terms.

STANDARDIZED TEST PRACTICE

64. ACT/SAT The first term of a geometric series is -1, and the common ratio is -3. How many terms are in the series if its sum is 182?

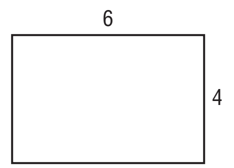
A 6

B 7

C 8

D 9

65. REVIEW Which set of dimensions corresponds to a rectangle similar to the one shown below?



F 3 units by 1 unit

G 12 units by 9 units

H 13 units by 8 units

J 18 units by 12 units

Spiral Review

Find the geometric means in each sequence. (Lesson 11-3)

66. $\frac{1}{24}$, __?__ , __?__ , __?__ , 54

67. -2, __?__ , __?__ , __?__ , __?__ , $-\frac{243}{16}$

Find the sum of each arithmetic series. (Lesson 11-2)

68. $50 + 44 + 38 + \cdots + 8$

69. $\sum_{n=1}^{12} (2n + 3)$

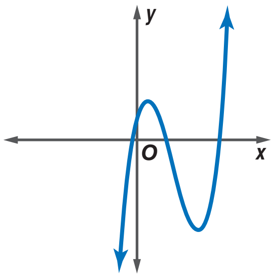
Solve each equation. Check your solutions. (Lesson 8-6)

70. $\frac{1}{y+1} - \frac{3}{y-3} = 2$

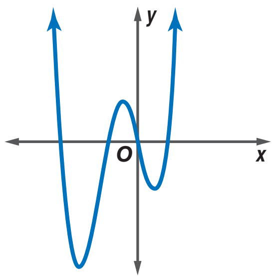
71. $\frac{6}{a-7} = \frac{a-49}{a^2-7a} + \frac{1}{a}$

Determine whether each graph represents an odd-degree polynomial function or an even-degree polynomial function. Then state how many real zeros each function has. (Lesson 6-4)

72.



73.



Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 5-3)

74. $3d^2 + 2d - 8$

75. $42pq - 35p + 18q - 15$

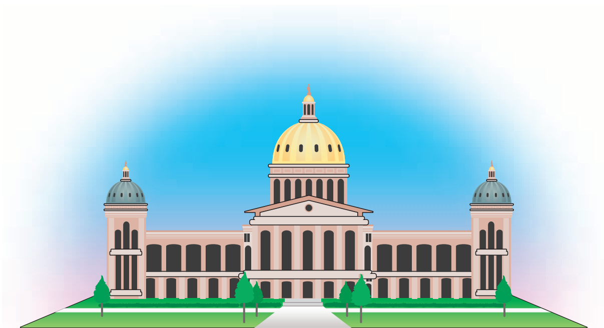
76. $13xyz + 3x^2z + 4k$

VOTING **For Exercises 77–79, use the table that shows the percent of the Iowa population of voting age that voted in each presidential election from 1984–2004.** (Lesson 2-5)

77. Draw a scatter plot in which x is the number of elections since the 1984 election.

78. Find a linear prediction equation.

79. Predict the percent of the Iowa voting age population that will vote in the 2012 election.



Year	1984	1988	1992	1996	2000	2004
Percent	76.0	75.7	72.4	80.2	73.2	75.0

Source: sos.state.ia.us

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate $\frac{a}{1-b}$ for the given values of a and b.** (Lesson 1-1)

80. $a = 1, b = \frac{1}{2}$

81. $a = 3, b = -\frac{1}{2}$

82. $a = \frac{1}{3}, b = -\frac{1}{3}$

83. $a = \frac{1}{2}, b = \frac{1}{4}$

84. $a = -1, b = 0.5$

85. $a = 0.9, b = -0.5$

11-5 Infinite Geometric Series

Main Ideas

- Find the sum of an infinite geometric series.
- Write repeating decimals as fractions.

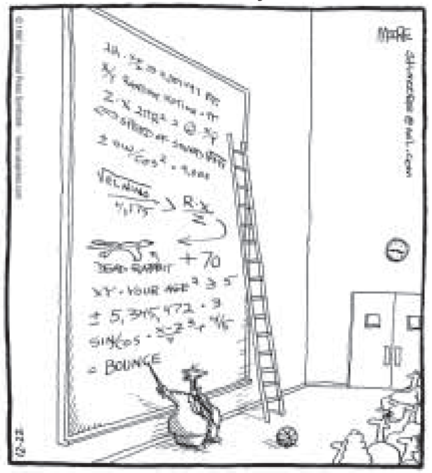
New Vocabulary

infinite geometric series
partial sum
convergent series

GET READY for the Lesson

Suppose you wrote a geometric series to find the sum of the heights of the rebounds of the ball on page 636. The series would have no last term because theoretically there is no last bounce of the ball. For every rebound of the ball, there is another rebound, 60% as high. Such a geometric series is called an **infinite geometric series.**

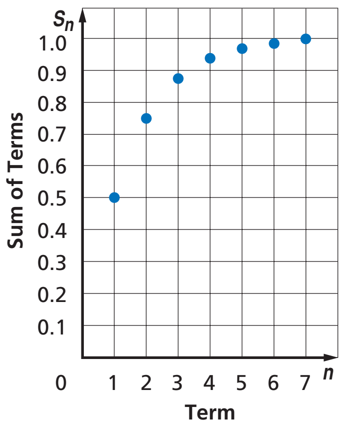
In the Bleachers By Steve Moore



"And that, ladies and gentlemen, is the way the ball bounces."

Infinite Geometric Series Consider the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. You have already learned how to find the sum S_n of the first n terms of a geometric series. For an infinite series, S_n is called a **partial sum** of the series. The table and graph show some values of S_n.

n	S_n
1	$\frac{1}{2}$ or 0.5
2	$\frac{3}{4}$ or 0.75
3	$\frac{7}{8}$ or 0.875
4	$\frac{15}{16}$ or 0.9375
5	$\frac{31}{32}$ or 0.96875
6	$\frac{63}{64}$ or 0.984375
7	$\frac{127}{128}$ or 0.9921875



Notice that as n increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which $|r| < 1$.

Let's look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.

Study Tip

Absolute Value

Recall that $|r| < 1$ means $-1 < r < 1$.

$$S_n = \frac{a_1 - a_1r^n}{1 - r} \quad \text{Sum of first } n \text{ terms of a finite geometric series}$$

$$= \frac{a_1}{1-r} - \frac{a_1r^n}{1-r} \quad \text{Write the fraction as a difference of fractions.}$$

If $-1 < r < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of an infinite geometric series will approach $\frac{a_1}{1-r} - \frac{a_1(0)}{1-r}$ or $\frac{a_1}{1-r}$. An infinite series that has a sum is called a **convergent series.**

Study Tip

Formula for Sum if $-1 < r < 1$

To convince yourself of this formula, make a table of the first ten partial sums of the geometric series with $r = \frac{1}{2}$ and $a_1 = 100$.

Term Number	Term	Partial Sum
1	100	100
2	50	150
3	25	175
⋮	⋮	⋮
10		

Complete the table and compare the sum that the series is approaching to that obtained by using the formula.

KEY CONCEPT — Sum of an Infinite Geometric Series

The sum S of an infinite geometric series with $-1 < r < 1$ is given by

$$S = \frac{a_1}{1-r}.$$

An infinite geometric series for which $|r| \geq 1$ does not have a sum. Consider the series $1 + 3 + 9 + 27 + 81 + \ldots$. In this series, $a_1 = 1$ and $r = 3$. The table shows some of the partial sums of this series. As n increases, S_n rapidly increases and has no limit.

n	S_n
5	121
10	29,524
15	7,174,453
20	1,743,392,200

EXAMPLE 1 Sum of an Infinite Geometric Series

Personal Tutor

Find the sum of each infinite geometric series, if it exists.

a. $\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \ldots$

Step 1 Find the value of r to determine if the sum exists.

$a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{8}$, so $r = \frac{\frac{3}{8}}{\frac{1}{2}}$ or $\frac{3}{4}$.

Since $\left|\frac{3}{4}\right| < 1$, the sum exists.

Step 2 Use the formula for the sum of an infinite geometric series.

$$S = \frac{a_1}{1-r} \quad \text{Sum formula}$$

$$= \frac{\frac{1}{2}}{1-\frac{3}{4}} \quad a_1 = \frac{1}{2}, r = \frac{3}{4}$$

$$= \frac{\frac{1}{2}}{\frac{1}{4}} \text{ or } 2 \quad \text{Simplify.}$$

b. $1 - 2 + 4 - 8 + \ldots$

$a_1 = 1$ and $a_2 = -2$, so $r = \frac{-2}{1}$ or -2. Since $|-2| \geq 1$, the sum does not exist.

CHECK Your Progress

1A. $3 + 9 + 27 + 51 + \ldots$

1B. $-3 + \frac{1}{3} - \frac{1}{27} + \ldots$

Online Personal Tutor at algebra2.com

You can use sigma notation to represent infinite series. An *infinity symbol* ∞ is placed above the Σ to indicate that a series is infinite.

EXAMPLE Infinite Series in Sigma Notation

2 **Evaluate $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1}$.**

$S = \frac{a_1}{1-r}$ Sum formula

$= \frac{24}{1-\left(-\frac{1}{5}\right)}$ $a_1 = 24, r = -\frac{1}{5}$

$= \frac{24}{\frac{6}{5}}$ or 20 Simplify.

CHECK Your Progress

2. Evaluate $\sum_{n=1}^{\infty} 11\left(\frac{1}{3}\right)^{n-1}$.

Repeating Decimals The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction.

EXAMPLE Write a Repeating Decimal as a Fraction

3 **Write $0.\overline{39}$ as a fraction.**

Method 1

$0.\overline{39} = 0.393939\ldots$

$= 0.39 + 0.0039 + 0.000039 + \ldots$

$= \frac{39}{100} + \frac{39}{10{,}000} + \frac{39}{1{,}000{,}000} + \ldots$

$S = \frac{a_1}{1-r}$ Sum formula

$= \frac{\frac{39}{100}}{1-\frac{1}{100}}$ $a_1 = \frac{39}{100}, r = \frac{1}{100}$

$= \frac{\frac{39}{100}}{\frac{99}{100}}$ Subtract.

$= \frac{39}{99}$ or $\frac{13}{33}$ Simplify.

Method 2

$S = 0.\overline{39}$ Label the given decimal.

$S = 0.393939\ldots$ Repeating decimal

$100S = 39.393939\ldots$ Multiply each side by 100.

$99S = 39$ Subtract the second equation from the third.

$S = \frac{39}{99}$ or $\frac{13}{33}$ Divide each side by 99.

Study Tip

Bar Notation

Remember that decimals with bar notation such as $0.\overline{2}$ and $0.\overline{47}$ represent 0.222222… and 0.474747…, respectively.

CHECK Your Progress

3. Write $0.\overline{47}$ as a fraction.

Math Online Extra Examples at algebra2.com

CHECK Your Understanding

Example 1 (p. 651)

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 36, r = \frac{2}{3}$

2. $a_1 = 18, r = -1.5$

3. $16 + 24 + 36 + \cdots$

4. $\frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \cdots$

5. CLOCKS Altovese's grandfather clock is broken. When she sets the pendulum in motion by holding it against the side of the clock and letting it go, it swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings before it stops?

Example 2 (p. 652)

Find the sum of each infinite geometric series, if it exists.

6. $\sum_{n=1}^{\infty} 6\,(-0.4)^{n-1}$

7. $\sum_{n=1}^{\infty} 40\left(\frac{3}{5}\right)^{n-1}$

8. $\sum_{n=1}^{\infty} 35\left(-\frac{3}{4}\right)^{n-1}$

9. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{8}\right)^{n-1}$

Example 3 (p. 652)

Write each repeating decimal as a fraction.

10. $0.\overline{5}$

11. $0.\overline{73}$

12. $0.\overline{175}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–22, 32–34	1
23–27	2
28–31	3

Find the sum of each infinite geometric series, if it exists.

13. $a_1 = 4, r = \frac{5}{7}$

14. $a_1 = 14, r = \frac{7}{3}$

15. $a_1 = 12, r = -0.6$

16. $a_1 = 18, r = 0.6$

17. $16 + 12 + 9 + \cdots$

18. $-8 - 4 - 2 - \cdots$

19. $12 - 18 + 24 - \cdots$

20. $18 - 12 + 8 - \cdots$

21. $1 + \frac{2}{3} + \frac{4}{9} + \cdots$

22. $\frac{5}{3} + \frac{25}{3} + \frac{125}{3} + \cdots$

23. $\sum_{n=1}^{\infty} 48\left(\frac{2}{3}\right)^{n-1}$

24. $\sum_{n=1}^{\infty} \left(\frac{3}{8}\right)\left(\frac{3}{4}\right)^{n-1}$

25. $\sum_{n=1}^{\infty} \frac{1}{2}(3)^{n-1}$

26. $\sum_{n=1}^{\infty} 10{,}000\left(\frac{1}{101}\right)^{n-1}$

27. $\sum_{n=1}^{\infty} \frac{1}{100}\left(\frac{101}{99}\right)^{n-1}$

Write each repeating decimal as a fraction.

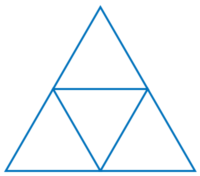
28. $0.\overline{7}$

29. $0.\overline{1}$

30. $0.\overline{36}$

31. $0.\overline{82}$

GEOMETRY For Exercises 32 and 33, refer to equilateral triangle *ABC*, which has a perimeter of 39 centimeters. If the midpoints of the sides are connected, a smaller equilateral triangle results. Suppose the process of connecting midpoints of sides and drawing new triangles is continued indefinitely.



32. Write an infinite geometric series to represent the sum of the perimeters of all of the triangles.

33. Find the sum of the perimeters of all of the triangles.

EXTRA PRACTICE
See pages 915, 936.

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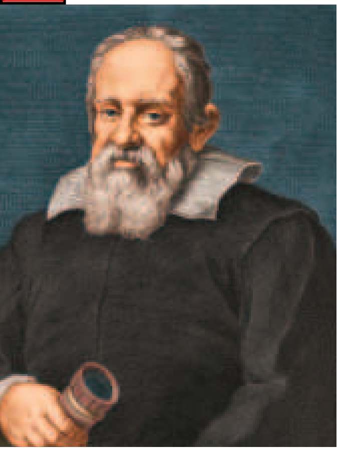


Real-World Link

Galileo Galilei performed experiments with wooden ramps and metal balls to study the physics of acceleration.

Source: galileoandeinstein.physics.virginia.edu

34. **PHYSICS** In a physics experiment, a steel ball on a flat track is accelerated and then allowed to roll freely. After the first minute, the ball has rolled 120 feet. Each minute the ball travels only 40% as far as it did during the preceding minute. How far does the ball travel?

Find the sum of each infinite geometric series, if it exists.

35. $\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \ldots$

36. $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \ldots$

37. $3 + 1.8 + 1.08 + \ldots$

38. $1 - 0.5 + 0.25 - \ldots$

39. $\sum_{n=1}^{\infty} 3(0.5)^{n-1}$

40. $\sum_{n=1}^{\infty} (1.5)(0.25)^{n-1}$

Write each repeating decimal as a fraction

41. $0.\overline{246}$

42. $0.\overline{427}$

43. $0.4\overline{5}$

44. $0.2\overline{31}$

45. **SCIENCE MUSEUM** An exhibit at a science museum offers visitors the opportunity to experiment with the motion of an object on a spring. One visitor pulled the object down and let it go. The object traveled a distance of 1.2 feet upward before heading back the other way. Each time the object changed direction, it moved only 80% as far as it did in the previous direction. Find the total distance the object traveled.

46. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.

47. The sum of an infinite geometric series is 125, and the value of r is 0.4. Find the first three terms of the series.

48. The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series.

49. The first term of an infinite geometric series is -8, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series.

H.O.T. Problems

50. **OPEN ENDED** Write the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ using sigma notation in two different ways.

51. **REASONING** Explain why $0.999999\ldots = 1$.

52. **FIND THE ERROR** Conrado and Beth are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \cdots$. Conrado says that the sum of the series is $-\frac{1}{7}$. Beth says that the series does not have a sum. Who is correct? Explain your reasoning.

Conrado

$$S = \frac{-\frac{1}{3}}{1 - \left(-\frac{4}{3}\right)}$$
$$= -\frac{1}{7}$$

53. **CHALLENGE** Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum S of a general infinite geometric series, multiply each side of the equation by r, and subtract equations.

54. ***Writing in Math*** Use the information on page 650 to explain how an infinite geometric series applies to a bouncing ball. Explain how to find the total distance traveled, both up and down, by the bouncing ball described on page 636.

Stock Montage/Getty Images

STANDARDIZED TEST PRACTICE

55. ACT/SAT What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$?

A 3

B 4

C 9

D 12

56. REVIEW What is the sum of the infinite geometric series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \dots$?

F $\frac{2}{3}$

G 1

H $1\frac{1}{3}$

J $1\frac{2}{3}$

Spiral Review

Find S_n for each geometric series described. (Lesson 11-4)

57. $a_1 = 1, a_6 = -243, r = -3$

58. $a_1 = 72, r = \frac{1}{3}, n = 7$

59. PHYSICS A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes? (Lesson 11-3)

Solve each equation or inequality. Check your solution. (Lesson 9-1)

60. $6^x = 216$

61. $2^{2x} = \frac{1}{8}$

62. $3^{x-2} \geq 27$

Simplify each expression. (Lesson 8-2)

63. $\frac{-2}{ab} + \frac{5}{a^2}$

64. $\frac{1}{x-3} - \frac{2}{x+1}$

65. $\frac{1}{x^2 + 6x + 8} + \frac{3}{x+4}$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers. (Lesson 5-3)

66. 6, −6

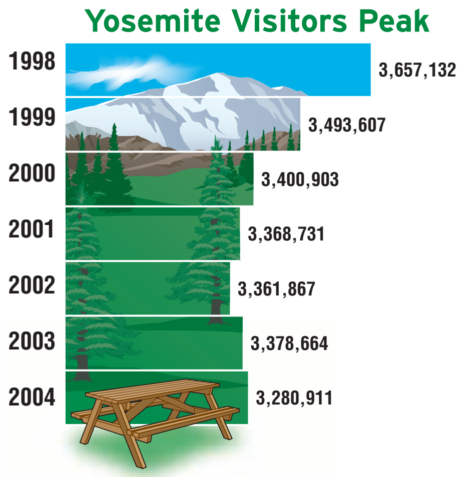
67. −2, −7

68. 6, 4

RECREATION For Exercises 69 and 70, refer to the graph at the right. (Lesson 2-3)

69. Find the average rate of change of the number of visitors to Yosemite National Park from 1998 to 2004.

70. Interpret your answer to Exercise 69.



Source: nps.gov

GET READY for the Next Lesson

PREREQUISITE SKILL Find each function value. (Lesson 2-1)

71. $f(x) = 2x, f(1)$

72. $g(x) = 3x - 3, g(2)$

73. $h(x) = -2x + 2, h(0)$

74. $f(x) = 3x - 1, f\left(\frac{1}{2}\right)$

75. $g(x) = x^2, g(2)$

76. $h(x) = 2x^2 - 4, h(0)$

CHAPTER 11

Mid-Chapter Quiz

Lessons 11-1 through 11-5

Find the indicated term of each arithmetic sequence. (Lesson 11-1)

1. $a_1 = 7, d = 3, n = 14$
2. $a_1 = 2, d = \frac{1}{2}, n = 8$

For Exercises 3 and 4, refer to the following information. (Lesson 11-1)

READING Amber makes a New Year's resolution to read 50 books by the end of the year.

3. By the end of February, Amber has read 9 books. If she reads 3 books each month for the rest of the year, will she meet her goal? Explain.
4. If Amber has read 10 books by the end of April, how many will she have to read on average each month in order to meet her goal?

5. **MULTIPLE CHOICE** The figures below show a pattern of filled squares and white squares that can be described by a relationship between 2 variables.

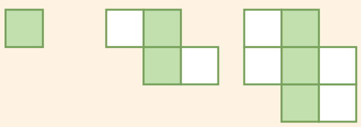


Figure 1 Figure 2 Figure 3

Which rule relates f, the number of filled squares, to w, the number of white squares? (Lesson 11-1)

A $w = f - 1$ **C** $f = \frac{1}{2}w - 1$

B $w = 2f - 2$ **D** $f = w - 1$

Find the sum of each arithmetic series described. (Lesson 11-2)

6. $a_1 = 5, a_n = 29, n = 11$
7. $6 + 12 + 18 + \cdots + 96$

8. **BANKING** Veronica has a savings account with $1500 dollars in it. At the end of each month, the balance in her account has increased by 0.25%. How much money will Veronica have in her savings account at the end of one year? (Lesson 11-3)

9. **GAMES** In order to help members of a group get to know each other, sometimes the group plays a game. The first person states his or her name and an interesting fact about himself or herself. The next person must repeat the first person's name and fact and then say his or her own. Each person must repeat the information for all those who preceded him or her. If there are 20 people in a group, what is the total number of times the names and facts will be stated? (Lesson 11-2)

10. Find a_7 for the geometric sequence 729, −243, 81, … . (Lesson 11-3)

Find the sum of each geometric series, if it exists. (Lessons 11-4 and 11-5)

11. $a_1 = 5, r = 3, n = 12$
12. $5 + 1 + \frac{1}{5} + \cdots$
13. $\sum_{n=1}^{6} 2(-3)^{n-1}$
14. $\sum_{n=1}^{\infty} 8\left(\frac{2}{3}\right)^{n-1}$
15. $\sum_{n=1}^{\infty} -13\left(\frac{1}{3}\right)^{n-1}$
16. $\sum_{n=1}^{\infty} \frac{1}{100}\left(\frac{10}{9}\right)^{n-1}$

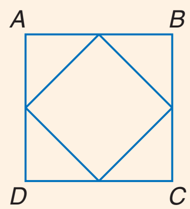
Write each repeating decimal as a fraction. (Lesson 11-5)

17. $0.\overline{17}$
18. $0.\overline{256}$
19. $1.\overline{27}$
20. $3.1\overline{5}$

GEOMETRY For Exercises 21 and 22, refer to square *ABCD*, which has a perimeter of 120 inches. (Lesson 11-5)

If the midpoints of the sides are connected, a smaller square results. Suppose the process of connecting midpoints of sides and drawing new squares is continued indefinitely.

21. Write an infinite geometric series to represent the sum of the perimeters of all of the squares.
22. Find the sum of the perimeters of all of the squares.



EXPLORE 11-6

Spreadsheet Lab

Amortizing Loans

When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest, and balance on a loan.

EXAMPLE

Marisela just bought a new sofa for $495. The store is letting her make monthly payments of $43.29 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by $1 + 0.0075$ or 1.0075 and then subtracting 43.29.

In a spreadsheet, use the column of numbers for the number of payments and use column B for the balance. Enter the interest rate and monthly payment in cells in column A so that they can be easily updated if the information changes.

Loans

	A	B
1	Interest rate	=495*(1+A2)−A5
2	0.0075	=B1*(1+A2)−A5
3		=B2*(1+A2)−A5
4	Monthly Payment	=B3*(1+A2)−A5
5	43.29	=B4*(1+A2)−A5
6		=B5*(1+A2)−A5

Sheet 1 Sheet 2 Sheet 3

The spreadsheet shows the formulas for the balances after each of the first six payments. After six months, Marisela still owes $253.04.

EXERCISES

1. Let b_n be the balance left on Marisela's loan after n months. Write an equation relating b_n and b_{n+1}.
2. What percent of Marisela's loan remains to be paid after half a year?
3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?
4. Suppose Marisela decides to pay $50 every month. How long would it take her to pay off the loan?
5. Suppose that, based on how much she can afford, Marisela will pay a variable amount each month in addition to the $43.29. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.
6. Jamie has a three-year, $12,000 car loan. The annual interest rate is 6%, and his monthly payment is $365.06. After twelve months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?

11-6 Recursion and Special Sequences

Main Ideas

- Recognize and use special sequences.
- Iterate functions.

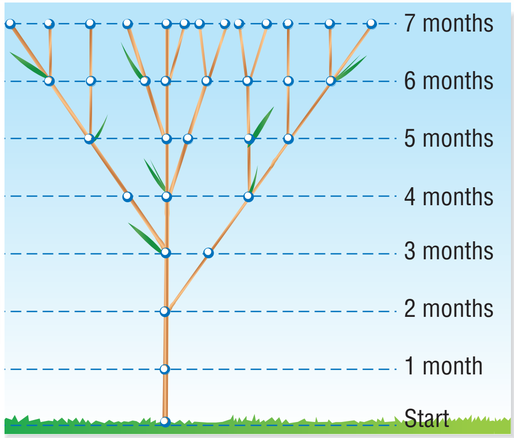
New Vocabulary

Fibonacci sequence
recursive formula
iteration

GET READY for the Lesson

A shoot on a sneezewort plant must grow for two months before it is strong enough to put out another shoot. After that, it puts out at least one shoot every month.

Month	1	2	3	4	5
Shoots	1	1	2	3	5



Special Sequences Notice that each term in the sequence is the sum of the two previous terms. For example, $8 = 3 + 5$ and $13 = 5 + 8$. This sequence is called the **Fibonacci sequence**, and it is found in many places in nature.

first term	a_1		1
second term	a_2		1
third term	a_3	$a_1 + a_2$	$1 + 1 = 2$
fourth term	a_4	$a_2 + a_3$	$1 + 2 = 3$
⋮	⋮	⋮	
nth term	a_n	$a_{n-2} + a_{n-1}$	

The formula $a_n = a_{n-2} + a_{n-1}$ is an example of a **recursive formula**. This means that each term is formulated from one or more previous terms.

EXAMPLE Use a Recursive Formula

1 **Find the first five terms of the sequence in which $a_1 = 4$ and $a_{n+1} = 3a_n - 2$, $n \geq 1$.**

$a_{n+1} = 3a_n - 2$ — Recursive formula

$a_{1+1} = 3a_1 - 2$ — $n = 1$

$a_2 = 3(4) - 2$ or 10 — $a_1 = 4$

$a_{2+1} = 3a_2 - 2$ — $n = 2$

$a_3 = 3(10) - 2$ or 28 — $a_2 = 10$

$a_{3+1} = 3a_3 - 2$ — $n = 3$

$a_4 = 3(28) - 2$ or 82 — $a_3 = 28$

$a_{4+1} = 3a_4 - 2$ — $n = 4$

$a_5 = 3(82) - 2$ or 244 — $a_4 = 82$

The first five terms of the sequence are 4, 10, 28, 82, and 244.

CHECK Your Progress

1. Find the first five terms of the sequence in which $a_1 = -1$ and $a_{n+1} = 2a_n + 4$, $n \geq 1$.

Personal Tutor

Real-World EXAMPLE Find and Use a Recursive Formula

2 **MEDICAL RESEARCH** **A pharmaceutical company is experimenting with a new drug. An experiment begins with 1.0×10^9 bacteria. A dose of the drug that is administered every four hours can kill 4.0×10^8 bacteria. Between doses of the drug, the number of bacteria increases by 50%.**

a. Write a recursive formula for the number of bacteria alive before each application of the drug.

Let b_n represent the number of bacteria alive just before the nth application of the drug. 4.0×10^8 of these will be killed by the drug, leaving $b_n - 4.0 \times 10^8$. The number b_{n+1} of bacteria before the next application will have increased by 50%. So $b_{n+1} = 1.5(b_n - 4.0 \times 10^8)$, or $1.5b_n - 6.0 \times 10^8$.

b. Find the number of bacteria alive before the fifth application.

Before the first application of the drug, there were 1.0×10^9 bacteria alive, so $b_1 = 1.0 \times 10^9$.

$b_{n+1} = 1.5b_n - 6.0 \times 10^8$ Recursive formula

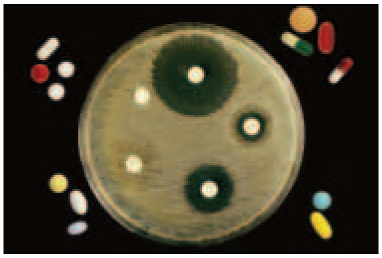
$b_{1+1} = 1.5b_1 - 6.0 \times 10^8$ $n = 1$
$b_2 = 1.5(1.0 \times 10^9) - 6.0 \times 10^8$
or 9.0×10^8

$b_{2+1} = 1.5b_2 - 6.0 \times 10^8$ $n = 2$
$b_3 = 1.5(9.0 \times 10^8) - 6.0 \times 10^8$
or 7.5×10^8

$b_{3+1} = 1.5b_3 - 6.0 \times 10^8$ $n = 3$
$b_4 = 1.5(7.5 \times 10^8) - 6.0 \times 10^8$
or 5.25×10^8

$b_{4+1} = 1.5b_4 - 6.0 \times 10^8$ $n = 4$
$b_5 = 1.5(5.25 \times 10^8) - 6.0 \times 10^8$
or 1.875×10^8

Before the fifth dose, there would be 1.875×10^8 bacteria alive.



Real-World Link

In 1928, Alexander Fleming found that penicillin mold could destroy certain types of bacteria. Production increases allowed the price of penicillin to fall from about \$20 per dose in 1943 to \$0.55 per dose in 1946.

Source: inventors.about.com

CHECK Your Progress

A stronger dose of the drug can kill 6.0×10^8 bacteria.

2A. Write a recursive formula for the number of bacteria alive before each dose of the drug.

2B. How many of the stronger doses of the drug will kill all the bacteria?

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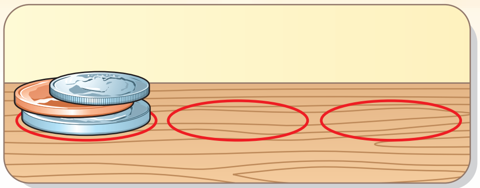
ALGEBRA LAB

Special Sequences

The object of the *Towers of Hanoi* game is to move a stack of *n* coins from one position to another in the fewest number a_n of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may not be placed on top of a dime.



(continued on the next page)

Dr. Dennis Drenner/Getty Images

MODEL AND ANALYZE

1. Draw three circles on a sheet of paper, as shown. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle?

2. Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.)

3. Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle?

MAKE A CONJECTURE

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number a_n of moves required to move a stack of n different sized coins.

Study Tip

Look Back
To review the **composition of functions**, see Lesson 7-5.

Iteration **Iteration** is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is $f \circ f(x)$ or $f(f(x))$. If you compose a function with itself two times, the result is $f \circ f \circ f(x)$ or $f(f(f(x)))$, and so on.

You can use iteration to recursively generate a sequence. Start with an initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(x_1)$ or $f(f(x_0))$, $x_3 = f(x_2)$ or $f(f(f(x_0)))$, and so on.

EXAMPLE Iterate a Function

3 **Find the first three iterates x_1, x_2, and x_3 of the function $f(x) = 2x + 3$ for an initial value of $x_0 = 1$.**

$x_1 = f(x_0)$	Iterate the function.	$x_3 = f(x_2)$	Iterate the function.
$= f(1)$	$x_0 = 1$	$= f(13)$	$x_2 = 13$
$= 2(1) + 3$ or 5	Simplify.	$= 2(13) + 3$ or 29	Simplify.
$x_2 = f(x_1)$	Iterate the function.		
$= f(5)$	$x_1 = 5$		
$= 2(5) + 3$ or 13	Simplify.		

The first three iterates are 5, 13, and 29.

CHECK Your Progress

3. Find the first four iterates, x_1, x_2, x_3, x_4, of the function $f(x) = x^2 - 2x - 1$ for an initial value of $x_0 = -1$.

CHECK Your Understanding

Example 1 (p. 658)

Find the first five terms of each sequence.

1. $a_1 = 12, a_{n+1} = a_n - 3$

2. $a_1 = -3, a_{n+1} = a_n + n$

3. $a_1 = 0, a_{n+1} = -2a_n - 4$

4. $a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3a_n$

Example 2 (p. 659)

BANKING **For Exercises 5 and 6, use the following information.**
Rita has deposited \$1000 in a bank account. At the end of each year, the bank posts 3% interest to her account, but then takes out a \$10 annual fee.

5. Let b_0 be the amount Rita deposited. Write a recursive equation for the balance b_n in her account at the end of n years.

6. Find the balance in the account after four years.

Example 3 (p. 660)

Find the first three iterates of each function for the given initial value.

7. $f(x) = 3x - 4, x_0 = 3$ **8.** $f(x) = -2x + 5, x_0 = 2$ **9.** $f(x) = x^2 + 2, x_0 = -1$

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–17	1
18–21	3
22–27	2

Find the first five terms of each sequence.

10. $a_1 = -6, a_{n+1} = a_n + 3$

11. $a_1 = 13, a_{n+1} = a_n + 5$

12. $a_1 = 2, a_{n+1} = a_n - n$

13. $a_1 = 6, a_{n+1} = a_n + n + 3$

14. $a_1 = 9, a_{n+1} = 2a_n - 4$

15. $a_1 = 4, a_{n+1} = 3a_n - 6$

16. If $a_0 = 7$ and $a_{n+1} = a_n + 12$ for $n \geq 0$, find the value of a_5.

17. If $a_0 = 1$ and $a_{n+1} = -2.1$ for $n \geq 0$, then what is the value of a_4?

Find the first three iterates of each function for the given initial value.

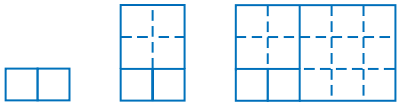
18. $f(x) = 9x - 2, x_0 = 2$

19. $f(x) = 4x - 3, x_0 = 2$

20. $f(x) = 3x + 5, x_0 = -4$

21. $f(x) = 5x + 1, x_0 = -1$

GEOMETRY **For Exercises 22–24, use the following information.**
Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process.



Step 1 Step 2 Step 3

22. Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with two original squares.

23. Write a recursive formula for the sequence of lengths added.

24. Identify the sequence in Exercise 23.

GEOMETRY **For Exercises 25–27, study the triangular numbers shown below.**

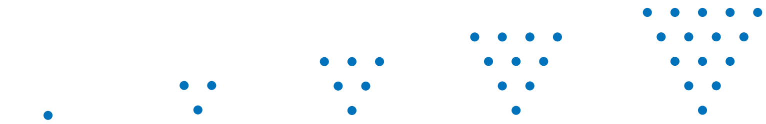


Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

25. Write a sequence of the first five triangular numbers.

26. Write a recursive formula for the nth triangular number t_n.

27. What is the 200th triangular number?

28. **LOANS** Miguel's monthly car payment is \$234.85. The recursive formula $b_n = 1.005b_{n-1} - 234.85$ describes the balance left on the loan after n payments. Find the balance of the \$10,000 loan after each of the first eight payments.

Real-World Career

Loan Officer

Loan officers help customers through the loan application process. Their work may require frequent travel.

Math Online For more information, go to algebra2.com.

EXTRA PRACTICE
See pages 915, 936.
Math Online
Self-Check Quiz at
algebra2.com

29. **ECONOMICS** If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function $c(x) = 1.02x$. Find the cost of a \$70 MP3 player in four years if the rate of inflation remains constant.

Find the first three iterates of each function for the given initial value.

30. $f(x) = 2x^2 - 5$, $x_0 = -1$
31. $f(x) = 3x^2 - 4$, $x_0 = 1$
32. $f(x) = 2x^2 + 2x + 1$, $x_0 = \frac{1}{2}$
33. $f(x) = 3x^2 - 3x + 2$, $x_0 = \frac{1}{3}$

H.O.T. Problems

34. **OPEN ENDED** Write a recursive formula for a sequence whose first three terms are 1, 1, and 3.

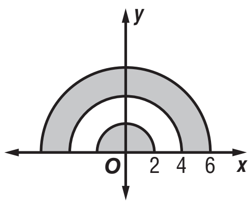
35. **REASONING** Is the statement $x_n \neq x_{n-1}$ *sometimes*, *always*, or *never* true if $x_n = f(x_{n-1})$? Explain.

36. **CHALLENGE** Are there a function $f(x)$ and an initial value x_0 such that the first three iterates, in order, are 4, 4, and 7? Explain.

37. ***Writing in Math*** Use the information on page 658 to explain how the Fibonacci sequence is illustrated in nature. Include the 13th term in the sequence, with an explanation of what it tells you about the plant described.

STANDARDIZED TEST PRACTICE

38. **ACT/SAT** The figure is made of three concentric semicircles. What is the total area of the shaded regions?



A 4π units^2
B 10π units^2
C 12π units^2
D 20π units^2

39. **REVIEW** If x is a real number, for what values of x is the equation $\frac{4x - 16}{4} = x - 4$ true?

F all values of x
G some values of x
H no values of x
J impossible to determine

Spiral Review

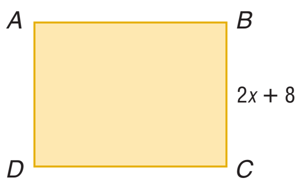
Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

40. $9 + 6 + 4 + \cdots$
41. $\frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$
42. $4 - \frac{8}{3} + \frac{16}{9} + \cdots$

Find the sum of each geometric series. (Lesson 11-4)

43. $2 - 10 + 50 - \cdots$ to 6 terms
44. $3 + 1 + \frac{1}{3} + \cdots$ to 7 terms

45. **GEOMETRY** The area of rectangle $ABCD$ is $6x^2 + 38x + 56$ square units. Its width is $2x + 8$ units. What is the length of the rectangle? (Lesson 6-3)



GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.**

46. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
47. $\frac{4 \cdot 3}{2 \cdot 1}$
48. $\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$

EXTEND 11-6

Algebra Lab
Fractals

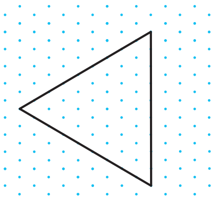
Fractals are sets of points that often involve intricate geometric shapes. Many fractals have the property that when small parts are magnified, the detail of the fractal is not lost. In other words, the magnified part is made up of smaller copies of itself. Such fractals can be constructed recursively.

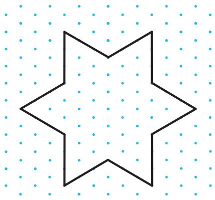
You can use isometric dot paper to draw stages of the construction of a fractal called the *von Koch snowflake*.

ACTIVITY

Stage 1 Draw an equilateral triangle with sides of length 9 units on the dot paper.

Stage 2 Now remove the middle third of each side of the triangle from Stage 1 and draw the other two sides of an equilateral triangle pointing outward.





Imagine continuing this process infinitely. The von Koch snowflake is the shape that these stages approach.

Model and Analyze the Results

1. Copy and complete the table. Draw stage 3, if necessary.

Stage	1	2	3	4
Number of Segments	3	8		
Length of each Segment	9	3		
Perimeter	27	36		

2. Write recursive formulas for the number s_n of segments in Stage n, the length ℓ_n of each segment in Stage n, and the perimeter P_n of Stage n.

3. Write nonrecursive formulas for s_n, ℓ_n, and P_n.

4. What is the perimeter of the von Koch snowflake? Explain.

5. Explain why the area of the von Koch snowflake can be represented by the infinite series $\frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + 3\sqrt{3} + \frac{4\sqrt{3}}{3} + \ldots$.

6. Find the sum of the series in Exercise 5. Explain your steps.

7. Do you think the results of Exercises 4 and 6 are contradictory? Explain.

11-7 The Binomial Theorem

Main Ideas

- Use Pascal's triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.

New Vocabulary

Pascal's triangle
Binomial Theorem
factorial

GET READY for the Lesson

According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. These sequences are listed below.

BBGG BGBG BGGB GBBG GBGB GGBB

Pascal's Triangle You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Expand a few powers of the binomial $b + g$.

$$\begin{aligned}
(b+g)^0 &= 1b^0g^0 \\
(b+g)^1 &= 1b^1g^0 + 1b^0g^1 \\
(b+g)^2 &= 1b^2g^0 + 2b^1g^1 + 1b^0g^2 \\
(b+g)^3 &= 1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3 \\
(b+g)^4 &= 1b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4
\end{aligned}$$

The coefficient 4 of the b^1g^3 term in the expansion of $(b + g)^4$ gives the number of sequences of births that result in one boy and three girls.

Here are some patterns in any binomial expansion of the form $(a + b)^n$.

1. There are $n + 1$ terms.
2. The exponent n of $(a + b)^n$ is the exponent of a in the first term and the exponent of b in the last term.
3. In successive terms, the exponent of a decreases by one, and the exponent of b increases by one.
4. The sum of the exponents in each term is n.
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

Real-World Link Although he did not discover it, Pascal's triangle is named for the French mathematician Blaise Pascal (1623–1662).

The coefficients form a pattern that is often displayed in a triangular formation. This is known as **Pascal's triangle**. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.

$$\begin{array}{lccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

EXAMPLE Use Pascal's Triangle

1 **Expand $(x + y)^7$.**

Write two more rows of Pascal's triangle. Then use the patterns of a binomial expansion and the coefficients to write the expansion.

$$\begin{array}{ccccccccccccccc} & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

$$(x + y)^7 = 1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7$$
$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

CHECK Your Progress

1. Expand $(c + d)^8$.

The Binomial Theorem Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

$$\begin{array}{ll} (a+b)^0 & 1 \\ (a+b)^1 & 1 \quad \frac{1}{1} \\ (a+b)^2 & 1 \quad \frac{2}{1} \quad \frac{2 \cdot 1}{1 \cdot 2} \\ (a+b)^3 & 1 \quad \frac{3}{1} \quad \frac{3 \cdot 2}{1 \cdot 2} \quad \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} \\ (a+b)^4 & 1 \quad \frac{4}{1} \quad \frac{4 \cdot 3}{1 \cdot 2} \quad \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \quad \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \end{array}$$

Eliminate common factors that are shown in color.

This pattern is summarized in the **Binomial Theorem**.

KEY CONCEPT *Binomial Theorem*

If n is a nonnegative integer, then $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \ldots + 1a^0b^n$.

Study Tip

Terms

The expansion of a binomial to the nth power has $n + 1$ terms. For example, $(a - b)^6$ has 7 terms.

EXAMPLE Use the Binomial Theorem

2 **Expand $(a - b)^6$.**

Use the sequence $1, \frac{6}{1}, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

$$(a - b)^6 = 1a^6(-b)^0 + \frac{6}{1}a^5(-b)^1 + \frac{6 \cdot 5}{1 \cdot 2}a^4(-b)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3(-b)^3 + \ldots$$
$$+ 1a^0(-b)^6$$
$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

Study Tip

Coefficients

Notice that in terms having the same coefficients, the exponents are reversed, as in $15a^4b^2$ and $15a^2b^4$.

CHECK Your Progress

2. Expand $(w + z)^5$.

Study Tip

Graphing Calculators

On a TI-83/84 Plus, the factorial symbol, !, is located on the MATH PRB menu.

The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product $4 \cdot 3 \cdot 2 \cdot 1$ is written 4! and is read *4 factorial*. In general, if n is a positive integer, then $n! = n(n-1)(n-2)(n-3) \ldots 2 \cdot 1$. *By definition, 0! = 1.*

EXAMPLE Factorials

3 **Evaluate $\frac{8!}{3!5!}$.**

$$\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}^{\,1}}{3 \cdot 2 \cdot 1 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\,1}}$$

Note that $8! = 8 \cdot 7 \cdot 6 \cdot 5!$, so $\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3!5!}$ or $\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$.

$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} \text{ or } 56$$

CHECK Your Progress

3. Evaluate $\frac{12!}{8!4!}$.

Study Tip

Missing Steps

If you don't understand a step like $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!}$, work it out on a piece of scrap paper.

$$\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{1 \cdot 2 \cdot 3 \cdot 3!} = \frac{6!}{3!3!}$$

The Binomial Theorem can be written in factorial notation and in sigma notation.

KEY CONCEPT *Binomial Theorem, Factorial Form*

$$(a+b)^n = \frac{n!}{n!0!}a^n b^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \ldots + \frac{n!}{0!n!}a^0 b^n$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}a^{n-k}b^k$$

EXAMPLE Use a Factorial Form of the Binomial Theorem

4 **Expand $(2x+y)^5$.**

$$(2x+y)^5 = \sum_{k=0}^{5} \frac{5!}{(5-k)!k!}(2x)^{5-k}y^k \quad \text{Binomial Theorem, factorial form}$$

$$= \frac{5!}{5!0!}(2x)^5y^0 + \frac{5!}{4!1!}(2x)^4y^1 + \frac{5!}{3!2!}(2x)^3y^2 + \frac{5!}{2!3!}(2x)^2y^3 + \frac{5!}{1!4!}(2x)^1y^4 + \frac{5!}{0!5!}(2x)^0y^5 \quad \text{Let } k = 0, 1, 2, 3, 4, \text{ and } 5.$$

$$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^5 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^4y + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}(2x)^3y^2 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}(2x)^2y^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}(2x)y^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}y^5$$

$$= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5 \quad \text{Simplify.}$$

CHECK Your Progress

4. Expand $(q - 3r)^4$.

Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, $k = 0$ for the first term, $k = 1$ for the second term, and so on. In general, the value of k is always one less than the number of the term you are finding.

Personal Tutor

EXAMPLE 5 Find a Particular Term

Find the fifth term in the expansion of $(p + q)^{10}$.

First, use the Binomial Theorem to write the expansion in sigma notation.

$$(p+q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10-k)!k!} p^{10-k} q^k$$

In the fifth term, $k = 4$.

$$\frac{10!}{(10-k)!k!} p^{10-k} q^k = \frac{10!}{(10-4)!4!} p^{10-4} q^4 \qquad k = 4$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} p^6 q^4 \qquad \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!4!} \text{ or } \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 210p^6q^4 \qquad \text{Simplify.}$$

CHECK Your Progress

5. Find the eighth term in the expansion of $(x - y)^{12}$.

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CHECK Your Understanding

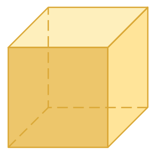
Examples 1, 2, 4 (pp. 665, 666)

Expand each power.

1. $(p + q)^5$ **2.** $(t + 2)^6$ **3.** $(x - 3y)^4$

Examples 2, 4 (pp. 665, 666)

4. GEOMETRY Write an expanded expression for the volume of the cube at the right.



3x + 2 cm

Example 3 (p. 666)

Evaluate each expression.

5. $8!$ **6.** $10!$

7. $\frac{13!}{9!}$ **8.** $\frac{12!}{2!10!}$

Example 5 (p. 667)

Find the indicated term of each expansion.

9. fourth term of $(a + b)^8$ **10.** fifth term of $(2a + 3b)^{10}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–16	1, 2, 4
17–20	3
21–26	5

Expand each power.

11. $(a - b)^3$ **12.** $(m + n)^4$ **13.** $(r + s)^8$

14. $(m - a)^5$ **15.** $(x + 3)^5$ **16.** $(a - 2)^4$

Evaluate each expression.

17. $9!$ **18.** $13!$ **19.** $\frac{9!}{7!}$ **20.** $\frac{7!}{4!}$

Find the indicated term of each expansion.

21. sixth term of $(x - y)^9$

22. seventh term of $(x + y)^{12}$

23. fourth term of $(x + 2)^7$

24. fifth term of $(a - 3)^8$

25. SCHOOL Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses?

Cross-Curricular Project

Math Online Pascal's triangle displays many patterns. Visit algebra2.com to continue work on your project.

26. INTRAMURALS Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two?

Expand each power.

27. $(2b - x)^4$

28. $(2a + b)^6$

29. $(3x - 2y)^5$

30. $(3x + 2y)^4$

31. $\left(\frac{a}{2} + 2\right)^5$

32. $\left(3 + \frac{m}{3}\right)^5$

Evaluate each expression.

33. $\frac{12!}{8!4!}$

34. $\frac{14!}{5!9!}$

Find the indicated term of each expansion.

35. fifth term of $(2a + 3b)^{10}$

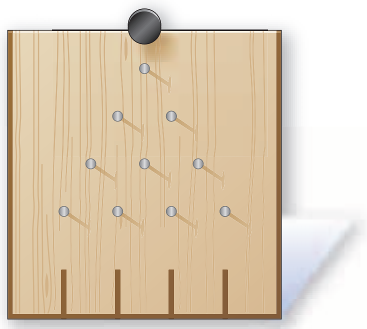
36. fourth term of $(2x + 3y)^9$

37. fourth term of $\left(x + \frac{1}{3}\right)^7$

38. sixth term of $\left(x - \frac{1}{2}\right)^{10}$

39. GENETICS The color of a particular flower may be either red, white, or pink. If the flower has two red alleles R, the flower is red. If the flower has two white alleles w, the flower is white. If the flower has one allele of each color, the flower will be pink. In a lab, two pink flowers are mated and eventually produce 1000 offspring. How many of the 1000 offspring will be pink?

40. GAMES The diagram shows the board for a game in which disks are dropped down a chute. A pattern of nails and dividers causes the disks to take various paths to the sections at the bottom. How many paths through the board lead to each bottom section?



EXTRA PRACTICE

See pages 916, 936.

Math Online

Self-Check Quiz at algebra2.com

H.O.T. Problems

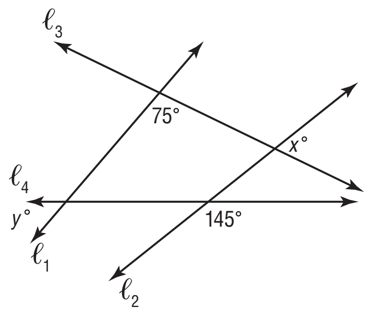
41. OPEN ENDED Write a power of a binomial for which the first term of the expansion is $625x^4$.

42. CHALLENGE Explain why $\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}$ without finding the value of any of the expressions.

43. *Writing in Math* Use the information on page 664 to explain how the power of a binomial describes the number of boys and girls in a family. Include the expansion of $(b + g)^5$ and an explanation of what it tells you about sequences of births of boys and girls in families with five children.

STANDARDIZED TEST PRACTICE

44. ACT/SAT If four lines intersect as shown, what is the value of $x + y$?



A 70 **B** 115 **C** 140 **D** 220

45. REVIEW $(2x - 2)^4 =$

F $16x^4 + 64x^3 - 96x^2 - 64x + 16$

G $16x^4 - 32x^3 - 192x^2 - 64x + 16$

H $16x^4 - 64x^3 + 96x^2 - 64x + 16$

J $16x^4 + 32x^3 - 192x^2 - 64x + 16$

Spiral Review

Find the first five terms of each sequence. (Lesson 11-6)

46. $a_1 = 7, a_{n+1} = a_n - 2$

47. $a_1 = 3, a_{n+1} = 2a_n - 1$

48. MINIATURE GOLF A wooden pole swings back and forth over the cup on a miniature golf hole. One player pulls the pole to the side and lets it go. Then it follows a swing pattern of 25 centimeters, 20 centimeters, 16 centimeters, and so on until it comes to rest. What is the total distance the pole swings before coming to rest? (Lesson 11-5)

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. (Lesson 10-6)

49. $x^2 - 6x - y^2 - 3 = 0$

50. $4y - x + y^2 = 1$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 9-4)

51. $\log_2 5$

52. $\log_3 10$

53. $\log_5 8$

Determine any vertical asymptotes and holes in the graph of each rational function. (Lesson 8-3)

54. $f(x) = \dfrac{1}{x^2 + 5x + 6}$

55. $f(x) = \dfrac{x + 2}{x^2 + 3x - 4}$

56. $f(x) = \dfrac{x^2 + 4x + 3}{x + 3}$

GET READY for the Next Lesson

PREREQUISITE SKILL **State whether each statement is *true* or *false* when $n = 1$. Explain.** (Lesson 1-1)

57. $1 = \dfrac{n(n + 1)}{2}$

58. $1 = \dfrac{(n + 1)(2n + 1)}{2}$

59. $1 = \dfrac{n^2 (n + 1)^2}{4}$

60. $3^n - 1$ is even.

61. $7^n - 3^n$ is divisible by 4.

62. $2^n - 1$ is prime.

11-8 Proof and Mathematical Induction

Main Ideas

- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

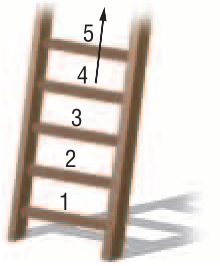
New Vocabulary

mathematical induction

inductive hypothesis

GET READY for the Lesson

Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?



Mathematical Induction **Mathematical induction** is used to prove statements about positive integers. This proof uses three steps.

KEY CONCEPT — *Mathematical Induction*

Step 1 Show that the statement is true for some positive integer n.

Step 2 Assume that the statement is true for some positive integer k, where $k \geq n$. This assumption is called the **inductive hypothesis.**

Step 3 Show that the statement is true for the next positive integer $k + 1$. If so, we can assume that the statement is true for any positive integer.

Study Tip

Step 1

In many cases, it will be helpful to let $n = 1$.

EXAMPLE 1 Summation Formula

Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Step 1 When $n = 1$, the left side of the given equation is 1^2 or 1.

The right side is $\dfrac{1(1+1)[2(1)+1]}{6}$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume $1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$ for a positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$$
\begin{aligned}
&1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{Add } (k+1)^2 \text{ to each side.} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{Add.} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{Factor.} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} && \text{Simplify.} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{Factor.} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}
$$

The last expression is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$. This proves the conjecture.

CHECK Your Progress

1. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Personal Tutor

EXAMPLE 2 Divisibility

Prove that $7^n - 1$ is divisible by 6 for all positive integers n.

Step 1 When $n = 1$, $7^n - 1 = 7^1 - 1$ or 6. Since 6 is divisible by 6, the statement is true for $n = 1$.

Step 2 Assume that $7^k - 1$ is divisible by 6 for some positive integer k. This means that there is a whole number r such that $7^k - 1 = 6r$.

Step 3 Show that the statement is true for $n = k + 1$.

$$
\begin{aligned}
7^k - 1 &= 6r && \text{Inductive hypothesis}\\
7^k &= 6r + 1 && \text{Add 1 to each side.}\\
7(7^k) &= 7(6r + 1) && \text{Multiply each side by 7.}\\
7^{k+1} &= 42r + 7 && \text{Simplify.}\\
7^{k+1} - 1 &= 42r + 6 && \text{Subtract 1 from each side.}\\
7^{k+1} - 1 &= 6(7r + 1) && \text{Factor.}
\end{aligned}
$$

Since r is a whole number, $7r + 1$ is a whole number. Therefore, $7^{k+1} - 1$ is divisible by 6. Thus, the statement is true for $n = k + 1$.

This proves that $7^n - 1$ is divisible by 6 for all positive integers n.

CHECK Your Progress

2. Prove that $10^n - 1$ is divisible by 9 for all positive integers n.

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Review Vocabulary

Counterexample
a specific case that shows that a statement is false (Lesson 1-2)

Counterexamples Of course, not every equation that you can write is true. You can show that an equation is not always true by finding a *counterexample*.

EXAMPLE 3 Counterexample

Find a counterexample for $1^4 + 2^4 + 3^4 + \cdots + n^4 = 1 + (4n - 4)^2$.

n	Left Side of Formula	Right Side of Formula	
1	1^4 or 1	$1 + [4(1) - 4]^2 = 1 + 0^2$ or 1	true
2	$1^4 + 2^4 = 1 + 16$ or 17	$1 + [4(2) - 4]^2 = 1 + 4^2$ or 17	true
3	$1^4 + 2^4 + 3^4 = 1 + 16 + 81$ or 98	$1 + [4(3) - 4]^2 = 1 + 64$ or 65	false

The value $n = 3$ is a counterexample for the equation.

CHECK Your Progress

3. Find a counterexample for the statement that $2n^2 + 11$ is prime for all positive integers n.

CHECK Your Understanding

Example 1 (pp. 670–671)

Prove that each statement is true for all positive integers.

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

2. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

3. PARTIES Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after n guests have arrived, a total of $\frac{n(n-1)}{2}$ handshakes have taken place.

Example 2 (p. 671)

Prove that each statement is true for all positive integers.

4. $4^n - 1$ is divisible by 3.

5. $5^n + 3$ is divisible by 4.

Example 3 (p. 672)

Find a counterexample for each statement.

6. $1 + 2 + 3 + \cdots + n = n^2$

7. $2^n + 3^n$ is divisible by 4.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8–11	1
12, 13	2
14, 15	1, 2
16–21	3

Prove that each statement is true for all positive integers.

8. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

9. $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n+1)}{2}$

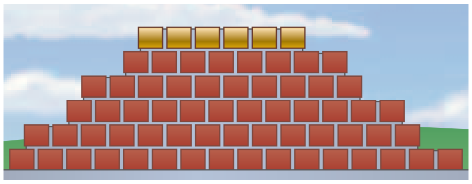
10. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

11. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

12. $8^n - 1$ is divisible by 7.

13. $9^n - 1$ is divisible by 8.

14. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top n rows is $n^2 + 5n$.



15. NATURE The terms of the Fibonacci sequence are found in many places in nature. The number of spirals of seeds in sunflowers are Fibonacci numbers, as are the number of spirals of scales on a pinecone. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ... Each element after the first two is found by adding the previous two terms. If f_n stands for the nth Fibonacci number, prove that $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$.

Find a counterexample for each statement.

16. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(3n-1)}{2}$

17. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = 12n^3 - 23n^2 + 12n$

18. $3^n + 1$ is divisible by 4.

19. $2^n + 2n^2$ is divisible by 4.

20. $n^2 - n + 11$ is prime.

21. $n^2 + n + 41$ is prime.

Prove that each statement is true for all positive integers.

22. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

23. $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$

24. $12^n + 10$ is divisible by 11.

25. $13^n + 11$ is divisible by 12.

26. ARITHMETIC SERIES Use mathematical induction to prove the formula $a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d] = \frac{n}{2}[2a_1 + (n-1)d]$ for the sum of an arithmetic series.

27. GEOMETRIC SERIES Use mathematical induction to prove the formula $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}$ for the sum of a finite geometric series.

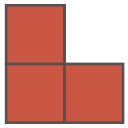
EXTRA PRACTICE

See pages 916, 936.

Math Online

Self-Check Quiz at algebra2.com

28. PUZZLES Show that a 2^n by 2^n checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right.



H.O.T. Problems

29. OPEN ENDED Write an expression of the form $b^n - 1$ that is divisible by 2 for all positive integers n.

30. CHALLENGE Refer to Example 2. Explain how to use the Binomial Theorem to show that $7^n - 1$ is divisible by 6 for all positive integers n.

31. *Writing in Math* Use the information on page 670 to explain how the concept of a ladder can help you prove statements about numbers.

STANDARDIZED TEST PRACTICE

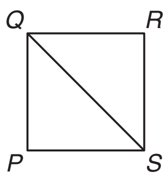
32. ACT/SAT $PQRS$ is a square. What is the ratio of the length of diagonal $\overline{QS}$ to the length of side $\overline{RS}$?

A 2

B $\sqrt{2}$

C 1

D $\frac{\sqrt{2}}{2}$



33. REVIEW The lengths of the bases of an isosceles trapezoid are 15 centimeters and 29 centimeters. If the perimeter of this trapezoid is 94 centimeters, what is the area?

F 500 cm^2

G 515 cm^2

H 528 cm^2

J 550 cm^2

Spiral Review

Expand each power. (Lesson 11-7)

34. $(x + y)^6$

35. $(a - b)^7$

36. $(2x + y)^8$

Find the first three iterates of each function for the given initial value. (Lesson 11-6)

37. $f(x) = 3x - 2, x_0 = 2$

38. $f(x) = 4x^2 - 2, x_0 = 1$

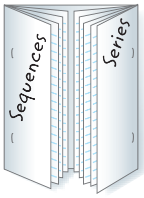
39. BIOLOGY Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas? (Lesson 9-2)

CHAPTER 11 Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES™ Study Organizer GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Arithmetic Sequences and Series (Lessons 11-1 and 11-2)

- The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is any positive integer.
- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Geometric Sequences and Series (Lessons 11-3 to 11-5)

- The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$, where n is any positive integer.
- The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1r^n}{1 - r}$, where $r \neq 1$.
- The sum S of an infinite geometric series with $-1 < r < 1$ is given by $S = \frac{a_1}{1 - r}$.

Recursion and Special Sequences (Lesson 11-6)

- In a recursive formula, each term is formulated from one or more previous terms.

The Binomial Theorem (Lesson 11-7)

- The Binomial Theorem:

$$(a + b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} b^k$$

Mathematical Induction (Lesson 11-8)

- Mathematical induction is a method of proof used to prove statements about the positive integers.

Key Vocabulary

arithmetic means (p. 624)
arithmetic sequence (p. 622)
arithmetic series (p. 629)
Binomial Theorem (p. 665)
common difference (p. 622)
common ratio (p. 636)
convergent series (p. 651)
factorial (p. 666)
Fibonacci sequence (p. 658)
geometric means (p. 638)
geometric sequence (p. 636)
geometric series (p. 643)
index of summation (p. 631)
inductive hypothesis (p. 670)
infinite geometric series (p. 650)
iteration (p. 660)
mathematical induction (p. 670)
partial sum (p. 650)
Pascal's triangle (p. 664)
recursive formula (p. 658)
sequence (p. 622)
series (p. 629)
sigma notation (p. 631)
term (p. 622)

Vocabulary Check

Choose the term from the list above that best completes each statement.

1. A(n) __________ of an infinite series is the sum of a certain number of terms.
2. If a sequence has a common ratio, then it is a(n) __________ .
3. Using __________ , the series 2 + 5 + 8 + 11 + 14 can be written as $\sum_{n=1}^{5} (3n - 1)$.
4. Eleven and 17 are two __________ between 5 and 23 in the sequence 5, 11, 17, 23.
5. Using the __________ , $(a - 2)^4$ can be expanded to $a^4 - 8a^3 + 24a^2 - 32a + 16$.
6. The __________ of the sequence 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$ is $\frac{2}{3}$.
7. The __________ 11 + 16.5 + 22 + 27.5 + 33 has a sum of 110.
8. A(n) __________ is expressed as $n! = n(n - 1)(n - 2) \ldots 2 \cdot 1$.

Lesson-by-Lesson Review

11–1 Arithmetic Sequences (pp. 622-628)

Find the indicated term of each arithmetic sequence.

9. $a_1 = 6, d = 8, n = 5$

10. $a_1 = -5, d = 7, n = 22$

11. $a_1 = 5, d = -2, n = 9$

12. $a_1 = -2, d = -3, n = 15$

Find the arithmetic means in each sequence.

13. -7, ____ , ____ , ____ , 9

14. 12, ____ , ____ , 4

15. 9, ____ , ____ , ____ , ____ , -6

16. 56, ____ , ____ , ____ , 28

17. **GLACIERS** The fastest glacier is recorded to have moved 12 kilometers every three months. If the glacier moved at a constant speed, how many kilometers did it move in one year?

Example 1 Find the 12th term of an arithmetic sequence if $a_1 = -17$ and $d = 4$.

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$a_{12} = -17 + (12 - 1)4$ $n = 12, a_1 = -17, d = 4$

$a_{12} = 27$ Simplify.

Example 2 Find the two arithmetic means between 4 and 25.

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$a_4 = 4 + (4 - 1)d$ $n = 4, a_1 = 4$

$25 = 4 + 3d$ $a_4 = 25$

$7 = d$ Simplify.

The arithmetic means are $4 + 7$ or 11 and $11 + 7$ or 18.

11–2 Arithmetic Series (pp. 629-635)

Find S_n for each arithmetic series.

18. $a_1 = 12, a_n = 117, n = 36$

19. $4 + 10 + 16 + \dots + 106$

20. $10 + 4 + (-2) + \dots + (-50)$

21. Evaluate $\sum_{n=2}^{13} (3n + 1)$.

22. **PATTERNS** On the first night of a celebration, a candle is lit and then blown out. The second night, a new candle and the candle from the previous night are lit and blown out. This pattern of lighting a new candle and all the candles from the previous nights is continued for seven nights. Find the total number of candle lightings.

Example 3 Find S_n for the arithmetic series with $a_1 = 34$, $a_n = 2$, and $n = 9$.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_9 = \frac{9}{2}(34 + 2)$ $n = 9, a_1 = 34, a_n = 2$

$= 162$ Simplify.

Example 4 Evaluate $\sum_{n=5}^{11} (2n - 3)$.

Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. There are 7 terms, $a_1 = 2(5) - 3$ or 7, and $a_7 = 2(11) - 3$ or 19.

$S_7 = \frac{7}{2}(19 + 7)$

$= 91$

11-3 Geometric Sequences (pp. 636-641)

Find the indicated term of each geometric sequence.

23. $a_1 = 2, r = 2, n = 5$

24. $a_1 = 7, r = 2, n = 4$

25. $a_1 = 243, r = -\frac{1}{3}, n = 5$

26. a_6 for $\frac{2}{3}, \frac{4}{3}, \frac{8}{3} \ldots$

Find the geometric means in each sequence.

27. 3, ____ , ____ , 24

28. 7.5, ____ , ____ , ____ , 120

29. SAVINGS Kathy has a savings account with a current balance of $5000. What would Kathy's account balance be after five years if she receives 3% interest annually?

Example 5 **Find the fifth term of a geometric sequence for which $a_1 = 7$ and $r = 3$.**

$a_n = a_1 \cdot r^{n-1}$ Formula for the *n*th term

$a_5 = 7 \cdot 3^{5-1}$ $n = 5, a_1 = 7, r = 3.$

$a_5 = 567$ The fifth term is 567.

Example 6 **Find two geometric means between 1 and 8.**

$a_n = a_1 \cdot r^{n-1}$ Formula for the *n*th term

$a_4 = 1 \cdot r^{4-1}$ $n = 4$ and $a_1 = 1$

$8 = r^3$ $a_4 = 8$

$2 = r$ Simplify.

The geometric means are 1(2) or 2 and 2(2) or 4.

11-4 Geometric Series (pp. 643-649)

Find S_n for each geometric series.

30. $a_1 = 12, r = 3, n = 5$

31. $4 - 2 + 1 - \ldots$ to 6 terms

32. $256 + 192 + 144 + \ldots$ to 7 terms

33. Evaluate $\sum_{n=1}^{5} \left(-\frac{1}{2}\right)^{n-1}$.

34. TELEPHONES Joe started a phone tree to give information about a party to his friends. Joe starts by calling 3 people. Then each of those 3 people calls 3 people, and each person who receives a call then calls 3 more people. How many people have been called after 4 layers of the phone tree? (*Hint:* Joe is considered the first layer.)

Example 7 **Find the sum of a geometric series for which $a_1 = 7$, $r = 3$, and $n = 14$.**

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ Sum formula

$S_{14} = \frac{7 - 7 \cdot 3^{14}}{1 - 3}$ $n = 14, a_1 = 7, r = 3$

$S_{14} = 16{,}740{,}388$ Use a calculator.

Example 8 **Evaluate $\sum_{n=1}^{5} \left(\frac{3}{4}\right)^{n-1}$.**

$S_5 = \frac{1\left[1 - \left(\frac{3}{4}\right)^5\right]}{1 - \frac{3}{4}}$ $n = 5, a_1 = 1, r = \frac{3}{4}$

$= \frac{\frac{781}{1024}}{\frac{1}{4}}$ $\frac{3}{4}^5 = \frac{243}{1024}$

$= \frac{781}{256}$

Mixed Problem Solving
For mixed problem-solving practice, see page 936.

11-5 Infinite Geometric Series (pp. 650-655)

Find the sum of each infinite geometric series, if it exists.

35. $a_1 = 6, r = \frac{11}{12}$

36. $\frac{1}{8} - \frac{3}{16} + \frac{9}{32} - \frac{27}{64} + \cdots$

37. $\sum_{n=1}^{\infty} -2\left(-\frac{5}{8}\right)^{n-1}$

38. GEOMETRY If the midpoints of the sides of $\triangle ABC$ are connected, a smaller triangle results. Suppose the process of connecting midpoints of sides and drawing new triangles is continued indefinitely. Find the sum of the perimeters of all of the triangles if the perimeter of $\triangle ABC$ is 30 centimeters.

Example 9 **Find the sum of the infinite geometric series for which $a_1 = 18$ and $r = -\frac{2}{7}$.**

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{18}{1 - \left(-\frac{2}{7}\right)}$ $a_1 = 18, r = -\frac{2}{7}$

$= \frac{18}{\frac{9}{7}}$ or 14 Simplify.

11-6 Recursion and Special Sequences (pp. 658-662)

Find the first five terms of each sequence.

39. $a_1 = -2, a_{n+1} = a_n + 5$

40. $a_1 = 3, a_{n+1} = 4a_n - 10$

Find the first three iterates of each function for the given initial value.

41. $f(x) = -2x + 3, x_0 = 1$

42. $f(x) = 7x - 4, x_0 = 2$

43. SAVINGS Toni has a savings account with a \$15,000 balance. She has a 4% interest rate that is compounded monthly. Every month Toni makes a \$1000 withdrawal from the account to cover her expenses. The recursive formula $b_n = 1.04b_{n-1} - 1000$ describes the balance in Toni's savings account after n months. Find the balance of Toni's account after the first four months. Round your answer to the nearest dollar.

Example 10 **Find the first five terms of the sequence in which $a_1 = 2$, $a_{n+1} = 2a_n - 1$.**

$a_{n+1} = 2a_n - 1$	Recursive formula
$a_{1+1} = 2a_1 - 1$	$n = 1$
$a_2 = 2(2) - 1$ or 3	$a_1 = 2$
$a_{2+1} = 2a_2 - 1$	$n = 2$
$a_3 = 2(3) - 1$ or 5	$a_2 = 3$
$a_{3+1} = 2a_3 - 1$	$n = 3$
$a_4 = 2(5) - 1$ or 9	$a_3 = 5$
$a_{4+1} = 2a_4 - 1$	$n = 4$
$a_5 = 2(9) - 1$ or 17	$a_4 = 9$

The first five terms of the sequence are 2, 3, 5, 9, and 17.

CHAPTER 11

Study Guide and Review

11-7 The Binomial Theorem

(pp. 664-669)

Expand each power.

44. $(x - 2)^4$

45. $(3r + s)^5$

Find each indicated term of each expansion.

46. fourth term of $(x + 2y)^6$

47. second term of $(4x - 5)^{10}$

48. **SCHOOL** Mr. Brown is giving a four-question multiple-choice quiz. Each question can be answered A, B, C, or D. How many ways could a student answer the questions using each answer A, B, C, or D once?

Example 11 **Expand $(a - 2b)^4$.**

$$(a - 2b)^4$$

$$= \sum_{k=0}^{4} \frac{4!}{(4-k)!k!} a^{4-k}(-2b)^k$$

$$= \frac{4!}{4!0!}a^4(-2b)^0 + \frac{4!}{3!1!}a^3(-2b)^1 + \frac{4!}{2!2!}a^2(-2b)^2 + \frac{4!}{1!3!}a^1(-2b)^3 + \frac{4!}{0!4!}a^0(-2b)^4$$

$$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$$

11-8 Proof and Mathematical Induction

(pp. 670-674)

Prove that each statement is true for all positive integers.

49. $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

50. $6^n - 1$ is divisible by 5.

51. $3^n - 1$ is divisible by 2.

52. $1 + 4 + 7 + \ldots (3n - 2) = \frac{n(3n-1)}{2}$

Find a counterexample for each statement.

53. $n^2 - n + 13$ is prime.

54. $13^n + 11$ is divisible by 24.

55. $9^{n+1} - 1$ is divisible by 16.

56. $n^2 + n + 1$ is prime.

Example 12 **Prove that $1 + 5 + 25 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for positive integers n.**

Step 1 When $n = 1$, the left side of the given equation is 1. The right side is $\frac{1}{4}(5^1 - 1)$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume that $1 + 5 + 25 + \cdots + 5^{k-1} = \frac{1}{4}(5^k - 1)$ for some positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$1 + 5 + 25 + \ldots + 5^{k-1} + 5^{(k+1)-1}$

$= \frac{1}{4}(5^k - 1) + 5^{(k+1)-1}$ — Add to each side.

$= \frac{1}{4}(5^k - 1) + 5^k$ — Simplify the exponent.

$= \frac{5^k - 1 + 4 \cdot 5^k}{4}$ — Common denominator

$= \frac{5 \cdot 5^k - 1}{4}$ — Distributive Property

$= \frac{1}{4}(5^{k+1} - 1)$ — $5^k = 5^{k+1}$

Thus, the equation is true for $n = k + 1$. The conjecture is proved.

CHAPTER 11

Practice Test

1. Find the next four terms of the arithmetic sequence 42, 37, 32, … .
2. Find the 27th term of an arithmetic sequence for which $a_1 = 2$ and $d = 6$.

3. **MULTIPLE CHOICE** What is the tenth term in the arithmetic sequence that begins 10, 5.6, 1.2, −3.2, … ?

   A −39.6

   B −29.6

   C 29.6

   D 39.6

4. Find the three arithmetic means between −4 and 16.
5. Find the sum of the arithmetic series for which $a_1 = 7$, $n = 31$, and $a_n = 127$.
6. Find the next two terms of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$.
7. Find the sixth term of the geometric sequence for which $a_1 = 5$ and $r = -2$.

8. **MULTIPLE CHOICE** Find the next term in the geometric sequence $8, 6, \frac{9}{2}, \frac{27}{8}, \ldots$.

   F $\frac{11}{8}$

   G $\frac{27}{16}$

   H $\frac{9}{4}$

   J $\frac{81}{32}$

9. Find the two geometric means between 7 and 189.

10. Find the sum of the geometric series for which $a_1 = 125$, $r = \frac{2}{5}$, and $n = 4$.

Find the sum of each series, if it exists.

11. $\sum_{k=3}^{15} (14 - 2k)$

12. $\sum_{n=1}^{\infty} \frac{1}{3}(-2)^{n-1}$

13. $91 + 85 + 79 + \cdots + (-29)$

14. $12 + (-6) + 3 + \left(-\frac{3}{2}\right) + \cdots$

Find the first five terms of each sequence.

15. $a_1 = 1, a_{n+1} = a_n + 3$
16. $a_1 = -3, a_{n+1} = a_n + n^2$

17. Find the first three iterates of $f(x) = x^2 - 3x$ for an initial value of $x_0 = 1$.
18. Expand $(2s - 3t)^5$.
19. What is the coefficient of the fifth term of $(r + 2q)^7$?
20. Find the third term of the expansion of $(x + y)^{10}$.

Prove that each statement is true for all positive integers.

21. $1 + 7 + 49 + \cdots + 7^{n-1} = \frac{1}{6}(7^n - 1)$
22. $14^n - 1$ is divisible by 13.

23. Find a counterexample for the following statement.
    The units digit of $7^n - 3$ is never 8.

24. **DESIGN** The pattern in a red and white brick wall starts with 20 red bricks on the bottom row. Each row contains 3 fewer red bricks than the row below. If the top row has no red bricks, how many rows are there and how many red bricks were used?

25. **RECREATION** One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?

CHAPTER 11

Standardized Test Practice

Cumulative, Chapters 1–11

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. How many 3-inch cubes can be placed completely inside a box that is 15 inches long, 12 inches wide, and 18 inches tall?

A 5

B 20

C 120

D 360

2. Using the table below, which expression can be used to determine the nth term of the sequence?

n	y
1	6
2	10
3	14
4	18

F $y = 6n$

G $y = n + 5$

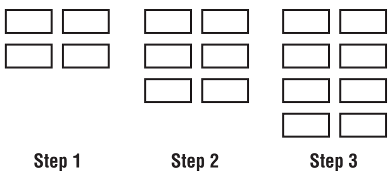
H $y = 2n + 1$

J $y = 2(2n + 1)$

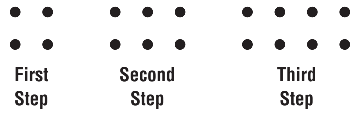
TEST-TAKING TIP

Question 2 Sometimes sketching the graph of a function can help you to see the relationship between n and y and answer the question.

3. GRIDDABLE The pattern of squares below continues infinitely, with more squares being added at each step. How many squares are in the tenth step?



4. The pattern of dots shown below continues infinitely, with more dots being added at each step.



Which expression can be used to determine the number of dots in the nth step?

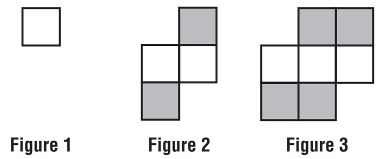
A $2n$

B $n(n + 2)$

C $n(n + 1)$

D $2(n + 1)$

5. The figures below show a pattern of dark tiles and white tiles that can be described by a relationship between two variables.



Which rule relates d, the number of dark tiles, to w, the number of white tiles?

F $d = 2w$

G $w = d - 1$

H $d = 2w - 2$

J $w = \frac{1}{2}d + 1$

6. Leland is renting an apartment. He looked at a 3-bedroom apartment for \$950 per month near the downtown area, and a 3-bedroom apartment for \$725 per month on the edge of town. About what percent of the cost of the downtown apartment is Leland saving by renting the apartment on the edge of town?

A 2%

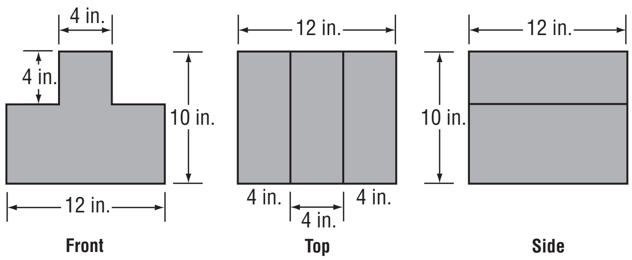
B 24%

C 31%

D 231%

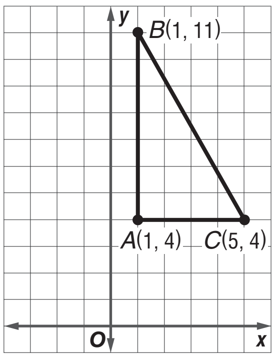
Preparing for Standardized Tests
For test-taking strategies and more practice, see pages 941–956.

7. What is the volume of a 3-dimensional object with the dimensions shown in the 3 views below?



F 864 in^3
G 1056 in^3
H 1248 in^3
J 1440 in^3

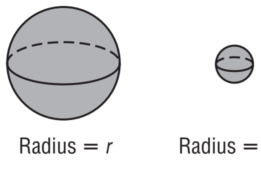
8. $\triangle ABC$ is graphed on the coordinate grid below.



Which set of coordinates represents the vertices of a triangle congruent to $\triangle ABC$?

A (−1, 7), (−1, 15), (3, 8)
B (2, 7), (2, 14), (3, 7)
C (4, 7), (4, 14), (7, 7)
D (−1, 7), (−1, 14), (3, 7)

9. The radius of the larger sphere shown below was multiplied by a factor of $\frac{1}{3}$ to produce the smaller sphere.



How does the volume of the smaller sphere compare to the volume of the larger sphere?

F The volume of the smaller sphere is $\frac{1}{9}$ as large.
G The volume of the smaller sphere is $\frac{1}{\pi^3}$ as large.
H The volume of the smaller sphere is $\frac{1}{27}$ as large.
J The volume of the smaller sphere is $\frac{1}{3}$ as large.

10. GRIDDABLE Marla is putting a binding around a square quilt. The length of the binding was 32 feet. Find the approximate length, in feet, of the diagonal of the square quilt. Round to one decimal place.

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. Kyla's annual salary is $50,000. Each year she gets a 6% raise.

a. To the nearest dollar, what will her salary be in four years?

b. To the nearest dollar, what will her salary be in 10 years?

NEED EXTRA HELP?											
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11
Go to Lesson or Page...	11-7	11-1	11-1	11-1	11-1	750	1-1	754	10-3	10-3	11-3