UNIT 5
Trigonometry

Focus

Trigonometry is used in navigation, physics, and construction, among other fields. In this unit, you will learn about trigonometric functions, graphs, and identities.

CHAPTER 13
Trigonometric Funtions

BIG Idea Understand and apply trigonometry to various problems.

BIG Idea Understand and apply the laws of sines and cosines.

CHAPTER 14
Trigonometric Graphs and Identities

BIG Idea Comprehend and manipulate the trigonometric functions, graphs and identities.

Cross-Curricular Project

Algebra and Physics

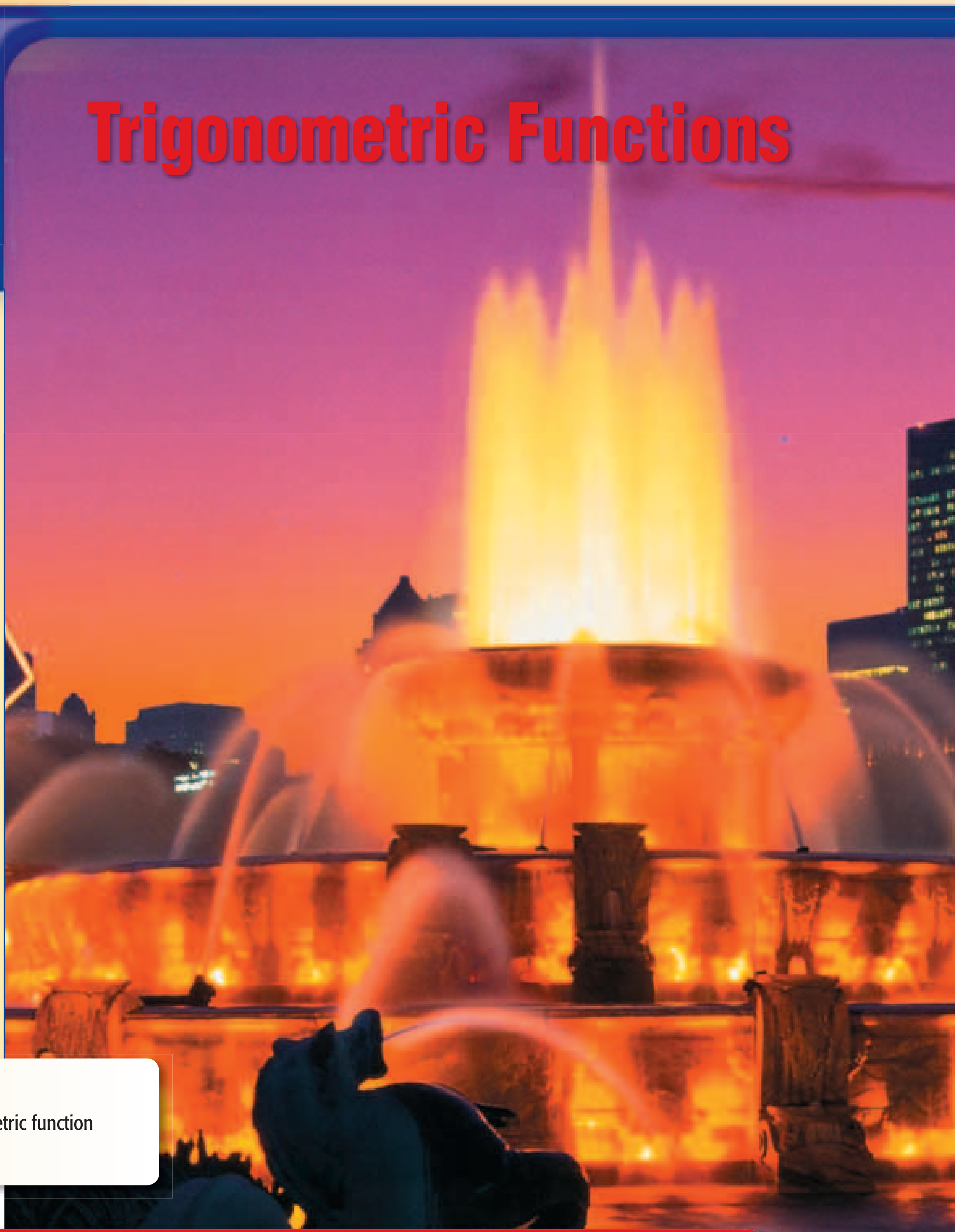
So, you want to be a rocket scientist? Have you ever built and launched a model rocket? If model rockets fascinate you, you may want to consider a career in the aerospace industry, such as aerospace engineering. The National Aeronautics and Space Administration (NASA) employs aerospace engineers and other people with expertise in aerospace fields. In this project, you will research applications of trigonometry as it applies to a possible career for you.

Math Online **Log on to algebra2.com to begin.**

Ed and Chris Kumler

CHAPTER 13

Trigonometric Functions



BIG Ideas

- Find values of trigonometric functions.
- Solve problems by using right triangle trigonometry.
- Solve triangles by using the Law of Sines and Law of Cosines.

Key Vocabulary

solve a right triangle (p. 762)

radian (p. 769)

Law of Sines (p. 786)

Law of Cosines (p. 793)

circular function (p. 800)

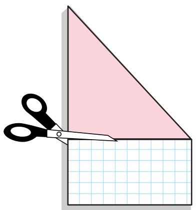
Real-World Link

Buildings Surveyors use a trigonometric function to find the heights of buildings.

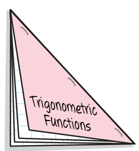
FOLDABLES Study Organizer

Trigonometric Functions Make this Foldable to help you organize your notes. Begin with one sheet of construction paper and two pieces of grid paper.

1. **Stack and Fold** on the diagonal. Cut to form a triangular stack.



2. **Staple** edge to form a book. Label Trigonometric Functions.



Bill Ross/CORBIS

GET READY for Chapter 13

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

Math Online Take the Online Readiness Quiz at algebra2.com.

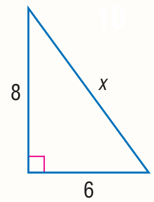
Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKQuiz

Find the value of x to the nearest tenth. (Prerequsite Skills, p. 881)

1.



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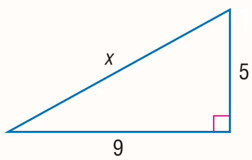

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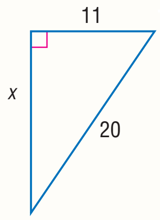

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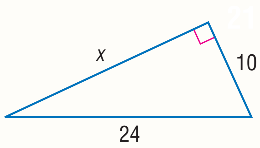
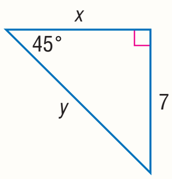


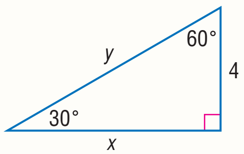
5. LADDER There is a window that is 10 feet high. You want to use a ladder to get up to the window; you decide to put the ladder 3 feet away from the wall. How long should the ladder be? (Prerequsite Skills, p. 881)

Find each missing measure. Write all radicals in simplest form. (Prerequsite Skill)

6.



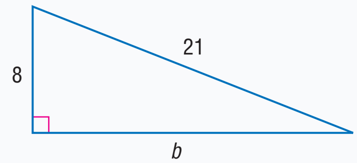
7.



8. KITES A kite is being flown at a 45° angle. The string of the kite is 20 feet long. How high is the kite? (Prerequsite Skill)

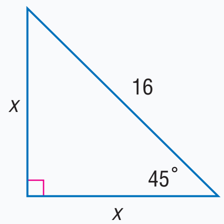
QUICKReview

Example 1 **Find the missing measure of the right triangle.**



$c^2 = a^2 + b^2$	Pythagorean Theorem
$21^2 = 8^2 + b^2$	Replace c with 21 and a with 8.
$441 = 64 + b^2$	Simplify.
$377 = b^2$	Subtract 64 from each side.
$19.4 \approx b$	Take the square root of each side.

Example 2 **Find the missing measures. Write all radicals in simplest form.**

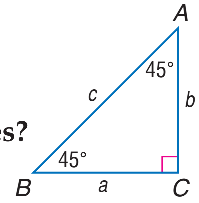


$x^2 + x^2 = 16^2$	Pythagorean Theorem
$2x^2 = 16^2$	Combine like terms.
$2x^2 = 256$	Simplify.
$x^2 = 128$	Divide each side by 2.
$x = \sqrt{128}$	Take the square root of each side.
$x = 8\sqrt{2}$	Simplify.

EXPLORE 13-1

Spreadsheet Lab
Special Right Triangles

ACTIVITY



The legs of a 45°-45°-90° triangle, *a* and *b*, are equal in measure. Use a spreadsheet to investigate the dimensions of 45°-45°-90° triangles. What patterns do you observe in the ratios of the side measures of these triangles?

=SQRT(A2^2+B2^2) =B2/A2 =B2/C2 =A2/C2

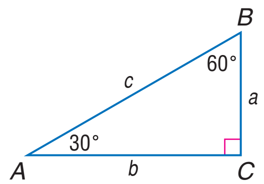
45-45-90 Triangles

◇	A	B	C	D	E	F
1	a	b		a/b	b/c	a/c
2	1	1	1.41421356	1	0.70710678	0.70710678
3	2	2	2.82842712	1	0.70710678	0.70710678
4	3	3	4.24264069	1	0.70710678	0.70710678
5	4	4	5.65685425	1	0.70710678	0.70710678
6	5	5	7.07106781	1	0.70710678	0.70710678
7						

Sheet 1 Sheet 2 Sheet 3

The spreadsheet shows the formula that will calculate the length of side *c*. The formula uses the Pythagorean Theorem in the form $c = \sqrt{a^2 + b^2}$. Since 45°-45°-90° triangles share the same angle measures, the triangles listed in the spreadsheet are all similar triangles. Notice that all of the ratios of side *b* to side *a* are 1. All of the ratios of side *b* to side *c* and of side *a* to side *c* are approximately 0.71.

Model and Analyze



For Exercises 1–3, use the spreadsheet for 30°-60°-90° triangles. If the measure of one leg of a right triangle and the measure of the hypotenuse are in a ratio of 1 to 2, then the acute angles of the triangle measure 30° and 60°.

30-60-90 Triangles

◇	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1		2			
3	2		4			
4	3		6			
5	4		8			
6	5		10			
7						

Sheet 1 Sheet 2 Sheet 3

1. Copy and complete the spreadsheet above.

2. Describe the relationship among the 30°-60°-90° triangles with the dimensions given.

3. What patterns do you observe in the ratios of the side measures of these triangles?

13-1 Right Triangle Trigonometry

Main Ideas

- Find values of trigonometric functions for acute angles.
- Solve problems involving right triangles.

New Vocabulary

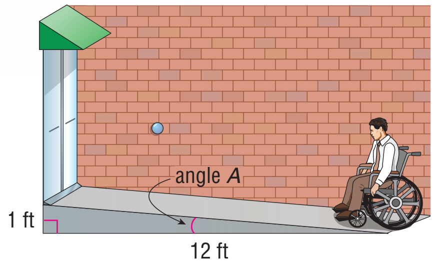
trigonometry
trigonometric functions
sine
cosine
tangent
cosecant
secant
cotangent
solve a right triangle
angle of elevation
angle of depression

Reading Math

Trigonometry
The word *trigonometry* is derived from two Greek words–*trigon* meaning triangle and *metra* meaning measurement.

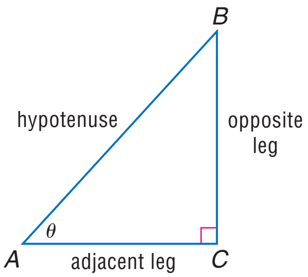
GET READY for the Lesson

The Americans with Disabilities Act (ADA) provides regulations designed to make public buildings accessible to all. Under this act, the slope of an entrance ramp designed for those with mobility disabilities must not exceed a ratio of 1 to 12. This means that for every 12 units of horizontal run, the ramp can rise or fall no more than 1 unit.



When viewed from the side, a ramp forms a right triangle. The slope of the ramp can be described by the *tangent* of the angle the ramp makes with the ground. In this example, the tangent of angle A is $\frac{1}{12}$.

Trigonometric Values The tangent of an angle is one of the ratios used in trigonometry. **Trigonometry** is the study of the relationships among the angles and sides of a right triangle.



Consider right triangle ABC in which the measure of acute angle A is identified by the Greek letter *theta*, θ. The sides of the triangle are the *hypotenuse*, the *leg opposite* θ, and the *leg adjacent to* θ.

Using these sides, you can define six **trigonometric functions: sine, cosine, tangent, cosecant, secant,** and **cotangent**. These functions are abbreviated sin, cos, tan, csc, sec, and cot, respectively.

KEY CONCEPT — *Trigonometric Functions*

If θ is the measure of an acute angle of a right triangle, *opp* is the measure of the leg opposite θ, *adj* is the measure of the leg adjacent to θ, and *hyp* is the measure of the hypotenuse, then the following are true.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Notice that the sine, cosine, and tangent functions are reciprocals of the cosecant, secant, and cotangent functions, respectively. Thus, the following are also true.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Study Tip

Memorize Trigonometric Ratios

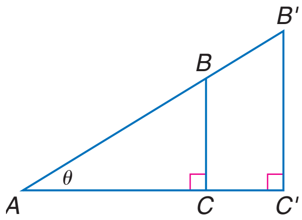
SOH-CAH-TOA is a mnemonic device for remembering the first letter of each word in the ratios for sine, cosine, and tangent.

$\sin \theta = \frac{\text{opp}}{\text{hyp}}$

$\cos \theta = \frac{\text{adj}}{\text{hyp}}$

$\tan \theta = \frac{\text{opp}}{\text{adj}}$

The domain of each of these trigonometric functions is the set of all acute angles θ of a right triangle. The values of the functions depend only on the measure of θ and not on the size of the right triangle. For example, consider $\sin \theta$ in the figure at the right.



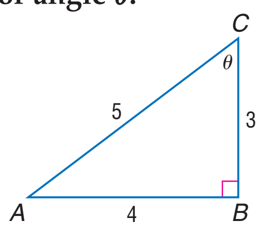
Using $\triangle ABC$: $\sin \theta = \frac{BC}{AB}$

Using $\triangle AB'C'$: $\sin \theta = \frac{B'C'}{AB'}$

The right triangles are similar because they share angle θ. Since they are similar, the ratios of corresponding sides are equal. That is, $\frac{BC}{AB} = \frac{B'C'}{AB'}$. Therefore, you will find the same value for $\sin \theta$ regardless of which triangle you use.

EXAMPLE Find Trigonometric Values

1 Find the values of the six trigonometric functions for angle θ.



For this triangle, the leg opposite θ is $\overline{AB}$, and the leg adjacent to θ is $\overline{CB}$. Recall that the hypotenuse is always the longest side of a right triangle, in this case $\overline{AC}$.

Use opp = 4, adj = 3, and hyp = 5 to write each trigonometric ratio.

$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$ $\quad\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$ $\quad\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$

$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$ $\quad\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$ $\quad\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$

CHECK Your Progress

1. Find the values of the six trigonometric functions for angle A in $\triangle ABC$ above.

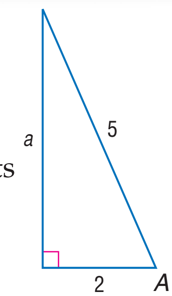
Throughout Unit 5, a capital letter will be used to denote both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase will be used to denote the side opposite that angle and its measure.

STANDARDIZED TEST EXAMPLE Use One Trigonometric Ratio to Find Another

2 If $\cos A = \frac{2}{5}$, find the value of $\tan A$.

A $\frac{5}{2}$ **B** $\frac{2\sqrt{21}}{21}$ **C** $\frac{\sqrt{21}}{2}$ **D** $\sqrt{21}$

Read the Test Item



Begin by drawing a right triangle and labeling one acute angle A. Since $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ and $\cos A = \frac{2}{5}$ in this case, label the adjacent leg 2 and the hypotenuse 5. This represents the simplest triangle for which $\cos A = \frac{2}{5}$.

Test-Taking Tip

Whenever necessary or helpful, draw a diagram of the situation.

Solve the Test Item

Use the Pythagorean Theorem to find a.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 2^2 = 5^2$	Replace b with 2 and c with 5.
$a^2 + 4 = 25$	Simplify.
$a^2 = 21$	Subtract 4 from each side.
$a = \sqrt{21}$	Take the square root of each side.

Now find tan A.

$\tan A = \frac{\text{opp}}{\text{adj}}$	Tangent ratio
$= \frac{\sqrt{21}}{2}$	Replace *opp* with $\sqrt{21}$ and *adj* with 2.

The answer is C.

CHECK Your Progress

2. If $\tan B = \frac{3}{7}$, find the value of $\sin B$.

F $\frac{7}{3}$ **G** $\frac{\sqrt{58}}{3}$ **H** $\frac{3\sqrt{58}}{58}$ **J** $\frac{\sqrt{58}}{7}$

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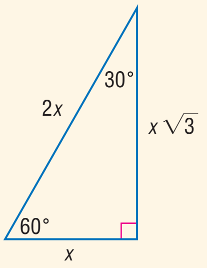
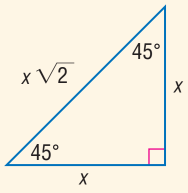
Angles that measure 30°, 45°, and 60° occur frequently in trigonometry. The table below gives the values of the six trigonometric functions for these angles. To remember these values, use the properties of 30°-60°-90° and 45°-45°-90° triangles.

KEY CONCEPT *Trigonometric Values for Special Angles*

30°-60°-90° Triangle



45°-45°-90° Triangle



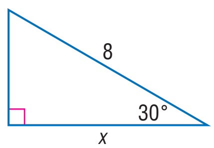
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

You will verify some of these values in Exercises 39 and 40.

Right Triangle Problems You can use trigonometric functions to solve problems involving right triangles.

EXAMPLE 3 Find a Missing Side Length of a Right Triangle

Write an equation involving sin, cos, or tan that can be used to find the value of x. Then solve the equation. Round to the nearest tenth.



The measure of the hypotenuse is 8. The side with the missing length is *adjacent* to the angle measuring 30°. The trigonometric function relating the adjacent side of a right triangle and the hypotenuse is the cosine function.

$$\cos \theta = \frac{\text{adj}}{\text{hyp}}$$ cosine ratio

$$\cos 30° = \frac{x}{8}$$ Replace θ with 30°, *adj* with x, and *hyp* with 8.

$$\frac{\sqrt{3}}{2} = \frac{x}{8}$$ $\cos 30° = \frac{\sqrt{3}}{2}$

$$4\sqrt{3} = x$$ Multiply each side by 8. The value of x is $4\sqrt{3}$ or about 6.9.

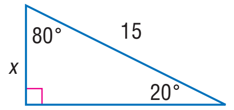
Study Tip

Common Misconception

The $\cos^{-1} x$ on a graphing calculator does *not* find $\frac{1}{\cos x}$. To find $\sec x$ or $\frac{1}{\cos x}$, find $\cos x$ and then use the $\boxed{x^{-1}}$ key.

CHECK Your Progress

3. Write an equation involving sin, cos, or tan that can be used to find the value of x. Then solve the equation. Round to the nearest tenth.



A calculator can be used to find the value of trigonometric functions for *any* angle, not just the special angles mentioned. Use $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$, and $\boxed{\text{TAN}}$ for sine, cosine, and tangent. Use these keys and the reciprocal key, $\boxed{x^{-1}}$, for cosecant, secant, and cotangent. Be sure your calculator is in degree mode.

Here are some calculator examples.

cos 46° **KEYSTROKES:** $\boxed{\text{COS}}$ 46 $\boxed{\text{ENTER}}$ 0.6946583705

cot 20° **KEYSTROKES:** $\boxed{\text{TAN}}$ 20 $\boxed{\text{ENTER}}$ $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$ 2.747477419

If you know the measures of any two sides of a right triangle or the measures of one side and one acute angle, you can determine the measures of all the sides and angles of the triangle. This process of finding the missing measures is known as **solving a right triangle**.

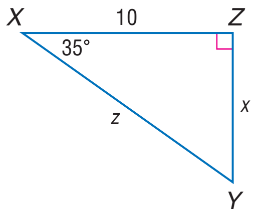
EXAMPLE Solve a Right Triangle

Study Tip

Error in Measurement

The value of z in Example 4 is found using the secant instead of using the Pythagorean Theorem. This is because the secant uses values given in the problem rather than calculated values.

4 **Solve $\triangle XYZ$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**



Find x and z.

$$\tan 35° = \frac{x}{10} \qquad \sec 35° = \frac{z}{10}$$

$$10 \tan 35° = x \qquad \frac{1}{\cos 35°} = \frac{z}{10}$$

$$7.0 \approx x \qquad \frac{1}{\cos 35°} = z$$

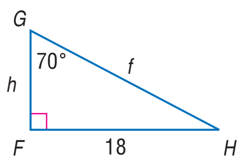
$$12.2 \approx z$$

Find Y. $35° + Y = 90°$ Angles X and Y are complementary.

$Y = 55°$ Therefore, $Y = 55°$, $x \approx 7.0$, and $z \approx 12.2$.

CHECK Your Progress

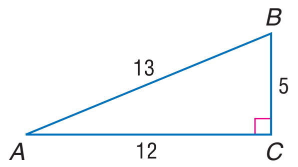
4. Solve $\triangle FGH$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



Use the inverse capabilities of your calculator to find the measure of an angle when one of its trigonometric ratios is known. For example, use the $\sin^{-1}$ function to find the measure of an angle when the sine of the angle is known. *You will learn more about inverses of trigonometric functions in Lesson 13-7.*

EXAMPLE Find Missing Angle Measures of Right Triangles

5 **Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**



You know the measures of the sides. You need to find A and B.

Find A. $\sin A = \frac{5}{13}$ $\quad$ $\sin A = \frac{\text{opp}}{\text{hyp}}$

Use a calculator and the [SIN^{-1}] function to find the angle whose sine is $\frac{5}{13}$.

KEYSTROKES: 2nd [SIN^{-1}] 5 ÷ 13) ENTER 22.61986495

To the nearest degree, $A \approx 23°$.

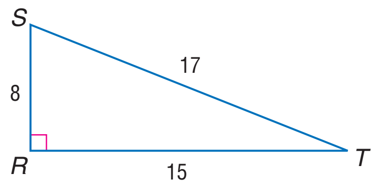
Find B. $23° + B \approx 90°$ Angles A and B are complementary.

$B \approx 67°$ Solve for B.

Therefore, $A \approx 23°$ and $B \approx 67°$.

CHECK Your Progress

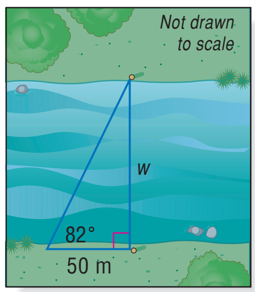
5. Solve $\triangle RST$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



Trigonometry has many practical applications. Among the most important is the ability to find distances that either cannot or are not easily measured directly.

Real-World EXAMPLE Indirect Measurement

6 **BRIDGE CONSTRUCTION In order to construct a bridge, the width of the river must be determined. Suppose a stake is planted on one side of the river directly across from a second stake on the opposite side. At a distance 50 meters to the left of the stake, an angle of 82° is measured between the stake, an angle of 82° is measured between the two stakes. Find the width of the river.**



Let w represent the width of the river at that location. Write an equation using a trigonometric function that involves the ratio of the distance w and 50.

$\tan 82° = \frac{w}{50}$ $\quad$ $\tan \theta = \frac{\text{opp}}{\text{adj}}$

$50 \tan 82° = w$ $\quad$ Multiply each side by 50.

$355.8 \approx w$ $\quad$ The width of the river is about 355.8 meters.

Real-World Link

There are an estimated 595,625 bridges in use in the United States.

Source: betterroads.com

CHECK Your Progress

6. John found two trees directly across from each other in a canyon. When he moved 100 feet from the tree on his side (parallel to the edge of the canyon), the angle formed by the tree on his side, John, and the tree on the other side was 70°. Find the distance across the canyon.

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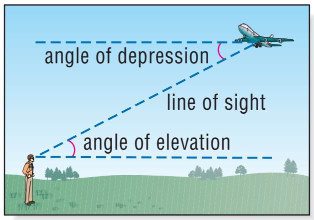
Study Tip

Angle of Elevation and Depression

The angle of elevation and the angle of depression are congruent since they are alternate interior angles of parallel lines.

Some applications of trigonometry use an angle of elevation or depression. In the figure at the right, the angle formed by the line of sight from the observer and a line parallel to the ground is called the **angle of elevation.** The angle formed by the line of sight from the plane and a line parallel to the ground is called the **angle of depression.**



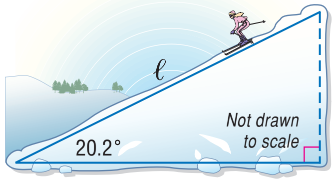
Real-World Link

The average annual snowfall in Alpine Meadows, California, is 495 inches. The longest designated run there is 2.5 miles.

Source: www.onthesnow.com

EXAMPLE Use an Angle of Elevation

7 **SKIING** **The Aerial run in Snowbird, Utah, has an angle of elevation of 20.2°. Its vertical drop is 2900 feet. Estimate the length of this run.**



Let ℓ represent the length of the run. Write an equation using a trigonometric function that involves the ratio of ℓ and 2900.

$\sin 20.2° = \dfrac{2900}{\ell}$ $\sin \theta = \dfrac{\text{opp}}{\text{hyp}}$

$\ell = \dfrac{2900}{\sin 20.2°}$ Solve for ℓ.

$\ell \approx 8398.5$ Use a calculator.

The length of the run is about 8399 feet.

CHECK Your Progress

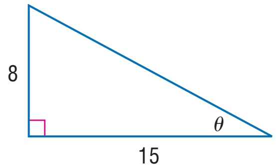
7. A ramp for unloading a moving truck has an angle of elevation of 32°. If the top of the ramp is 4 feet above the ground, estimate the length of the ramp.

CHECK Your Understanding

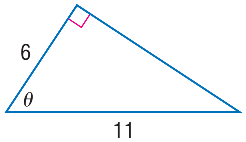
Example 1 (p. 760)

Find the values of the six trigonometric functions for angle θ.

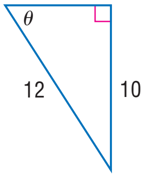
1.



2.



3.



Example 2 (pp. 760–761)

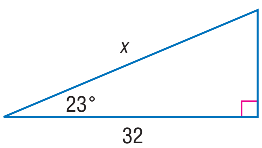
4. **STANDARDIZED TEST PRACTICE** If $\tan \theta = 3$, find the value of $\sin \theta$. **B**

A $\dfrac{3}{10}$ B $\dfrac{3\sqrt{10}}{10}$ C $\dfrac{10}{3}$ D $\dfrac{1}{3}$

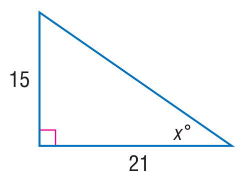
Examples 3, 5 (pp. 761–763)

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and angles to the nearest degree.

5.



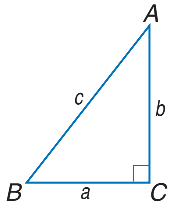
6.



John P. Kelly/Getty Images

Examples 4, 5
(pp. 762–763)

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



7. $A = 45°, b = 6$

8. $B = 56°, c = 6$

9. $b = 7, c = 18$

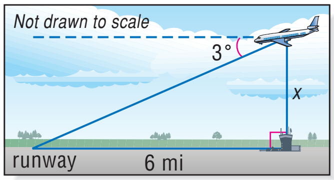
10. $a = 14, b = 13$

Example 6
(p. 763)

11. BRIDGES Tom wants to build a rope bridge between his tree house and Roy's tree house. Suppose Tom's tree house is directly behind Roy's tree house. At a distance of 20 meters to the left of Tom's tree house, an angle of 52° is measured between the two tree houses. Find the length of the rope bridge.

Example 7
(p. 764)

12. AVIATION When landing, a jet will average a 3° angle of descent. What is the altitude x, to the nearest foot, of a jet on final descent as it passes over an airport beacon 6 miles from the start of the runway?

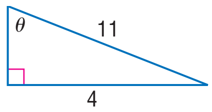


Exercises

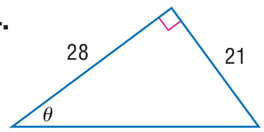
HOMEWORK HELP	
For Exercises	See Examples
12–14	1, 2
15–18	3
21–26	4
19, 20	5
27, 28	6, 7

Find the values of the six trigonometric functions for angle θ.

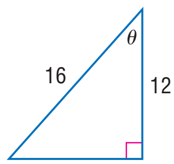
13.



14.

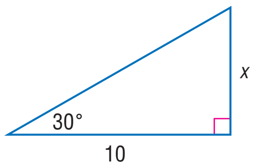


15.

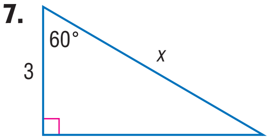


Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and angles to the nearest degree.

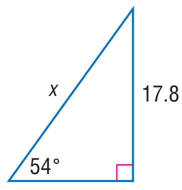
16.



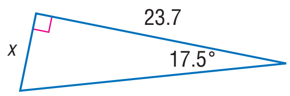
17.



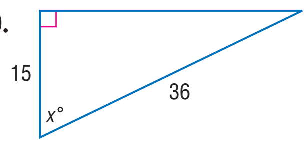
18.



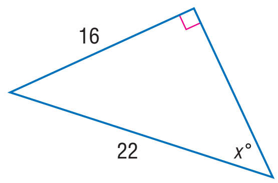
19.



20.



21.



Real-World Career

Surveyor

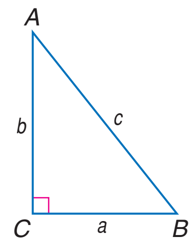
Land surveyors manage survey parties that measure distances, directions, and angles between points, lines, and contours on Earth's surface.

Math Online

For more information, go to algebra2.com.

SuperStock

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



22. $A = 16°, c = 14$

23. $B = 27°, b = 7$

24. $A = 34°, a = 10$

25. $B = 15°, c = 25$

26. $B = 30°, b = 11$

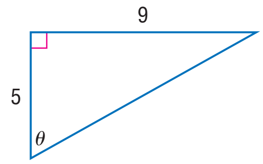
27. $A = 45°, c = 7\sqrt{2}$

28. SURVEYING A surveyor stands 100 feet from a building and sights the top of the building at a 55° angle of elevation. Find the height of the building.

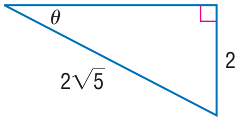
29. TRAVEL In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls?

Find the values of the six trigonometric functions for angle θ.

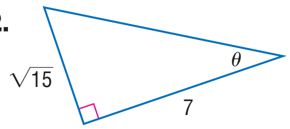
30.



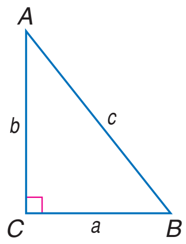
31.



32.



Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



33. $B = 18°, a = \sqrt{15}$

34. $A = 10°, b = 15$

35. $b = 6, c = 13$

36. $a = 4, c = 9$

37. $\tan B = \frac{7}{8}, b = 7$

38. $\sin A = \frac{1}{3}, a = 5$

39. Using the 30°-60°-90° triangle shown in the lesson, verify each value.

a. $\sin 30° = \frac{1}{2}$ **b.** $\cos 30° = \frac{\sqrt{3}}{2}$ **c.** $\sin 60° = \frac{\sqrt{3}}{2}$

40. Using the 45°-45°-90° triangle shown in the lesson, verify each value.

a. $\sin 45° = \frac{\sqrt{2}}{2}$ **b.** $\cos 45° = \frac{\sqrt{2}}{2}$ **c.** $\tan 45° = 1$

Cross-Curricular Project

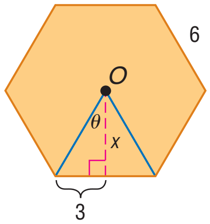
Math Online You can use the tangent ratio to determine the maximum height of a rocket. Visit algebra2.com to continue work on your project.

EXERCISE For Exercises 41 and 42, use the following information.
A preprogrammed workout on a treadmill consists of intervals walking at various rates and angles of incline. A 1% incline means 1 unit of vertical rise for every 100 units of horizontal run.

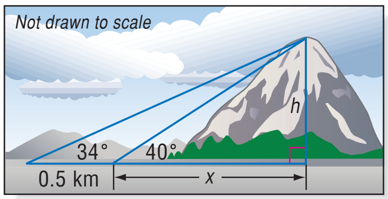
41. At what angle, with respect to the horizontal, is the treadmill bed when set at a 10% incline? Round to the nearest degree.

42. If the treadmill bed is 40 inches long, what is the vertical rise when set at an 8% incline?

43. GEOMETRY Find the area of the regular hexagon with point O as its center. (*Hint*: First find the value of x.)



44. GEOLOGY A geologist measured a 40° of elevation to the top of a mountain. After moving 0.5 kilometer farther away, the angle of elevation was 34°. How high is the top of the mountain? (*Hint*: Write a system of equations in two variables.)



EXTRA PRACTICE
See pages 920, 938.
Math Online Self-Check Quiz at algebra2.com

H.O.T. Problems

45. **OPEN ENDED** Draw two right triangles $\triangle ABC$ and $\triangle DEF$ for which $\sin A = \sin D$. What can you conclude about $\triangle ABC$ and $\triangle DEF$? Justify your reasoning.

46. **REASONING** Find a counterexample to the statement *It is always possible to solve a right triangle.*

47. **CHALLENGE** Explain why the sine and cosine of an acute angle are never greater that 1, but the tangent of an acute angle may be greater than 1.

48. *Writing in Math* Use the information on page 759 to explain how trigonometry is used in building construction. Include an explanation as to why the ratio of vertical rise to horizontal run on an entrance ramp is the tangent of the angle the ramp makes with the horizontal.

STANDARDIZED TEST PRACTICE

49. **ACT/SAT** If the secant of angle θ is $\frac{25}{7}$, what is the sine of angle θ?

A $\frac{5}{25}$

B $\frac{7}{25}$

C $\frac{24}{25}$

D $\frac{25}{7}$

50. **REVIEW** A person holds one end of a rope that runs through a pulley and has a weight attached to the other end. Assume the weight is directly beneath the pulley. The section of rope between the pulley and the weight is 12 feet long. The rope bends through an angle of 33 degrees in the pulley. How far is the person from the weight?

F 7.8 ft

G 10.5 ft

H 12.9 ft

J 14.3 ft

Spiral Review

Determine whether each situation would produce a random sample. Write *yes* or *no* and explain your answer (Lesson 12-9)

51. surveying band members to find the most popular type of music at your school

52. surveying people coming into a post office to find out what color cars are most popular

Find each probability if a coin is tossed 4 times (Lesson 12-8)

53. P(exactly 2 heads)

54. P(4 heads)

55. P(at least 1 head)

Solve each equation (Lesson 6-6)

56. $y^4 - 64 = 0$

57. $x^5 - 5x^3 + 4x = 0$

58. $d + \sqrt{d} - 132 = 0$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each product. Include the appropriate units with your answer.** (Lesson 6-1)

59. $5 \text{ gallons}\left(\frac{4 \text{ quarts}}{1 \text{ gallon}}\right)$

60. $6.8 \text{ miles}\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right)$

61. $\left(\frac{2 \text{ square meters}}{5 \text{ dollars}}\right)30 \text{ dollars}$

62. $\left(\frac{4 \text{ liters}}{5 \text{ minutes}}\right)60 \text{ minutes}$

13-2 Angles and Angle Measure

Main Ideas

- Change radian measure to degree measure and vice versa.
- Identify coterminal angles.

New Vocabulary

initial side
terminal side
standard position
unit circle
radian
coterminal angles

GET READY for the Lesson

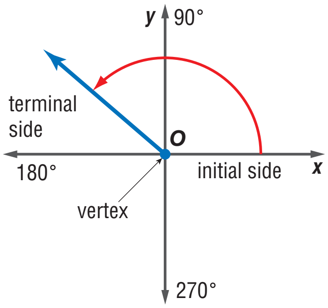
The Ferris wheel at Navy Pier in Chicago has a 140-foot diameter and 40 gondolas equally spaced around its circumference. The average angular velocity ω of one of the gondolas is given by $\omega = \frac{\theta}{t}$ where θ is the angle through which the gondola has revolved after a specified amount of time t. For example, if a gondola revolves through an angle of 225° in 40 seconds, then its average angular velocity is 225° ÷ 40 or about 5.6° per second.

ANGLE MEASUREMENT What does an angle measuring 225° look like? In Lesson 13-1, you worked only with acute angles, those measuring between 0° and 90°, but angles can have *any* real number measurement.

Reading Math

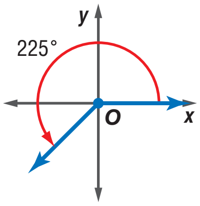
Angle of Rotation In trigonometry, an angle is sometimes referred to as an *angle of rotation.*

On a coordinate plane, an angle may be generated by the rotation of two rays that share a fixed endpoint at the origin. One ray, called the **initial side** of the angle, is fixed along the positive x-axis. The other ray, called the **terminal side** of the angle, can rotate about the center. An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis is said to be in **standard position**.

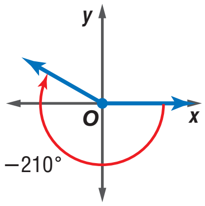


The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side.

Positive Angle Measure
counterclockwise

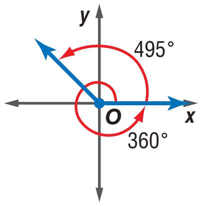


Negative Angle Measure
clockwise



Concepts in Motion
Animation algebra2.com

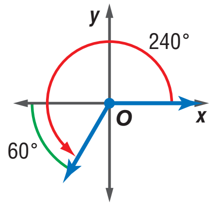
When terminal sides rotate, they may sometimes make one or more revolutions. An angle whose terminal side has made exactly one revolution has a measure of 360°.



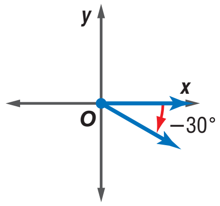
EXAMPLE 1 Draw an Angle in Standard Position

Draw an angle with the given measure in standard position.

a. 240° $240° = 180° + 60°$
Draw the terminal side of the angle 60° counterclockwise past the negative x-axis.



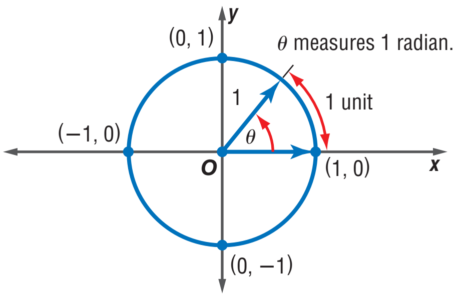
b. −30° The angle is negative.
Draw the terminal side of the angle 30° clockwise from the positive x-axis.

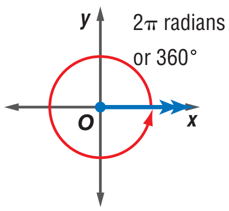


CHECK Your Progress

1A. 450° **1B.** −110°

Another unit used to measure angles is a radian. The definition of a radian is based on the concept of a **unit circle**, which is a circle of radius 1 unit whose center is at the origin of a coordinate system. One **radian** is the measure of an angle θ in standard position whose rays intercept an arc of length 1 unit on the unit circle.





The circumference of any circle is $2\pi r$, where r is the radius measure. So the circumference of a unit circle is $2\pi(1)$ or 2π units. Therefore, an angle representing one complete revolution of the circle measures 2π radians. This same angle measures 360°. Therefore, the following equation is true.

$$2\pi \text{ radians} = 360°$$

As with degrees, the measure of an angle in radians is positive if its rotation is counterclockwise. The measure is negative if the rotation is clockwise.

To change angle measures from radians to degrees or vice versa, solve the equation above in terms of both units.

$$2\pi \text{ radians} = 360^\circ \qquad\qquad 2\pi \text{ radians} = 360^\circ$$

$$\frac{2\pi \text{ radians}}{2\pi} = \frac{360^\circ}{2\pi} \qquad\qquad \frac{2\pi \text{ radians}}{360} = \frac{360^\circ}{360}$$

$$1 \text{ radian} = \frac{180^\circ}{\pi} \qquad\qquad \frac{\pi \text{ radians}}{180} = 1^\circ$$

1 radian is about 57 degrees. 1 degree is about 0.0175 radian.

These equations suggest a method for converting between radian and degree measure.

Reading Math

Radian Measure The word *radian* is usually omitted when angles are expressed in radian measure. Thus, when no units are given for an angle measure, radian measure is implied.

KEY CONCEPT *Radian and Degree Measure*

- To rewrite the radian measure of an angle in degrees, multiply the number of radians by $\frac{180^\circ}{\pi \text{ radians}}$.
- To rewrite the degree measure of an angle in radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180^\circ}$.

EXAMPLE **Convert Between Degree and Radian Measure**

2 **Rewrite the degree measure in radians and the radian measure in degrees.**

a. 60°

$$60^\circ = 60\cancel{^\circ}\left(\frac{\pi \text{ radians}}{180\cancel{^\circ}}\right)$$

$$= \frac{60\pi}{180} \text{ or } \frac{\pi}{3} \text{ radians}$$

b. $-\frac{7\pi}{4}$

$$-\frac{7\pi}{4} = \left(-\frac{7\cancel{\pi}}{4}\ \cancel{\text{radians}}\right)\left(\frac{180^\circ}{\cancel{\pi}\ \cancel{\text{radians}}}\right)$$

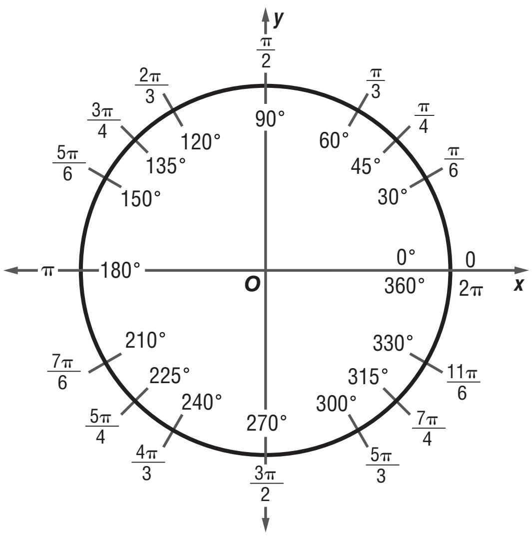
$$= -\frac{1260^\circ}{4} \text{ or } -315^\circ$$

CHECK Your Progress

2A. 190°

2B. $\frac{3\pi}{8}$

You will find it useful to learn equivalent degree and radian measures for the special angles shown in the diagram at the right. This diagram is more easily learned by memorizing the equivalent degree and radian measures for the first quadrant and for 90°. All of the other special angles are multiples of these angles.



Concepts in Motion

Interactive Lab algebra2.com

EXAMPLE Measure an Angle in Degrees and Radians

3 TIME Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 1:00 P.M. to 3:00 P.M.

The numbers on a clock divide it into 12 equal parts with 12 equal angles. The angle from 1 to 3 on the clock represents $\frac{2}{12}$ or $\frac{1}{6}$ of a complete rotation of 360°. $\frac{1}{6}$ of 360° is 60°.

Since the rotation is clockwise, the angle through which the hour hand rotates is negative. Therefore, the angle measures $-60°$.

60° has an equivalent radian measure of $\frac{\pi}{3}$. So the equivalent radian measure of $-60°$ is $-\frac{\pi}{3}$.

Real-World Link

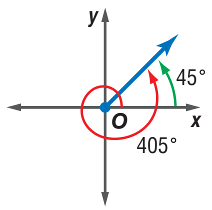
The clock tower in the United Kingdom Parliament House was opened in 1859. The copper minute hand in each of the four clocks of the tower is 4.2 meters long, 100 kilograms in mass, and travels a distance of about 190 kilometers a year.

Source: parliament.uk/index.cfm

CHECK Your Progress

3. How long does it take for a minute hand on a clock to pass through 2.5π radians?

COTERMINAL ANGLES If you graph a 405° angle and a 45° angle in standard position on the same coordinate plane, you will notice that the terminal side of the 405° angle is the same as the terminal side of the 45° angle. When two angles in standard position have the same terminal sides, they are called **coterminal angles**.



Notice that $405° - 45° = 360°$. In degree measure, coterminal angles differ by an integral multiple of 360°. You can find an angle that is coterminal to a given angle by adding or subtracting a multiple of 360°. In radian measure, a coterminal angle is found by adding or subtracting a multiple of 2π.

EXAMPLE Find Coterminal Angles

4 Find one angle with positive measure and one angle with negative measure coterminal with each angle.

a. 240°

A positive angle is $240° + 360°$ or $600°$.

A negative angle is $240° - 360°$ or $-120°$.

b. $\frac{9\pi}{4}$

A positive angle is $\frac{9\pi}{4} + 2\pi$ or $\frac{17\pi}{4}$. $\quad \frac{9\pi}{4} + \frac{8\pi}{4} = \frac{17\pi}{4}$

A negative angle is $\frac{9\pi}{4} - 2(2\pi)$ or $-\frac{7\pi}{4}$. $\quad \frac{9\pi}{4} + \left(\frac{-16\pi}{4}\right) = \frac{-7\pi}{4}$

Study Tip

Coterminal Angles

Notice in Example 4b that it is necessary to subtract a multiple of 2π to find a coterminal angle with negative measure.

CHECK Your Progress

4A. 15° **4B.** $-\frac{\pi}{4}$

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Example 1 (p. 769)

Draw an angle with the given measure in standard position.

1. 70° **2.** 300° **3.** 570° **4.** $-45°$

Example 2 (p. 770)

Rewrite each degree measure in radians and each radian measure in degrees.

5. 130° **6.** $-10°$ **7.** 485°

8. $\frac{3\pi}{4}$ **9.** $-\frac{\pi}{6}$ **10.** $\frac{19\pi}{3}$

Example 3 (pp. 770–771)

ASTRONOMY For Exercises 11 and 12, use the following information.
Earth rotates on its axis once every 24 hours.

11. How long does it take Earth to rotate through an angle of 315°?

12. How long does it take Earth to rotate through an angle of $\frac{\pi}{6}$?

Example 4 (p. 771)

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

13. 60° **14.** 425° **15.** $\frac{\pi}{3}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
16–19	1
20–27	2
28–33	4
34, 35	3

Draw an angle with the given measure in standard position.

16. 235° **17.** 270° **18.** 790° **19.** 380°

Rewrite each degree measure in radians and each radian measure in degrees.

20. 120° **21.** 60° **22.** $-15°$ **23.** $-225°$

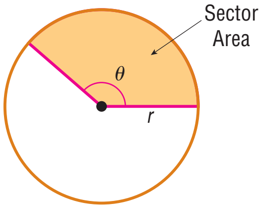
24. $\frac{5\pi}{6}$ **25.** $\frac{11\pi}{4}$ **26.** $-\frac{\pi}{4}$ **27.** $-\frac{\pi}{3}$

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

28. 225° **29.** 30° **30.** $-15°$

31. $\frac{3\pi}{4}$ **32.** $\frac{7\pi}{6}$ **33.** $-\frac{5\pi}{4}$

GEOMETRY For Exercises 34 and 35, use the following information.
A *sector* is a region of a circle that is bounded by a central angle θ and its intercepted arc. The area A of a sector with radius r and central angle θ is given by $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.



34. Find the area of a sector with a central angle of $\frac{4\pi}{3}$ radians in a circle whose radius measures 10 inches.

35. Find the area of a sector with a central angle of 150° in a circle whose radius measures 12 meters.

Draw an angle with the given measure in standard position.

36. $-150°$ **37.** $-50°$ **38.** π **39.** $-\frac{2\pi}{3}$

Rewrite each degree measure in radians and each radian measure in degrees.

40. $660°$ **41.** $570°$ **42.** $158°$ **43.** $260°$

44. $\frac{29\pi}{4}$ **45.** $\frac{17\pi}{6}$ **46.** 9 **47.** 3

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

48. $-140°$ **49.** $368°$ **50.** $760°$

51. $-\frac{2\pi}{3}$ **52.** $\frac{9\pi}{2}$ **53.** $\frac{17\pi}{4}$

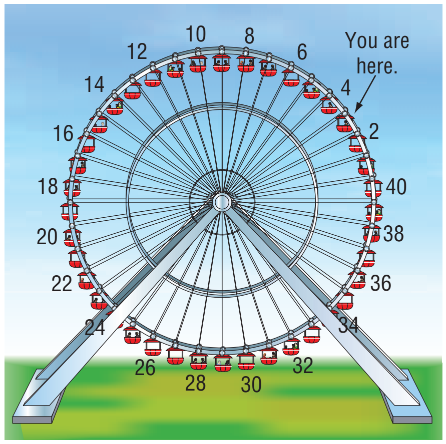
54. DRIVING Some sport-utility vehicles (SUVs) use 15-inch radius wheels. When driven 40 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and the nearest radian.

Real-World Link

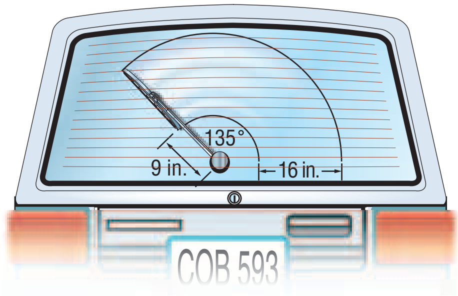
Vehicle tires are marked with numbers and symbols that indicate the specifications of the tire, including its size and the speed the tire can safely travel.

Source: usedtire.com

55. ENTERTAINMENT Suppose the gondolas on the Navy Pier Ferris Wheel were numbered from 1 through 40 consecutively in a counterclockwise fashion. If you were sitting in gondola number 3 and the wheel were to rotate counterclockwise through $\frac{47\pi}{10}$ radians, which gondola used to be in the position that you are in now?



56. CARS Use the Area of a Sector Formula in Exercises 34 and 35 to find the area swept by the rear windshield wiper of the car shown at the right.



EXTRA PRACTICE

See pages 920, 938.

Math Online

Self-Check Quiz at algebra2.com

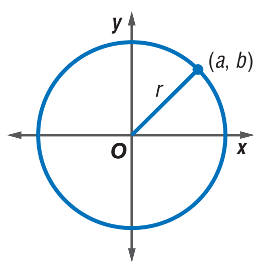
H.O.T. Problems

57. OPEN ENDED Draw and label an example of an angle with negative measure in standard position. Then find an angle with positive measure that is coterminal with this angle.

58. CHALLENGE A line with positive slope makes an angle of $\frac{\pi}{2}$ radians with the positive x-axis at the point (2, 0). Find an exact equation for this line.

59. CHALLENGE If (a, b) is on a circle that has radius r and center at the origin, prove that each of the following points is also on this circle.

a. $(a, -b)$ **b.** (b, a) **c.** $(b, -a)$



60. REASONING Express $\frac{1}{8}$ of a revolution in degrees.

61. *Writing in Math* Use the information on page 768 to explain how angles can be used to describe circular motion. Include an explanation of the significance of angles of more than 180° in terms of circular motion, an explanation of the significance of angles with negative measure in terms of circular motion, and an interpretation of a rate of more than 360° per minute.

STANDARDIZED TEST PRACTICE

62. ACT/SAT Choose the radian measure that is equal to 56°.

A $\frac{\pi}{15}$

B $\frac{7\pi}{45}$

C $\frac{14\pi}{45}$

D $\frac{\pi}{3}$

63. REVIEW Angular velocity is defined by the equation $\omega = \frac{\theta}{t}$, where θ is usually expressed in radians and t represents time. Find the angular velocity in radians per second of a point on a bicycle tire if it completes 2 revolutions in 3 seconds.

F $\frac{\pi}{3}$

G $\frac{\pi}{2}$

H $\frac{2\pi}{3}$

J $\frac{4\pi}{3}$

Spiral Review

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-1)

64. $A = 34°, b = 5$

65. $B = 68°, b = 14.7$

66. $B = 55°, c = 16$

67. $a = 0.4, b = 0.4\sqrt{3}$

Find the margin of sampling error. (Lesson 12-9)

68. $p = 72\%, n = 100$

69. $p = 50\%, n = 200$

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities. (Lesson 12-2)

70. choosing an arrangement of 5 CDs from your 30 favorite CDs

71. choosing 3 different types of snack foods out of 7 at the store to take on a trip

Find $[g \circ h](x)$ and $[h \circ g](x)$. (Lesson 7-1)

72. $g(x) = 2x$
$h(x) = 3x - 4$

73. $g(x) = 2x + 5$
$h(x) = 2x^2 - 3x + 9$

GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify each expression.** (Lesson 7-5)

74. $\frac{2}{\sqrt{3}}$

75. $\frac{3}{\sqrt{5}}$

76. $\frac{4}{\sqrt{6}}$

77. $\frac{5}{\sqrt{10}}$

78. $\frac{\sqrt{7}}{\sqrt{2}}$

79. $\frac{\sqrt{5}}{\sqrt{8}}$

EXTEND 13-2

Algebra Lab
Investigating Regular Polygons Using Trigonometry

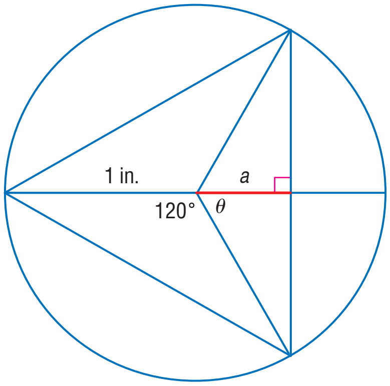
ACTIVITY

- Use a compass to draw a circle with a radius of one inch. Inscribe an equilateral triangle inside of the circle. To do this, use a protractor to measure three angles of 120° at the center of the circle, since $\frac{360°}{3} = 120°$. Then connect the points where the sides of the angles intersect the circle using a straightedge.
- The **apothem** of a regular polygon is a segment that is drawn from the center of the polygon perpendicular to a side of the polygon. Use the cosine of angle θ to find the length of an apothem, labeled a in the diagram below.

Analyze the Results

1. Make a table like the one shown below and record the length of the apothem of the equilateral triangle.

Number of Sides, n	θ	a
3	60	
4	45	
5		
6		
7		
8		
9		
10		



Inscribe each regular polygon named in the table in a circle of radius one inch. Copy and complete the table.

2. What do you notice about the measure of θ as the number of sides of the inscribed polygon increases?

3. What do you notice about the values of a?

4. **MAKE A CONJECTURE** Suppose you inscribe a 20-sided regular polygon inside a circle. Find the measure of angle θ.

5. Write a formula that gives the measure of angle θ for a polygon with n sides.

6. Write a formula that gives the length of the apothem of a regular polygon inscribed in a circle of radius one inch.

7. How would the formula you wrote in Exercise 6 change if the radius of the circle was not one inch?

13-3 Trigonometric Functions of General Angles

Main Ideas

- Find values of trigonometric functions for general angles.
- Use reference angles to find values of trigonometric functions.

New Vocabulary

quadrantal angle
reference angle

GET READY for the Lesson

A skycoaster consists of a large arch from which two steel cables hang and are attached to riders suited together in a harness. A third cable, coming from a larger tower behind the arch, is attached with a ripcord. Riders are hoisted to the top of the larger tower, pull the ripcord, and then plunge toward Earth. They swing through the arch, reaching speeds of more than 60 miles per hour. After the first several swings of a certain skycoaster, the angle θ of the riders from the center of the arch is given by $\theta = 0.2 \cos (1.6t)$, where t is the time in seconds after leaving the bottom of their swing.

Trigonometric Functions and General Angles In Lesson 13-1, you found values of trigonometric functions whose domains were the set of all acute angles, angles between 0 and $\frac{\pi}{2}$, of a right triangle. For $t > 0$ in the equation above, you must find the cosine of an angle greater than $\frac{\pi}{2}$. In this lesson, we will extend the domain of trigonometric functions to include angles of *any* measure.

KEY CONCEPT — Trigonometric Functions, θ in Standard Position

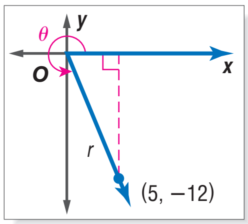
Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side of θ. Using the Pythagorean Theorem, the distance r from the origin to P is given by $r = \sqrt{x^2 + y^2}$. The trigonometric functions of an angle in standard position may be defined as follows.

$\sin \theta = \frac{y}{r}$ $\quad$ $\cos \theta = \frac{x}{r}$ $\quad$ $\tan \theta = \frac{y}{x}, x \neq 0$

$\csc \theta = \frac{r}{y}, y \neq 0$ $\quad$ $\sec \theta = \frac{r}{x}, x \neq 0$ $\quad$ $\cot \theta = \frac{x}{y}, y \neq 0$

EXAMPLE Evaluate Trigonometric Functions for a Given Point

1 **Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point $(5, -12)$.**



From the coordinates, you know that $x = 5$ and $y = -12$. Use the Pythagorean Theorem to find r.

courtesy of Skycoaster of Florida

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} && \text{Pythagorean Theorem} \\
&= \sqrt{5^2 + (-12)^2} && \text{Replace } x \text{ with 5 and } y \text{ with } 2\text{—}12. \\
&= \sqrt{169} \text{ or } 13 && \text{Simplify.}
\end{aligned}
$$

Now, use $x = 5$, $y = -12$, and $r = 13$ to write the ratios.

$$\sin\theta = \frac{y}{r} = \frac{-12}{13} \text{ or } -\frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{5}{13} \qquad \tan\theta = \frac{y}{x} = -\frac{12}{5} \text{ or } -\frac{12}{5}$$

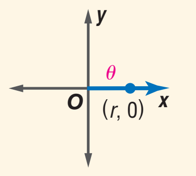
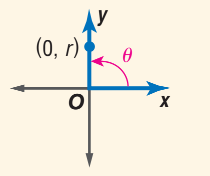
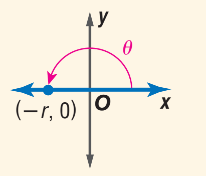
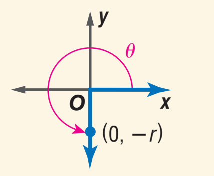
$$\csc\theta = \frac{r}{y} = \frac{13}{-12} \text{ or } -\frac{13}{12} \qquad \sec\theta = \frac{r}{x} = \frac{13}{5} \qquad \cot\theta = \frac{x}{y} = \frac{5}{-12} \text{ or } -\frac{5}{12}$$

CHECK Your Progress

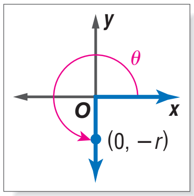
1. Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point $(-8, -15)$.

If the terminal side of angle θ lies on one of the axes, θ is called a **quadrantal angle.** The quadrantal angles are 0°, 90°, 180°, and 270°. Notice that for these angles either x or y is equal to 0. Since division by zero is undefined, two of the trigonometric values are undefined for each quadrantal angle.

KEY CONCEPT *Quadrantal Angles*

$\theta = 0°$ or 0 radians	$\theta = 90°$ or $\frac{\pi}{2}$ radians	$\theta = 180°$ or π radians	$\theta = 270°$ or $\frac{3\pi}{2}$ radians
			

EXAMPLE Quadrantal Angles



2 **Find the values of the six trigonometric functions for an angle in standard position that measures 270°.**

When $\theta = 270°$, $x = 0$ and $y = -r$.

$$\sin\theta = \frac{y}{r} = \frac{-r}{r} \text{ or } -1 \qquad \cos\theta = \frac{x}{r} = \frac{0}{r} \text{ or } 0 \qquad \tan\theta = \frac{y}{x} = \frac{-r}{0} \text{ or undefined}$$

$$\csc\theta = \frac{r}{y} = \frac{r}{-r} \text{ or } -1 \qquad \sec\theta = \frac{r}{x} = \frac{r}{0} \text{ or undefined} \qquad \cot\theta = \frac{x}{y} = \frac{0}{-r} \text{ or } 0$$

CHECK Your Progress

2. Find the values of the six trigonometric functions for an angle in standard position that measures 180°.

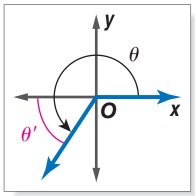
Reading Math

Theta Prime θ' is read *theta prime*.

Concepts in Motion

Animation
algebra2.com

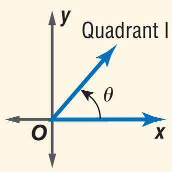
Reference Angles To find the values of trigonometric functions of angles greater than 90° (or less than 0°), you need to know how to find the measures of reference angles. If θ is a nonquadrantal angle in standard position, its **reference angle**, θ, is defined as the acute angle formed by the terminal side of θ and the x-axis.



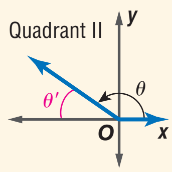
You can use the rule below to find the reference angle for any nonquadrantal angle θ where $0° < \theta < 360°$ (or $0 < \theta < 2\pi$).

KEY CONCEPT — *Reference Angle Rule*

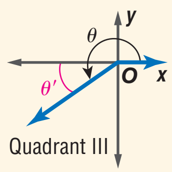
For any nonquadrantal angle θ, $0° < \theta < 360°$ (or $0 < \theta < 2\pi$), its reference angle θ' is defined as follows.



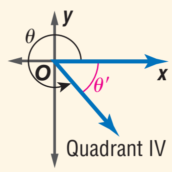
$\theta' = \theta$



$\theta = 180° - \theta$
$(\theta' = \pi - \theta)$



$\theta' = \theta - 180°$
$(\theta' = \theta - \pi)$



$= 360° - \theta$
$(\theta' = 2\pi - \theta)$

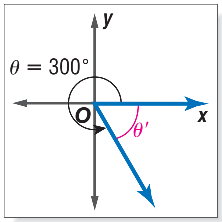
If the measure of θ is greater than 360° or less than 0°, its reference angle can be found by associating it with a coterminal angle of positive measure between 0° and 360°.

EXAMPLE **Find the Reference Angle for a Given Angle**

3 **Sketch each angle. Then find its reference angle.**

a. 300°

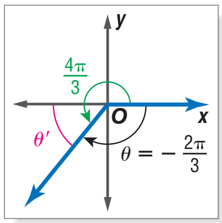
Because the terminal side of 300° lies in Quadrant IV, the reference angle is $360° - 300°$ or $60°$



b. $-\frac{2\pi}{3}$

A coterminal angle of $-\frac{2\pi}{3}$ is $2\pi - \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Because the terminal side of this angle lies in Quadrant III, the reference angle is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.



CHECK Your Progress

3A. $-200°$ **3B.** $\frac{2\pi}{3}$

To use the reference angle θ' to find a trigonometric value of θ, you need to know the sign of that function for an angle θ. From the function definitions, these signs are determined by x and y, since r is always positive. Thus, the sign of each trigonometric function is determined by the quadrant in which the terminal side of θ lies.

The chart summarizes the signs of the trigonometric functions for each quadrant.

	Quadrant			
Function	**I**	**II**	**III**	**IV**
$\sin\theta$ or $\csc\theta$	+	+	–	–
$\cos\theta$ or $\sec\theta$	+	–	–	+
$\tan\theta$ or $\cot\theta$	+	–	+	–

Use the following steps to find the value of a trigonometric function of any angle θ.

Step 1 Find the reference angle θ'.

Step 2 Find the value of the trigonometric function for θ'.

Step 3 Using the quadrant in which the terminal side of θ lies, determine the sign of the trigonometric function value of θ.

Study Tip

Look Back

To review **trigonometric values of angles measuring 30°, 45°, and 60°**, see Lesson 13-1.

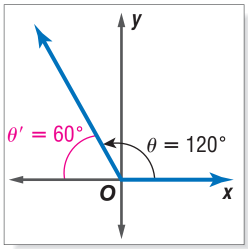
EXAMPLE Use a Reference Angle to Find a Trigonometric Value

4 **Find the exact value of each trigonometric function.**

a. $\sin 120°$

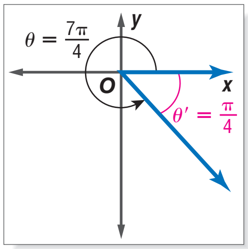
Because the terminal side of 120° lies in Quadrant II, the reference angle θ' is $180° - 120°$ or 60°. The sine function is positive in Quadrant II, so

$\sin 120° = \sin 60°$ or $\frac{\sqrt{3}}{2}$.



b. $\cot \frac{7\pi}{4}$

Because the terminal side of $\frac{7\pi}{4}$ lies in Quadrant IV, the reference angle θ' is $2\pi - \frac{7\pi}{4}$ or $\frac{\pi}{4}$. The cotangent function is negative in Quadrant IV.



$$\cot\frac{7\pi}{4} = -\cot\frac{\pi}{4}$$

$= -\cot 45°$ $\frac{\pi}{4}$ radians =5 45°

$= -1$ cot 45° =5 1

CHECK Your Progress

4A. $\cos 135°$

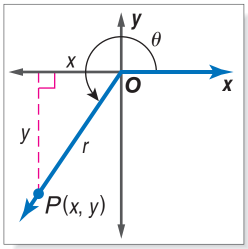
4B. $\tan\frac{5\pi}{6}$

If you know the quadrant that contains the terminal side of θ in standard position and the exact value of one trigonometric function of θ, you can find the values of the other trigonometric functions of θ using the function definitions.

EXAMPLE 5 Quadrant and One Trigonometric Value of θ

Suppose θ is an angle in standard position whose terminal side is in Quadrant III and $\sec\theta = -\frac{4}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

Draw a diagram of this angle, labeling a point $P(x, y)$ on the terminal side of θ. Use the definition of secant to find the values of x and r.



$\sec\theta = -\frac{4}{3}$ Given

$\frac{r}{x} = -\frac{4}{3}$ Definition of secant

Since x is negative in Quadrant III and r is always positive, $x = -3$ and $r = 4$. Use these values and the Pythagorean Theorem to find y.

$x^2 + y^2 = r^2$ Pythagorean Theorem

$(-3)^2 + y^2 = 4^2$ Replace x with -3 and r with 4.

$y^2 = 16 - 9$ Simplify. Then subtract 9 from each side.

$y = \pm\sqrt{7}$ Simplify. Then take the square root of each side.

$y = -\sqrt{7}$ y is negative in Quadrant III.

Use $x = -3$, $y = -\sqrt{7}$, and $r = 4$ to write the remaining trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{7}}{4} \text{ or } -\frac{\sqrt{7}}{4}$$

$$\cos\theta = \frac{x}{r} = -\frac{3}{4}$$

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{7}}{-3} \text{ or } \frac{\sqrt{7}}{3}$$

$$\csc\theta = \frac{r}{y} = \frac{4}{-\sqrt{7}} \text{ or } -\frac{4\sqrt{7}}{7}$$

$$\cot\theta = \frac{x}{y} = \frac{-3}{-\sqrt{7}} \text{ or } \frac{3\sqrt{7}}{7}$$

CHECK Your Progress

5. Suppose θ is an angle in standard position whose terminal side is in Quadrant IV and $\tan\theta = -\frac{2}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

Just as an exact point on the terminal side of an angle can be used to find trigonometric function values, trigonometric function values can be used to find the exact coordinates of a point on the terminal side of an angle.

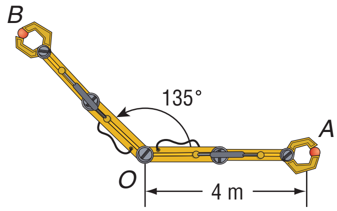
Real-World Link

RoboCup is an annual event in which teams from all over the world compete in a series of soccer matches in various classes according to the size and intellectual capacity of their robot. The robots are programmed to react to the ball and communicate with each other.

Source: www.robocup.org

Real-World EXAMPLE — Find Coordinates Given a Radius and an Angle

6 ROBOTICS In a robotics competition, a robotic arm 4 meters long is to pick up an object at point A and release it into a container at point B. The robot's arm is programmed to rotate through an angle of precisely 135° to accomplish this task. What is the new position of the object relative to the pivot point O?



With the pivot point at the origin and the angle through which the arm rotates in standard position, point A has coordinates (4, 0). The reference angle θ' for 135° is 180° – 135° or 45°.

Let the position of point B have coordinates (x, y). Then, use the definitions of sine and cosine to find the value of x and y. The value of r is the length of the robotic arm, 4 meters. Because B is in Quadrant II, the cosine of 135° is negative.

$\cos 135° = \frac{x}{r}$	cosine ratio	$\sin 135° = \frac{y}{r}$	sine ratio
$-\cos 45° = \frac{x}{4}$	$180° - 135° =5\ 45°$	$\sin 45° = \frac{y}{4}$	$180° - 35° = 45°$
$-\frac{\sqrt{2}}{2} = \frac{x}{4}$	$\cos 45° = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} = \frac{y}{4}$	$\sin 45° = \frac{\sqrt{2}}{2}$
$-2\sqrt{2} = x$	Solve for x.	$2\sqrt{2} = y$	Solve for y.

The exact coordinates of B are $(-2\sqrt{2}, 2\sqrt{2})$. Since $2\sqrt{2}$ is about 2.83, the object is about 2.83 meters to the left of the pivot point and about 2.83 meters in front of the pivot point.

CHECK Your Progress

6. After releasing the object in the container at point B, the arm must rotate another 75°. What is the new position of the end of the arm relative to the pivot point O?

Online **Personal Tutor at** algebra2.com

CHECK Your Understanding

Example 1 (pp. 776–777)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

1. $(-15, 8)$ **2.** $(-3, 0)$ **3.** $(4, 4)$

Examples 2, 4 (pp. 777, 779)

Find the exact value of each trigonometric function.

4. $\sin 300°$ **5.** $\cos 180°$ **6.** $\tan \frac{5\pi}{3}$ **7.** $\sec \frac{7\pi}{6}$

Example 3 (p. 778)

Sketch each angle. Then find its reference angle.

8. 235° **9.** $\frac{7\pi}{4}$ **10.** $-240°$

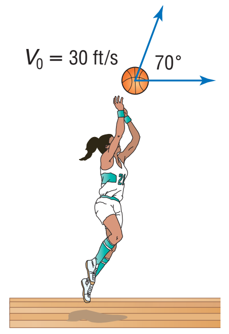
Example 5 (p. 780)

Suppose θ is an a ngle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

11. $\cos \theta = -\frac{1}{2}$, Quadrant II **12.** $\cot \theta = -\frac{\sqrt{2}}{2}$, Quadrant IV

Example 6 (p. 781)

13. BASKETBALL The maximum height H in feet that a basketball reaches after being shot is given by the formula $H = \frac{V_0^2 (\sin \theta)^2}{64}$, where V_0 represents the initial velocity and θ represents the degree measure of the angle that the path of the basketball makes with the ground. Find the maximum height reached by a ball shot with an initial velocity of 30 feet per second at an angle of 70°.



Exercises

HOMEWORK HELP	
For Exercises	See Examples
14–17	1
18–25	2, 4
26–29	3
30–33	5
34–36	6

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

14. $(7, 24)$ **15.** $(2, 1)$ **16.** $(5, -8)$ **17.** $(4, -3)$

18. $(0, -6)$ **19.** $(-1, 0)$ **20.** $\left(\sqrt{2}, -\sqrt{2}\right)$ **21.** $\left(-\sqrt{3}, -\sqrt{6}\right)$

Find the exact value of each trigonometric function.

22. $\sin 240°$ **23.** $\sec 120°$ **24.** $\tan 300°$ **25.** $\cot 510°$

26. $\csc 5400°$ **27.** $\cos \frac{11\pi}{3}$ **28.** $\cot \left(-\frac{5\pi}{6}\right)$ **29.** $\sin \frac{3\pi}{4}$

30. $\sec \frac{3\pi}{2}$ **31.** $\csc \frac{17\pi}{6}$ **32.** $\cos (-30°)$ **33.** $\tan \left(-\frac{5\pi}{4}\right)$

Sketch each angle. Then find its reference angle.

34. $315°$ **35.** $240°$ **36.** $\frac{5\pi}{4}$ **37.** $\frac{5\pi}{6}$

38. $-210°$ **39.** $-125°$ **40.** $\frac{13\pi}{7}$ **41.** $-\frac{2\pi}{3}$

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

42. $\cos \theta = \frac{3}{5}$, Quadrant IV **43.** $\tan \theta = -\frac{1}{5}$, Quadrant II

44. $\sin \theta = \frac{1}{3}$, Quadrant II **45.** $\cot \theta = \frac{1}{2}$, Quadrant III

Real-World Link

If a major league pitcher throws a pitch at 95-miles per hour, it takes only about 4-tenths of a second for the ball to travel the 60-feet, 6-inches from the pitcher's mound to home plate. In that time, the hitter must decide whether to swing at the ball and if so, when to swing.

Source: exploratorium.edu

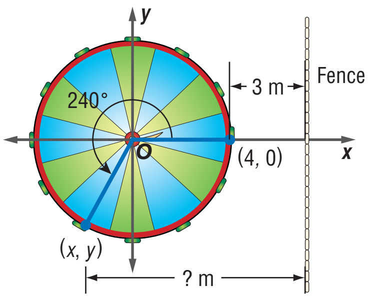
BASEBALL For Exercises 46 and 47, use the following information.

The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground.

46. If the ball was hit with an initial velocity of 80 feet per second at an angle of 30°, how far was it hit?

47. Which angle will result in the greatest distance? Explain your reasoning.

48. CAROUSELS Anthony's little brother gets on a carousel that is 8 meters in diameter. At the start of the ride, his brother is 3 meters from the fence to the ride. How far will his brother be from the fence after the carousel rotates 240°?



49. SKYCOASTING Mikhail and Anya visit a local amusement park to ride a skycoaster. After the first several swings, the angle the skycoaster makes with the vertical is modeled by $\theta = 0.2 \cos \pi t$, with θ measured in radians and t measured in seconds. Determine the measure of the angle for $t = 0$, 0.5, 1, 1.5, 2, 2.5, and 3 in both radians and degrees.

EXTRA PRACTICE
See pages 920, 938

Math Online
Self-Check Quiz at algebra2.com

50. NAVIGATION Ships and airplanes measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 2\theta$ feet, where θ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is 60°.

H.O.T. Problems

51. OPEN ENDED Give an example of an angle for which the sine is negative and the tangent is positive.

52. REASONING Determine whether the following statement is *true* or *false*. If true, explain your reasoning. If false, give a counterexample.

The values of the secant and tangent functions for any quadrantal angle are undefined.

53. *Writing in Math* Use the information on page 776 to explain how you can model the position of riders on a skycoaster.

STANDARDIZED TEST PRACTICE

54. ACT/SAT If the cotangent of angle θ is 1, then the tangent of angle θ is

A -1.   **C** 1.

B 0.   **D** 3.

55. REVIEW Which angle has a tangent and cosine that are both negative?

F 110°   **H** 210°

G 180°   **J** 340°

Spiral Review

Rewrite each degree measure in radians and each radian measure in degrees. (Lesson 13–2)

56. 90°   **57.** $\frac{5\pi}{3}$ **58.** 5

59. LITERATURE In one of *Grimm's Fairy Tales*, Rumpelstiltskin has the ability to spin straw into gold. Suppose on the first day, he spun 5 pieces of straw into gold, and each day thereafter he spun twice as much. How many pieces of straw would he have spun into gold by the end of the week? (Lesson 11-4)

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

60. $3x - 4y = 13$
$-2x + 5y = -4$

61. $5x + 7y = 1$
$3x + 5y = 3$

62. $2x + 3y = -2$
$-6x + y = -34$

GET READY for the Next Lesson

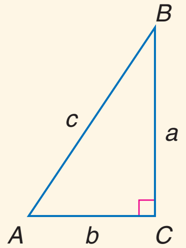
PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.** (Lesson 13-1)

63. $\frac{a}{\sin 32°} = \frac{8}{\sin 65°}$ **64.** $\frac{b}{\sin 45°} = \frac{21}{\sin 100°}$ **65.** $\frac{c}{\sin 60°} = \frac{3}{\sin 75°}$

Chapter 13 Mid-Chapter Quiz

Lessons 13-1 through 13-3

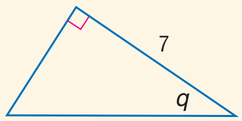
Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-1)



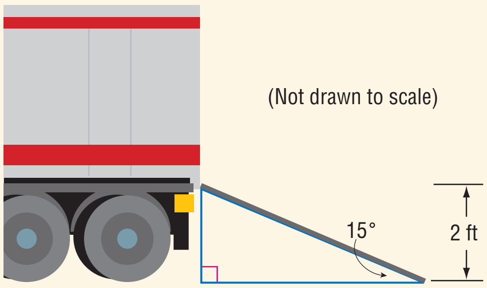
1. $A = 48°$, b 12
2. $a = 18$, $c = 21$

3. Draw an angle measuring $-60°$ in standard position. (Lesson 13-1)

4. Find the values of the six trigonometric functions for angle θ in the triangle at the right. (Lesson 13-1)



5. **TRUCKS** The tailgate of a moving truck is 2 feet above the ground. The incline of the ramp used for loading the truck is 15° as shown. Find the length of the ramp to the nearest tenth of a foot. (Lesson 13-1)



Rewrite each degree measure in radians and each radian measure in degrees. (Lesson 13-2)

6. 190°
7. 450°
8. $\frac{7\pi}{6}$
9. $-\frac{11\pi}{5}$

Find one angle with positive measure and one angle with negative measure coterminal with each angle. (Lesson 13-2)

10. $-55°$
11. $\frac{11\pi}{3}$

SUNDIAL For Exercises 12 and 13, use the following information. (Lesson 13-2)
A sector is a region of a circle that is bounded by a central angle θ and its intercepted arc. The area A of a sector with radius r and central angle θ is given by $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.

12. Find the shaded area of a sundial with a central angle of $\frac{3\pi}{4}$ radians and a radius that measures 6 inches.
13. Find the sunny area of a sundial with a central angle of 270° with a radius measuring 10 inches.

14. Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the point $(-2, 3)$. (Lesson 13-3)

15. Find the exact value of $\csc \frac{5\pi}{3}$. (Lesson 13-3)

16. **NAVIGATION** Airplanes and ships measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 2\theta$ feet, where θ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is 120°. (Lesson 13-3)

17. **MULTIPLE CHOICE** Suppose θ is an angle in standard position with $\sin \theta > 0$. In which quadrant(s) does the terminal side of θ lie? (Lesson 13-3)

    A I
    B II
    C III
    D I and II

13-4 Law of Sines

Main Ideas

- Solve problems by using the Law of Sines.
- Determine whether a triangle has one, two, or no solutions.

New Vocabulary

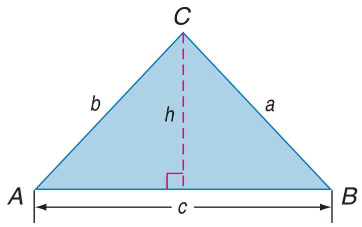
Law of Sines

GET READY for the Lesson

You know how to find the area of a triangle when the base and the height are known. Using this formula, the area of $\triangle ABC$ below is $\frac{1}{2}ch$. If the height h of this triangle were not known, you could still find the area given the measures of angle A and the length of side b.

$$\sin A = \frac{h}{b} \rightarrow h = b \sin A$$

By combining this equation with the area formula, you can find a new formula for the area of the triangle.



$$\text{Area} = \frac{1}{2}ch \rightarrow \text{Area} = \frac{1}{2}c(b \sin A)$$

Law of Sines You can find two other formulas for the area of the triangle above in a similar way.

Study Tip

Area Formulas

These formulas allow you to find the area of any triangle when you know the measures of two sides and the included angle.

KEY CONCEPT — *Area of a Triangle*

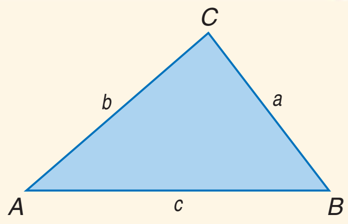
Words The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.

Symbols

$\text{area} = \frac{1}{2}bc \sin A$

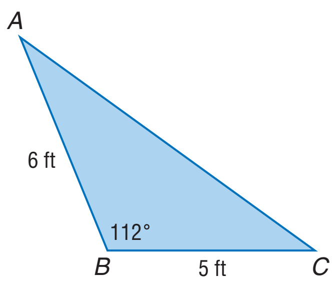
$\text{area} = \frac{1}{2}ac \sin B$

$\text{area} = \frac{1}{2}ab \sin C$



EXAMPLE Find the Area of a Triangle

1 Find the area of $\triangle ABC$ to the nearest tenth.



In this triangle, $a = 5$, $c = 6$, and $B = 112°$. Choose the second formula because you know the values of its variables.

$\text{Area} = \frac{1}{2}ac \sin B$	Area formula
$= \frac{1}{2}(5)(6) \sin 112°$	Replace a with 5, c with 6, and B with 112°.
≈ 13.9	To the nearest tenth, the area is 13.9 square feet.

CHECK Your Progress

1. Find the area of $\triangle ABC$ to the nearest tenth if $A = 31°$, $b = 18$ m, and $c = 22$ m.

All of the area formulas for $\triangle ABC$ represent the area of the same triangle. So, $\frac{1}{2}bc \sin A$, $\frac{1}{2}ac \sin B$, and $\frac{1}{2}ab \sin C$ are all equal. You can use this fact to derive the **Law of Sines**.

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$ Set area formulas equal to each other.

$$\frac{\frac{1}{2}bc \sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac \sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}abc}$$ Divide each expression by $\frac{1}{2}abc$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ Simplify.

Study Tip

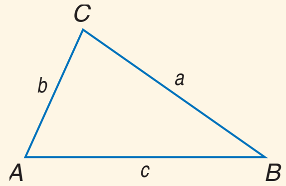
Alternate Representations

The Law of Sines may also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

KEY CONCEPT — *Law of Sines*

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C respectively. Then,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



The Law of Sines can be used to write three different equations.

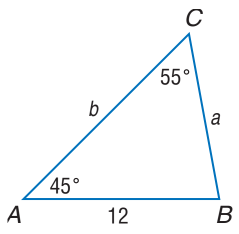
$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

In Lesson 13-1, you learned how to solve right triangles. To solve *any* triangle, you can apply the Law of Sines if you know

- the measures of two angles and any side or
- the measures of two sides and the angle opposite one of them.

EXAMPLE Solve a Triangle Given Two Angles and a Side

2 **Solve $\triangle ABC$.**



You are given the measures of two angles and a side. First, find the measure of the third angle.

$45° + 55° + B = 180°$ The sum of the angle measures of a triangle is 180°.

$B = 80°$ $180 - (45 + 55) = 80$

Now use the Law of Sines to find a and b. Write two equations, each with one variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin C}{c}$
$\frac{\sin 45°}{a} = \frac{\sin 55°}{12}$	Replace A with 45°, B with 80°, C with 55°, and c with 12.	$\frac{\sin 80°}{b} = \frac{\sin 55°}{12}$
$a = \frac{12 \sin 45°}{\sin 55°}$	Solve for the variable.	$b = \frac{12 \sin 80°}{\sin 55°}$
$a \approx 10.4$	Use a calculator.	$b \approx 14.4$

Therefore, $B = 80°$, $a \approx 10.4$, and $b \approx 14.4$.

CHECK Your Progress

2. Solve $\triangle FGH$ if $m\angle G = 80°$, $m\angle H = 40°$, and $g = 14$.

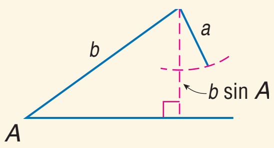
One, Two, or No Solutions When solving a triangle, you must analyze the data you are given to determine whether there is a solution. For example, if you are given the measures of two angles and a side, as in Example 2, the triangle has a unique solution. However, if you are given the measures of two sides and the angle opposite one of them, a single solution may not exist. One of the following will be true.

- No triangle exists, and there is no solution.
- Exactly one triangle exists, and there is one solution.
- Two triangles exist, and there are two solutions.

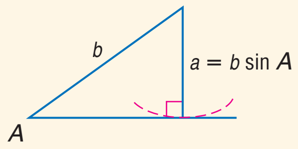
KEY CONCEPT — Possible Triangles Given Two Sides and One Opposite Angle

Suppose you are given a, b, and A for a triangle.

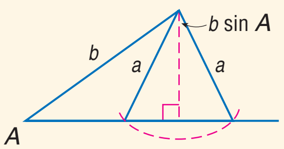
A Is Acute ($A < 90°$).



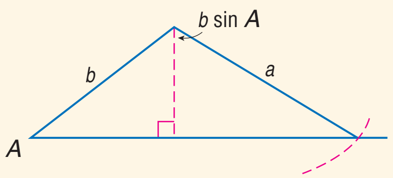
$a < b \sin A$
no solution



$a = b \sin A$
one solution

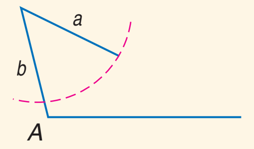


$b > a > b \sin A$
two solutions

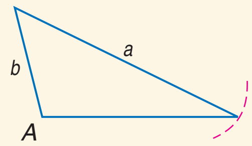


$a \geq b$
one solution

A Is Right or Obtuse ($A \geq 90°$).



$a \leq b$
no solution



$a > b$
one solution

EXAMPLE 3 One Solution

In $\triangle ABC$, $A = 118°$, $a = 20$, and $b = 17$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Because angle A is obtuse and $a > b$, you know that one solution exists.

Use the Law of Sines to find B.

$\frac{\sin B}{17} = \frac{\sin 118°}{20}$ Law of Sines

$\sin B = \frac{17 \sin 118°}{20}$ Multiply each side by 17.

$\sin B \approx 0.7505$ Use a calculator.

$B \approx 49°$ Use the $\sin^{-1}$ function.

Use the Law of Sines again to find c.

$\frac{\sin 13}{c} = \frac{\sin 118°}{20}$ Law of Sines

$c = \frac{20 \sin 13°}{\sin 118°}$ or about 5.1

Therefore, $B \approx 49°$, $C \approx 13°$, and $c \approx 5.1$.

The measure of angle C is approximately $180 - (118 + 49)$ or $13°$.

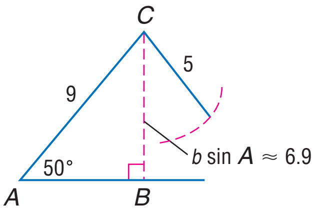
CHECK Your Progress

3. In $\triangle ABC$, $B = 95°$, $b = 19$, and $c = 12$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

EXAMPLE No Solution

4 **In $\triangle ABC$, $A = 50°$, $a = 5$, and $b = 9$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.**

Since angle A is acute, find $b \sin A$ and compare it with a.



$b \sin A = 9 \sin 50°$ Replace b with 9 and A with 50°.

≈ 6.9 Use a calculator.

Since $5 < 6.9$, there is no solution.

CHECK Your Progress

4. In $\triangle ABC$, $B = 95°$, $b = 10$, and $c = 12$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Study Tip

***A* Is Acute**

We compare $b \sin A$ to a because $b \sin A$ is the minimum distance from C to $\overline{AB}$ when A is acute.

When two solutions for a triangle exist, it is called the *ambiguous case.*

EXAMPLE Two Solutions

5 **In $\triangle ABC$, $A = 39°$, $a = 10$, and $b = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.**

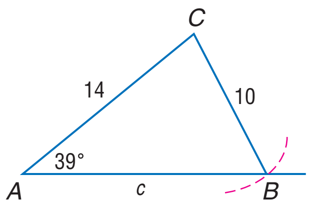
Since angle A is acute, find $b \sin A$ and compare it with a.

$b \sin A = 14 \sin 39°$ Replace b with 14 and A with 39°.

≈ 8.81 Use a calculator.

Since $14 > 10 > 8.81$, there are two solutions. Thus, there are two possible triangles to be solved.

Case 1 Acute Angle B



First, use the Law of Sines to find B.

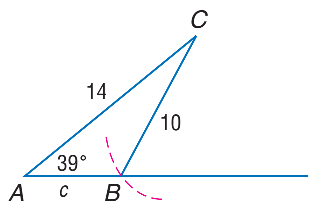
$$\frac{\sin B}{14} = \frac{\sin 39°}{10}$$

$$\sin B = \frac{14 \sin 39°}{10}$$

$$\sin B = 0.8810$$

$$B \approx 62°$$

Case 2 Obtuse Angle B



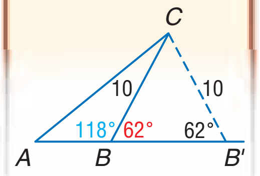
To find B, you need to find an obtuse angle whose sine is also 0.8810. To do this, subtract the angle given by your calculator, 62°, from 180°. So B is approximately $180 - 62$ or 118°.

The measure of angle C is approximately $180 - (39 + 118)$ or 23°.

Study Tip

Alternate Method

Another way to find the obtuse angle in Case 2 of Example 5 is to notice in the figure below that $\triangle CBB'$ is isosceles. Since the base angles of an isosceles triangle are always congruent and $m\angle B' = 62°$, $m\angle CBB' = 62°$. Also, $\angle ABC$ and $m\angle CBB'$ are supplementary. Therefore, $m\angle ABC = 180° - 62°$ or 118°.



The measure of angle C is approximately $180 - (39 + 62)$ or $79°$.

$$\frac{\sin 79°}{c} = \frac{\sin 39°}{10}$$
$$c = \frac{10 \sin 79°}{\sin 39°}$$
$$c \approx 15.6$$

Therefore, $B \approx 62°$, $C \approx 79°$, and $c \approx 15.6$.

Use the Law of Sines to find c.

$$\frac{\sin 23°}{c} = \frac{\sin 39°}{10}$$
$$c = \frac{10 \sin 23°}{\sin 39°}$$
$$c \approx 6.2$$

Therefore, $B \approx 118°$, $C \approx 23°$, and $c \approx 6.2$.

CHECK Your Progress

5. In $\triangle ABC$, $A = 44°$, $b = 19$, and $a = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

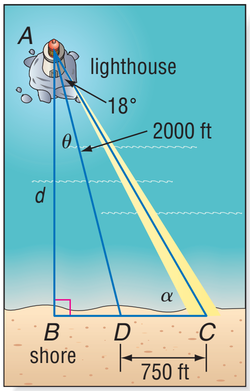
Real-World Link

Standing 208 feet tall, the Cape Hatteras Lighthouse in North Carolina is the tallest lighthouse in the United States.

Source: www.oldcapehatteraslighthouse.com

Real-World EXAMPLE — Use the Law of Sines to Solve a Problem

6 **LIGHTHOUSES The light on a lighthouse revolves counterclockwise at a steady rate of one revolution per minute. The beam strikes a point on the shore that is 2000 feet from the lighthouse. Three seconds later, the light strikes a point 750 feet further down the shore. To the nearest foot, how far is the lighthouse from the shore?**



Because the lighthouse makes one revolution every 60 seconds, the angle through which the light revolves in 3 seconds is $\frac{3}{60}(360°)$ or $18°$.

Use the Law of Sines to find the measure of angle α.

$\frac{\sin \alpha}{2000} = \frac{\sin 18°}{750}$	Law of Sines
$\sin \alpha = \frac{2000 \sin 18°}{750}$	Multiply each side by 2000.
$\sin \alpha \approx 0.8240$	Use a calculator.
$\alpha \approx 55°$	Use the $\sin^{-1}$ function.

Use this angle measure to find the measure of angle θ.

$\alpha + m\angle BAC = 90°$	Angles α and $\angle BAC$ are complementary.
$55° + (\theta + 18°) \approx 90°$	$\alpha < 55°$ and $m\angle BAC = \theta + 18°$
$\theta \approx 17°$	Solve for θ.

To find the distance from the lighthouse to the shore, solve $\triangle ABD$ for d.

$\cos \theta = \frac{AB}{AD}$	Cosine ratio
$\cos 17° \approx \frac{d}{2000}$	$\theta = 17°$ and $AD = 2000$
$d \approx 2000 \cos 17°$	Solve for d.
$d \approx 1913$	Use a calculator.

To the nearest foot, it is 1913 feet from the lighthouse to the shore.

CHECK Your Progress

6. The beam of light from another lighthouse strikes the shore 3000 feet away. Three seconds later, the beam strikes 1200 feet farther down the shore. To the nearest foot, how far is this lighthouse from the shore?

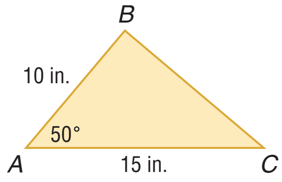
Online **Personal Tutor at algebra2.com**

CHECK Your Understanding

Example 1 (p. 785)

Find the area of $\triangle ABC$ to the nearest tenth.

1.



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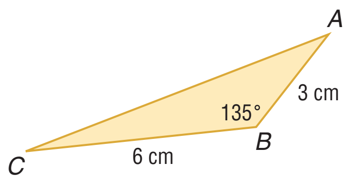
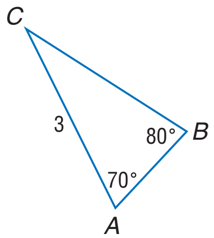


Example 2 (pp. 786–787)

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

3.



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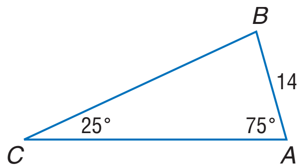

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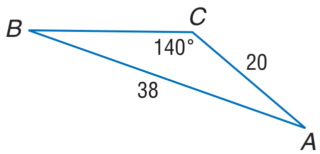


Examples 3–5 (pp. 787–789)

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

6. $A = 123°, a = 12, b = 23$

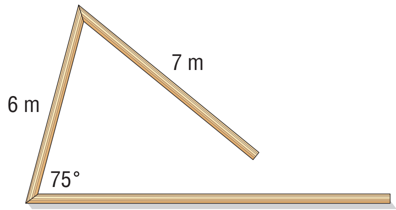
7. $A = 30°, a = 3, b = 4$

8. $A = 55°, a = 10, b = 5$

9. $A = 145°, a = 18, b = 10$

Example 6 (p. 789)

10. WOODWORKING Latisha is to join a 6-meter beam to a 7-meter beam so the angle opposite the 7-meter beam measures 75°. To what length should Latisha cut the third beam in order to form a triangular brace? Round to the nearest tenth.



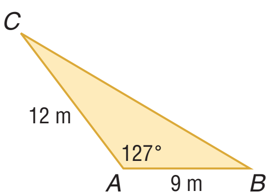
Exercises

HOMEWORK HELP

For Exercises	See Examples
11–16	1
17–30	2–5
31, 32	6

Find the area of $\triangle ABC$ to the nearest tenth.

11.



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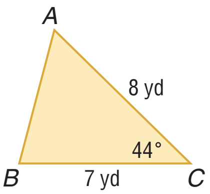


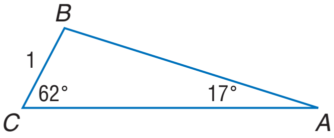
13. $B = 85°, c = 23 \text{ ft}, a = 50 \text{ ft}$

14. $A = 60°, b = 12 \text{ cm}, c = 12 \text{ cm}$

15. $C = 136°, a = 3 \text{ m}, b = 4 \text{ m}$

16. $B = 32°, a = 11 \text{ mi}, c = 5 \text{ mi}$

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

17.


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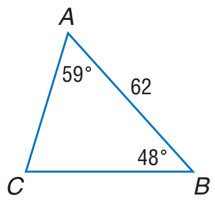

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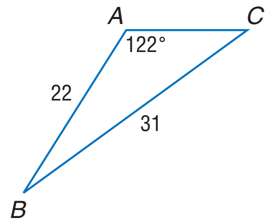

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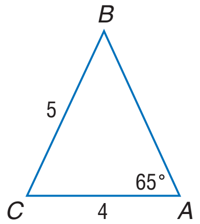

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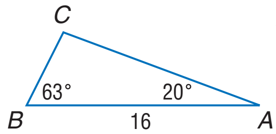

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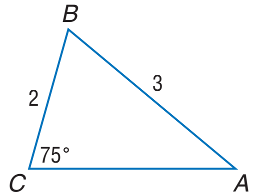


Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

23. $A = 124°, a = 1, b = 2$

24. $A = 99°, a = 2.5, b = 1.5$

25. $A = 33°, a = 2, b = 3.5$

26. $A = 68°, a = 3, b = 5$

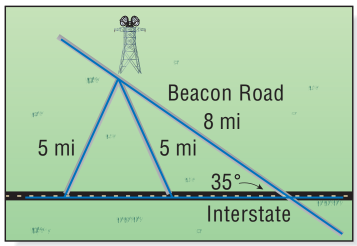
27. $A = 30°, a = 14, b = 28$

28. $A = 61°, a = 23, b = 8$

29. $A = 52°, a = 190, b = 200$

30. $A = 80°, a = 9, b = 9.1$

31. **RADIO** A radio station providing local tourist information has its transmitter on Beacon Road, 8 miles from where it intersects with the interstate highway. If the radio station has a range of 5 miles, between what two distances from the intersection can cars on the interstate tune in to hear this information?



32. **FORESTRY** Two forest rangers, 12 miles from each other on a straight service road, both sight an illegal bonfire away from the road. Using their radios to communicate with each other, they determine that the fire is between them. The first ranger's line of sight to the fire makes an angle of 38° with the road, and the second ranger's line of sight to the fire makes a 63° angle with the road. How far is the fire from each ranger?

Real-World Link

Hot-air balloons range in size from approximately 54,000 cubic feet to over 250,000 cubic feet.

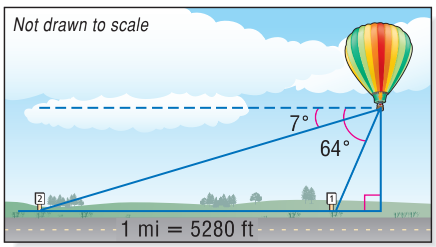
Source: www.unicorn-ballon.com

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

33. $A = 50°, a = 2.5, c = 3$

34. $B = 18°, C = 142°, b = 20$

35. **BALLOONING** As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts the angles of depression are 64° and 7°. How high is the balloon to the nearest foot?



EXTRA PRACTICE

See pages 921, 938.

Math Online

Self-Check Quiz at algebra2.com

H.O.T. Problems

36. OPEN ENDED Give an example of a triangle that has two solutions by listing measures for A, a, and b, where a and b are in centimeters. Then draw both cases using a ruler and protractor.

37. FIND THE ERROR Dulce and Gabe are finding the area of $\triangle ABC$ for $A = 64°$, $a = 15$ meters, and $b = 8$ meters using the sine function. Who is correct? Explain your reasoning.

Dulce	Gabe
$\text{Area} = \frac{1}{2}(15)(8)\sin 64°$	$\text{Area} = \frac{1}{2}(15)(8)\sin 87.4°$
$\approx 53.9 \text{ m}^2$	$\approx 59.9 \text{ m}^2$

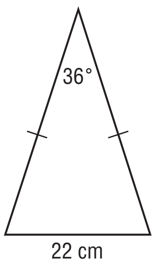
38. REASONING Determine whether the following statement is *sometimes*, *always* or *never* true. Explain your reasoning.

If given the measure of two sides of a triangle and the angle opposite one of them, you will be able to find a unique solution.

39. *Writing in Math* Use the information on page 785 to explain how trigonometry can be used to find the area of a triangle.

STANDARDIZED TEST PRACTICE

40. ACT/SAT Which of the following is the perimeter of the triangle shown?



A 49.0 cm **C** 91.4 cm

B 66.0 cm **D** 93.2 cm

41. REVIEW The longest side of a triangle is 67 inches. Two angles have measures of 47° and 55°. What is the length of the shortest leg of the triangle?

F 50.1 in. **H** 60.1 in.

G 56.1 in. **J** 62.3 in.

Spiral Review

Find the exact value of each trigonometric function. (Lesson 13-3)

42. $\cos 30°$ **43.** $\cot\left(\frac{\pi}{3}\right)$ **44.** $\csc\left(\frac{\pi}{4}\right)$

Find one angle with positive measure and one angle with negative measure coterminal with each angle. (Lesson 13-2)

45. $300°$ **46.** $47°$ **47.** $\frac{5\pi}{3}$

48. AERONAUTICS A rocket rises 20 feet in the first second, 60 feet in the second second, and 100 feet in the third second. If it continues at this rate, how many feet will it rise in the 20th second? (Lesson 11-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.** (Lesson 13-1)

49. $a^2 = 3^2 + 5^2 - 2(3)(5)\cos 85°$ **50.** $c^2 = 12^2 + 10^2 - 2(12)(10)\cos 40°$

51. $7^2 = 11^2 + 9^2 - 2(11)(9)\cos B°$ **52.** $13^2 = 8^2 + 6^2 - 2(8)(6)\cos A°$

13-5 Law of Cosines

Main Ideas

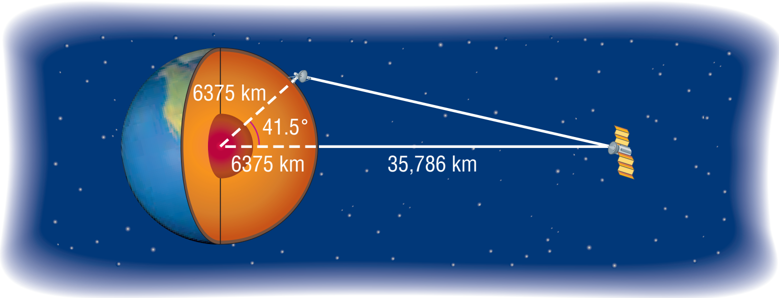
- Solve problems by using the Law of Cosines.
- Determine whether a triangle can be solved by first using the Law of Sines or the Law of Cosines.

New Vocabulary

Law of Cosines

GET READY for the Lesson

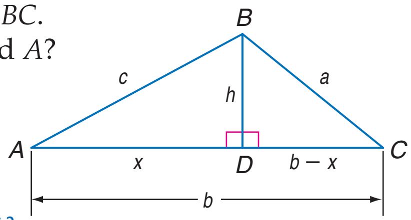
A satellite in a *geosynchronous orbit* about Earth appears to remain stationary over one point on the equator. A receiving dish for the satellite can be directed at one spot in the sky. The satellite orbits 35,786 kilometers above the equator at 87°W longitude. The city of Valparaiso, Indiana, is located at approximately 87°W longitude and 41.5°N latitude.



If the radius of Earth is about 6375 kilometers, you can use trigonometry to determine the angle at which to direct the receiver.

Law of Cosines Problems such as this, in which you know the measures of two sides and the included angle of a triangle, cannot be solved using the Law of Sines. You can solve problems such as this by using the **Law of Cosines**.

To derive the Law of Cosines, consider $\triangle ABC$. What relationship exists between a, b, c, and A?



$a^2 = (b - x)^2 + h^2$	Use the Pythagorean Theorem for $\triangle DBC$.
$= b^2 - 2bx + x^2 + h^2$	Expand $(b - x)^2$.
$= b^2 - 2bx + c^2$	In $\triangle ADB$, $c^2 = x^2 + h^2$.
$= b^2 - 2b(c \cos A) + c^2$	$\cos A = \frac{x}{c}$, so $x = c \cos A$.
$= b^2 + c^2 - 2bc \cos A$	Commutative Property

Study Tip

You can apply the Law of Cosines to a triangle if you know the measures of two sides and the included angle, or the measures of three sides.

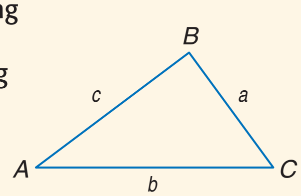
KEY CONCEPT — Law of Cosines

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides, and opposite angles with measures A, B, and C, respectively. Then the following equations are true.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

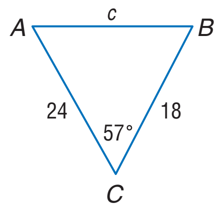
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



EXAMPLE 1 Solve a Triangle Given Two Sides and Included Angle

Solve $\triangle ABC$.



Begin by using the Law of Cosines to determine c.

$c^2 = a^2 + b^2 - 2ab \cos C$ — Law of Cosines

$c^2 = 18^2 + 24^2 - 2(18)(24) \cos 57°$ — $a = 18$, $b = 24$, and $C = 57°$

$c^2 \approx 429.4$ — Simplify using a calculator.

$c \approx 20.7$ — Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle A.

$\frac{\sin A}{a} = \frac{\sin C}{c}$ — Law of Sines

$\frac{\sin A}{18} \approx \frac{\sin 57°}{20.7}$ — $a = 18$, $C = 57°$, and $c < 20.7$

$\sin A \approx \frac{18 \sin 57°}{20.7}$ — Multiply each side by 18.

$\sin A \approx 0.7293$ — Use a calculator.

$A \approx 47°$ — Use the $\sin^{-1}$ function.

The measure of angle B is approximately $180° - (57° + 47°)$ or $76°$. Therefore, $c \approx 20.7$, $A \approx 47°$, and $B \approx 76°$.

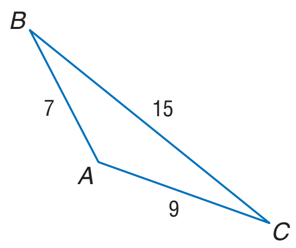
> **Study Tip**
>
> **Alternative Method**
>
> After finding the measure of c in Example 1, the Law of Cosines could be used again to find a second angle.

CHECK Your Progress

1. Solve $\triangle FGH$ if $m\angle G = 82°$, $f = 6$, and $h = 4$.

EXAMPLE 2 Solve a Triangle Given Three Sides

Solve $\triangle ABC$.



Use the Law of Cosines to find the measure of the largest angle first, angle A.

$a^2 = b^2 + c^2 - 2bc \cos A$ — Law of Cosines

$15^2 = 9^2 + 7^2 - 2(9)(7) \cos A$ — $a = 15$, $b = 9$, and $c = 7$

$15^2 - 9^2 - 7^2 = -2(9)(7) \cos A$ — Subtract 9^2 and 7^2 from each side.

$\frac{5^2 - 9^2 - 7^2}{-2(9)(7)} = \cos A$ — Divide each side by $-2(9)(7)$.

$-0.7540 \approx \cos A$ — Use a calculator.

$139° \approx A$ — Use the $\cos^{-1}$ function.

You can use the Law of Sines to find the measure of angle B.

$\frac{\sin B}{b} = \frac{\sin A}{a}$ — Law of Sines

$\frac{\sin B}{9} \approx \frac{\sin 139°}{15}$ — $b = 9$, $A \approx 139°$, and $a = 15$

$\sin B \approx \frac{9 \sin 139°}{15}$ — Multiply each side by 9.

$\sin B \approx 0.3936$ — Use a calculator.

$B \approx 23°$ — Use the $\sin^{-1}$ function.

The measure of angle C is approximately $180° - (139° + 23°)$ or $18°$. Therefore, $A \approx 139°$, $B \approx 23°$, and $C \approx 18°$.

Study Tip

Sides and Angles

When solving triangles, remember that the angle with the greatest measure is always opposite the longest side. The angle with the least measure is always opposite the shortest side.

CHECK Your Progress

2. Solve $\triangle FGH$ if $f = 2$, $g = 11$, and $h = 1$.

Personal Tutor at algebra2.com

Choose the Method To solve a triangle that is *oblique*, or having no right angle, you need to know the measure of at least one side and any two other parts. If the triangle has a solution, then you must decide whether to begin solving by using the Law of Sines or the Law of Cosines. Use the chart to help you choose.

CONCEPT SUMMARY *Solving an Oblique Triangle*

Given	Begin by Using
two angles and any side	Law of Sines
two sides and an angle opposite one of them	Law of Sines
two sides and their included angle	Law of Cosines
three sides	Law of Cosines

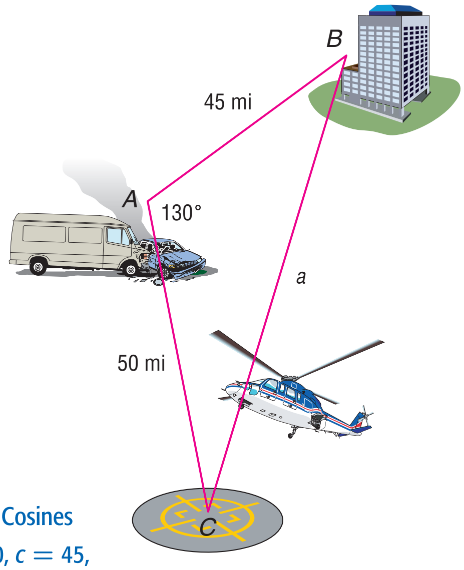
Real-World Link

Medical evacuation (Medevac) helicopters provide quick transportation from areas that are difficult to reach by any other means. These helicopters can cover long distances and are primary emergency vehicles in locations where there are few hospitals.

Source: The Helicopter Education Center

Real-World EXAMPLE Apply the Law of Cosines

3 **EMERGENCY MEDICINE** **A medical rescue helicopter has flown from its home base at point *C* to pick up an accident victim at point *A* and then from there to the hospital at point *B*. The pilot needs to know how far he is now from his home base so he can decide whether to refuel before returning. How far is the hospital from the helicopter's base?**



You are given the measures of two sides and their included angle, so use the Law of Cosines to find a.

$a^2 = b^2 + c^2 - 2bc \cos A$ — Law of Cosines

$a^2 = 50^2 + 45^2 - 2(50)(45) \cos 130°$ — $b = 50$, $c = 45$, and $A = 130°$.

$a^2 \approx 7417.5$ — Use a calculator to simplify.

$a \approx 86.1$ — Take the square root of each side.

The distance between the hospital and the helicopter base is approximately 86.1 miles.

CHECK Your Progress

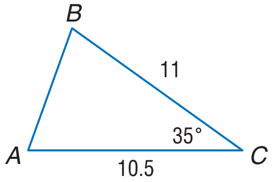
3. As part of training to run a marathon, Amelia ran 6 miles in one direction. She then turned and ran another 9 miles. The two legs of her run formed an angle of 79°. How far was Amelia from her starting point at the end of the 9-mile leg of her run?

CHECK Your Understanding

Examples 1, 2
(pp. 794–795)

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

1.



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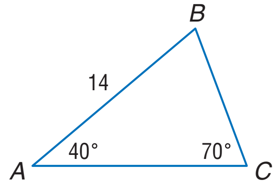


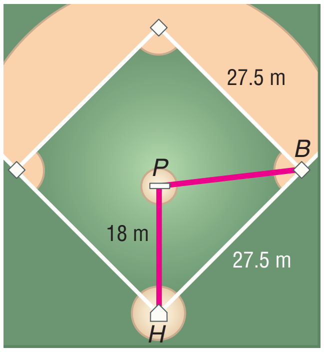
3. $A = 42°, b = 57, a = 63$

4. $a = 5, b = 12, c = 13$

Example 3
(p. 795)

BASEBALL **For Exercises 5 and 6, use the following information.**

In Australian baseball, the bases lie at the vertices of a square 27.5 meters on a side and the pitcher's mound is 18 meters from home plate.



5. Find the distance from the pitcher's mound to first base.

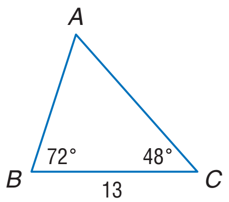
6. Find the angle between home plate, the pitcher's mound, and first base.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
7–18	1, 2
19, 20	3

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

7.



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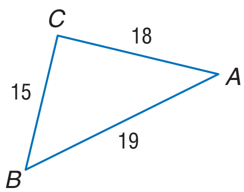

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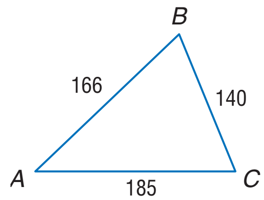
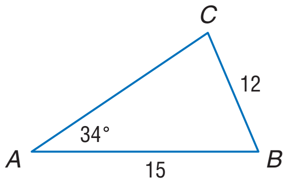
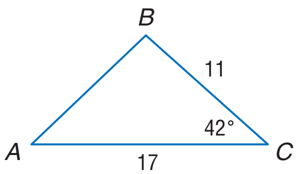


10.



11.



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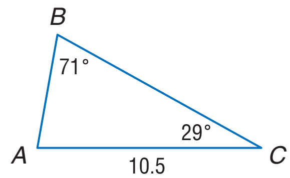


13. $a = 20, c = 24, B = 47°$

14. $a = 345, b = 648, c = 442$

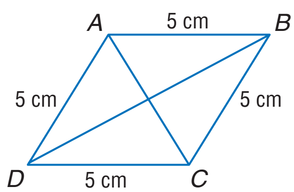
15. $A = 36°, a = 10, b = 19$

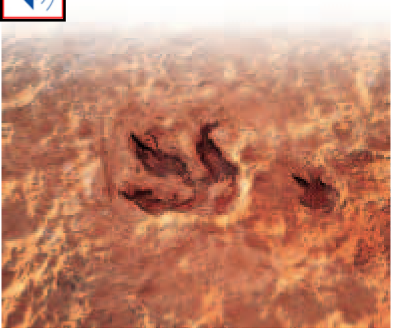
16. $A = 25°, B = 78°, a = 13.7$

17. $a = 21.5, b = 16.7, c = 10.3$

18. $a = 16, b = 24, c = 41$

19. **GEOMETRY** In rhombus $ABCD$, the measure of $\angle ADC$ is 52°. Find the measures of diagonals $\overline{AC}$ and $\overline{BD}$ to the nearest tenth.





Real-World Link

At digs such as the one at the Glen Rose formation in Texas, anthropologists study the footprints made by dinosaurs millions of years ago. *Locomoter* parameters, such as pace and stride, taken from these prints can be used to describe how a dinosaur once moved.

Source: Mid-America Paleontology Society

20. SURVEYING Two sides of a triangular plot of land have lengths of 425 feet and 550 feet. The measure of the angle between those sides is 44.5°. Find the perimeter and area of the plot.

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

21. $a = 8, b = 24, c = 18$

22. $B = 19°, a = 51, c = 61$

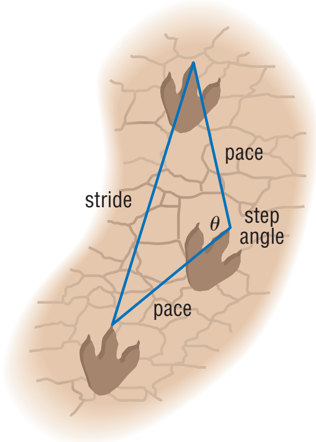
23. $A = 56°, B = 22°, a = 12.2$

24. $a = 4, b = 8, c = 5$

25. $a = 21.5, b = 13, C = 38°$

26. $A = 40°, b = 7, a = 6$

DINOSAURS For Exercises 27–29, use the diagram at the right.



27. An anthropologist examining the footprints made by a bipedal (two-footed) dinosaur finds that the dinosaur's average pace was about 1.60 meters and average stride was about 3.15 meters. Find the step angle θ for this dinosaur.

28. Find the step angle θ made by the hindfeet of a herbivorous dinosaur whose pace averages 1.78 meters and stride averages 2.73 meters.

29. An efficient walker has a step angle that approaches 180°, meaning that the animal minimizes "zig-zag" motion while maximizing forward motion. What can you tell about the motion of each dinosaur from its step angle?

EXTRA PRACTICE

See pages 921, 938.

Math Online

Self-Check Quiz at algebra2.com

30. AVIATION A pilot typically flies a route from Bloomington to Rockford, covering a distance of 117 miles. In order to avoid a storm, the pilot first flies from Bloomington to Peoria, a distance of 42 miles, then turns the plane and flies 108 miles on to Rockford. Through what angle did the pilot turn the plane over Peoria?

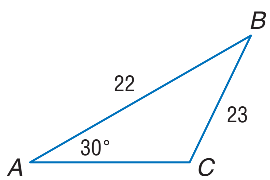
H.O.T. Problems

31. REASONING Explain how to solve a triangle by using the Law of Cosines if the lengths of

a. three sides are known.

b. two sides and the measure of the angle between them are known.

32. FIND THE ERROR Mateo and Amy are deciding which method, the Law of Sines or the Law of Cosines, should be used first to solve $\triangle ABC$.



Mateo	Amy
Begin by using the Law of Sines, since you are given two sides and an angle opposite one of them.	Begin by using the Law of Cosines, since you are given two sides and their included angle.

Who is correct? Explain your reasoning.

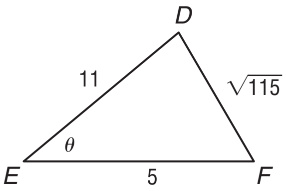
33. **OPEN ENDED** Give an example of a triangle that can be solved by first using the Law of Cosines.

34. **CHALLENGE** Explain how the Pythagorean Theorem is a special case of the Law of Cosines.

35. ***Writing in Math*** Use the information on page 793 to explain how you can determine the angle at which to install a satellite dish. Include an explanation of how, given the latitude of a point on Earth's surface, you can determine the angle at which to install a satellite dish at the same longitude.

STANDARDIZED TEST PRACTICE

36. **ACT/SAT** In $\triangle DEF$, what is the value of θ to the nearest degree?



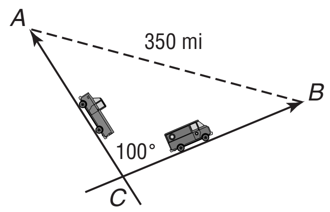
A 26°

B 74°

C 80°

D 141°

37. **REVIEW** Two trucks, A and B, start from the intersection C of two straight roads at the same time. Truck A is traveling twice as fast as truck B and after 4 hours, the two trucks are 350 miles apart. Find the approximate speed of truck B in miles per hour.



F 35 G 37 H 57 J 73

Spiral Review

38. **SANDBOX** Mr. Blackwell is building a triangular sandbox. He is to join a 3-meter beam to a 4 meter beam so the angle opposite the 4-meter beam measures 80°. To what length should Mr. Blackwell cut the third beam in order to form the triangular sandbox? Round to the nearest tenth. (Lesson 13-4)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. (Lesson 13-3)

39. $(5, 12)$ 40. $(4, 7)$ 41. $(\sqrt{10}, \sqrt{6})$

Solve each equation or inequality. (Lesson 9-5)

42. $e^x + 5 = 9$ 43. $4e^x - 3 > -1$ 44. $\ln(x + 3) = 2$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find one angle with positive measure and one angle with negative measure coterminal with each angle.** (Lesson 13-2)

45. 45° 46. 30° 47. 180°

48. $\frac{\pi}{2}$ 49. $\frac{7\pi}{6}$ 50. $\frac{4\pi}{3}$

13-6 Circular Functions

Main Ideas

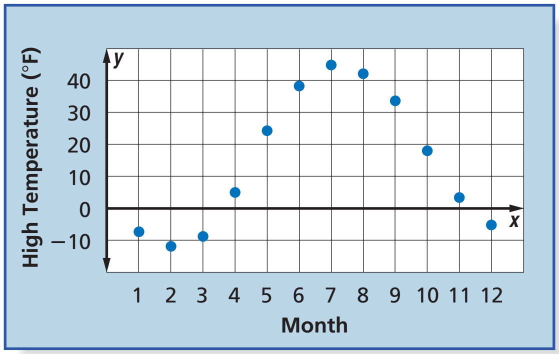
- Define and use the trigonometric functions based on the unit circle.
- Find the exact values of trigonometric functions of angles.

New Vocabulary

circular function
periodic
period

GET READY for the Lesson

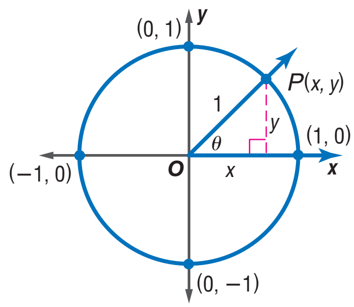
The average high temperatures, in degrees Fahrenheit, for Barrow, Alaska, are given in the table at the right. With January assigned a value of 1, February a value of 2, March a value of 3, and so on, these data can be graphed as shown below. This pattern of temperature fluctuations repeats after a period of 12 months.



MONTH	HIGH TEMP. (°F)
Jan	-7.4
Feb	-11.8
March	-9.0
April	4.7
May	24.2
June	38.3
July	45.0
Aug	42.3
Sept	33.8
Oct	18.1
Nov	3.5
Dec	-5.2

Source: www.met.utah.edu

Unit Circle Definitions From your work with reference angles, you know that the values of trigonometric functions also repeat. For example, sin 30° and sin 150° have the same value, $\frac{1}{2}$. In this lesson, we will further generalize the functions by defining them in terms of the unit circle.

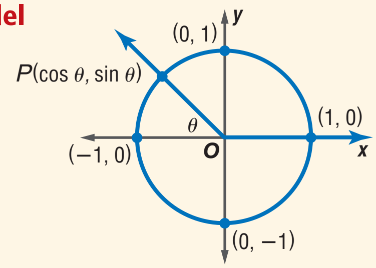


Consider an angle θ in standard position. The terminal side of the angle intersects the unit circle at a unique point, $P(x, y)$. Recall that $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$. Since $P(x, y)$ is on the unit circle, $r = 1$. Therefore, $\sin \theta = y$ and $\cos \theta = x$.

KEY CONCEPT — Definition of Sine and Cosine

Words If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of P can be written as $P(\cos \theta, \sin \theta)$.

Model



Since there is exactly one point $P(x, y)$ for any angle θ, the relations $\cos \theta = x$ and $\sin \theta = y$ are functions of θ. Because they are both defined using a unit circle, they are often called **circular functions**.

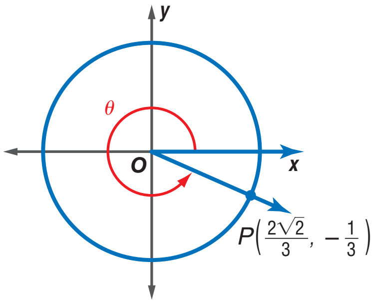
Study Tip

Remembering Relationships

To help you remember that $x = \cos \theta$ and $y = \sin \theta$, notice that alphabetically x comes before y and cosine comes before sine.

EXAMPLE 1 Find Sine and Cosine Given Point on Unit Circle

Given an angle θ in standard position, if $P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$ lies on the terminal side and on the unit circle, find $\sin \theta$ and $\cos \theta$.



$P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = P(\cos \theta, \sin \theta)$,

so $\sin \theta = -\frac{1}{3}$ and $\cos \theta = \frac{2\sqrt{2}}{3}$.

CHECK Your Progress

1. Given an angle θ in standard position, if $P\left(\frac{\sqrt{6}}{5}, \frac{\sqrt{19}}{5}\right)$ lies on the terminal side and on the unit circle, find $\sin \theta$ and $\cos \theta$.

GRAPHING CALCULATOR LAB

Sine and Cosine on the Unit Circle

Press MODE **and highlight Degree and Par. Then use the following range values to set up a viewing window:** TMIN = 0, TMAX = 360, TSTEP = 15, XMIN = −2.4, XMAX = 2.35, XSCL = 0.5, YMIN = −1.5, YMAX = 1.55, YSCL = 0.5.

Press Y= **to define the unit circle with $X_{1T} = \cos T$ and $Y_{1T} = \sin T$.**

Press GRAPH **. Use the** TRACE **function to move around the circle.**

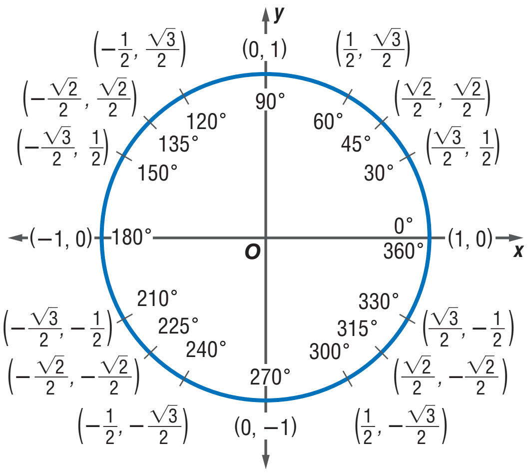
THINK AND DISCUSS

1. What does T represent? What do the x- and y-values represent?

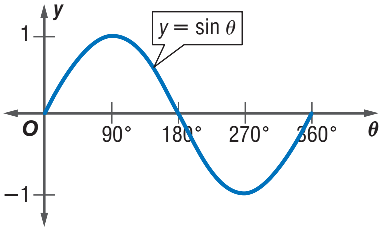
2. Determine the sine and cosine of the angles whose terminal sides lie at 0°, 90°, 180°, and 270°.

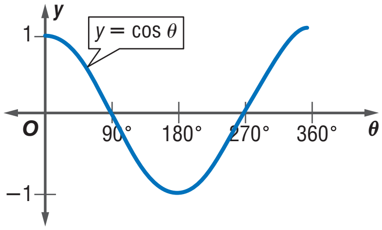
3. How do the values of sine change as you move around the unit circle? How do the values of cosine change?

The exact values of the sine and cosine functions for specific angles are summarized using the definition of sine and cosine on the unit circle at the right.

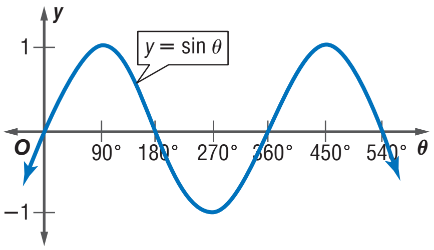


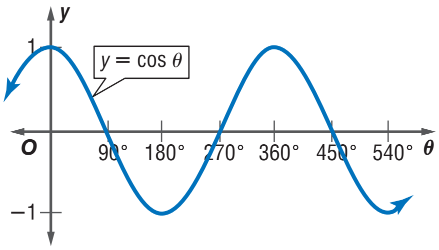
This same information is presented on the graphs of the sine and cosine functions below, where the horizontal axis shows the values of θ and the vertical axis shows the values of $\sin \theta$ or $\cos \theta$.





Periodic Functions Notice in the graph above that the values of sine for the coterminal angles 0° and 360° are both 0. The values of cosine for these angles are both 1. Every 360° or 2π radians, the sine and cosine functions repeat their values. So, we can say that the sine and cosine functions are **periodic**, each having a **period** of 360° or 2π radians.





KEY CONCEPT — *Periodic Function*

A function is called periodic if there is a number a such that $f(x) = f(x + a)$ for all x in the domain of the function. The least positive value of a for which $f(x) = f(x + a)$ is called the period of the function.

For the sine and cosine functions, $\cos (x + 360°) = \cos x$, and $\sin (x + 360°) = \sin x$. In radian measure, $\cos (x + 2\pi) = \cos x$, and $\sin (x + 2\pi) = \sin x$. Therefore, the period of the sine and cosine functions is 360° or 2π.

EXAMPLE Find the Value of a Trigonometric Function

2 **Find the exact value of each function.**

a. **cos 675°**

$$\cos 675° = \cos (315° + 360°)$$
$$= \cos 315°$$
$$= \frac{\sqrt{2}}{2}$$

b. **sin** $\left(-\frac{5\pi}{6}\right)$

$$\sin \left(-\frac{5\pi}{6}\right) = \sin \left(-\frac{5\pi}{6} + 2\pi\right)$$
$$= \sin \frac{7\pi}{6}$$
$$= -\frac{1}{2}$$

CHECK Your Progress

2A. $\cos \left(-\frac{3\pi}{4}\right)$

2B. $\sin 420°$

Personal Tutor at algebra2.com

When you look at the graph of a periodic function, you will see a repeating pattern: a shape that repeats over and over as you move to the right on the x-axis. The period is the distance along the x-axis from the beginning of the pattern to the point at which it begins again.

Many real-world situations have characteristics that can be described with periodic functions.

Real-World EXAMPLE Find the Value of a Trigonometric Function

Real-World Link

The Ferris Wheel was designed by bridge builder George W. Ferris in 1893. It was designed to be the landmark of the World's Fair in Chicago in 1893.

Source: National Academy of Sciences

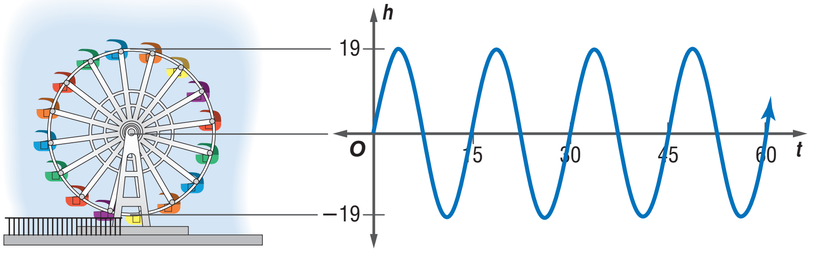
3 **FERRIS WHEEL** **As you ride a Ferris wheel, the height that you are above the ground varies periodically as a function of time. Consider the height of the center of the wheel to be the starting point. A particular wheel has a diameter of 38 feet and travels at a rate of 4 revolutions per minute.**

a. Identify the period of this function.

Since the wheel makes 4 complete counterclockwise rotations every minute, the period is the time it takes to complete one rotation, which is $\frac{1}{4}$ of a minute or 15 seconds.

b. Make a graph in which the horizontal axis represents the time t in seconds and the vertical axis represents the height h in feet in relation to the starting point.

Your height is 0 feet at the starting point. Since the diameter of the wheel is 38 feet, the wheel reaches a maximum height of $\frac{38}{2}$ or 19 feet above the starting point and a minimum of 19 feet below the starting point.



Because the period of the function is 15 seconds, the pattern of the graph repeats in intervals of 15 seconds on the x-axis.

CHECK Your Progress

A new model of the Ferris wheel travels at a rate of 5 revolutions per minute and has a diameter of 44 feet.

3A. What is the period of this function?

3B. Graph the function.

CHECK Your Understanding

Example 1 (p. 800)

If the given point P is located on the unit circle, find sin θ and cos θ.

1. $P\left(\frac{5}{13}, -\frac{12}{13}\right)$

2. $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Example 2 (p. 801)

Find the exact value of each function.

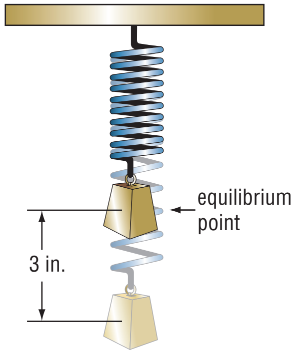
3. $\sin -240°$

4. $\cos \frac{10\pi}{3}$

Example 3 (p. 802)

PHYSICS For Exercises 5 and 6, use the following information.

The motion of a weight on a spring varies periodically as a function of time. Suppose you pull the weight down 3 inches from its equilibrium point and then release it. It bounces above the equilibrium point and then returns below the equilibrium point in 2 seconds.



5. Find the period of this function.

6. Graph the height of the spring as a function of time.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
7–12	1
13–18	2
19–38	3

The given point P is located on the unit circle. Find sin θ and cos θ.

7. $P\left(-\frac{3}{5}, \frac{4}{5}\right)$

8. $P\left(-\frac{12}{13}, -\frac{5}{13}\right)$

9. $P\left(\frac{8}{17}, \frac{15}{17}\right)$

10. $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

11. $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

12. $P(0.6, 0.8)$

Find the exact value of each function.

13. $\sin 690°$

14. $\cos 750°$

15. $\cos 5\pi$

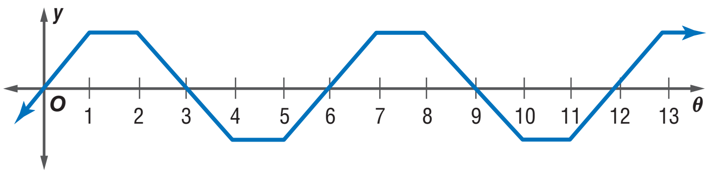
16. $\sin\left(\frac{14\pi}{6}\right)$

17. $\sin\left(-\frac{3\pi}{2}\right)$

18. $\cos(-225°)$

Determine the period of each function.

19.



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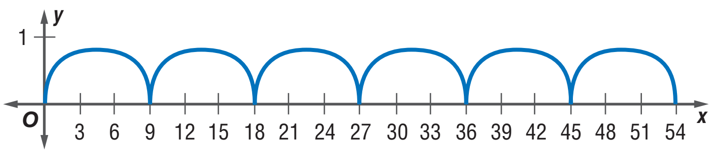

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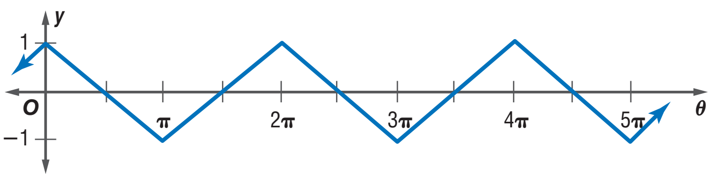
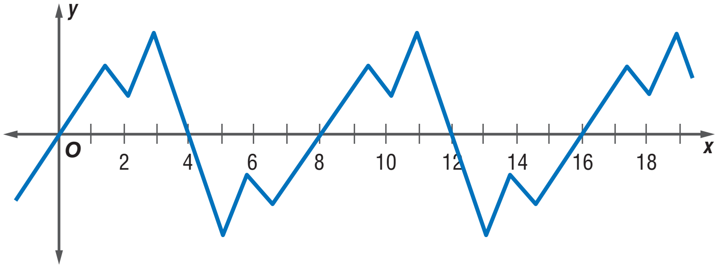


Determine the period of the function.

22.



Real-World Link

Most guitars have six strings. The frequency at which one of these strings vibrates is controlled by the length of the string, the amount of tension on the string, the weight of the string, and the springiness of the strings' material.

Source: www.howstuffworks.com

GUITAR **For Exercises 23 and 24, use the following information.**
When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz.

23. Find the period of this function.

24. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit and the minimum distance below this position have a value of 1 unit.

Find the exact value of each function.

25. $\dfrac{\cos 60° + \sin 30°}{4}$

26. $3(\sin 60°)(\cos 30°)$

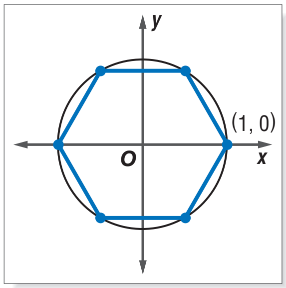
27. $\sin 30° - \sin 60°$

28. $\dfrac{4 \cos 330° + 2 \sin 60°}{3}$

29. $12(\sin 150°)(\cos 150°)$

30. $(\sin 30°)^2 + (\cos 30°)^2$

31. **GEOMETRY** A regular hexagon is inscribed in a unit circle centered at the origin. If one vertex of the hexagon is at (1, 0), find the exact coordinates of the remaining vertices.



32. **BIOLOGY** In a certain area of forested land, the population of rabbits R increases and decreases periodically throughout the year. If the population can be modeled by $R = 425 + 200 \sin\left[\frac{\pi}{365}(d - 60)\right]$, where d represents the dth day of the year, describe what happens to the population throughout the year.

SLOPE **For Exercises 33–38, use the following information.**
Suppose the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$.

33. What is the slope of $\overline{OP}$?

34. Which of the six trigonometric functions is equal to the slope of $\overline{OP}$?

35. What is the slope of any line perpendicular to $\overline{OP}$?

36. Which of the six trigonometric functions is equal to the slope of any line perpendicular to $\overline{OP}$?

37. Find the slope of $\overline{OP}$ when $\theta = 60°$.

38. If $\theta = 60°$, find the slope of the line tangent to circle O at point P.

EXTRA PRACTICE
See pages 921, 938.

Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. OPEN ENDED Give an example of a situation that could be described by a periodic function. Then state the period of the function.

40. WHICH ONE DOESN'T BELONG? Identify the expression that does not belong with the other three. Explain your reasoning.

$\sin 90°$	$\tan \frac{\pi}{4}$	$\cos 180°$	$\csc \frac{\pi}{2}$

41. CHALLENGE Determine the domain and range of the functions $y = \sin \theta$ and $y = \cos \theta$.

42. *Writing in Math* If the formula for the temperature T in degrees Fahrenheit of a city t months into the year is given by $T = 50 + 25 \sin\left(\frac{\pi}{6}t\right)$, explain how to find the average temperature and the maximum and minimum predicted over the year.

STANDARDIZED TEST PRACTICE

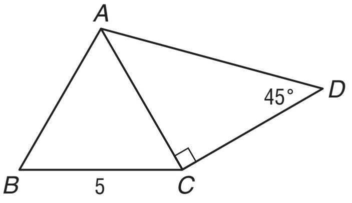
43. ACT/SAT If $\triangle ABC$ is an equilateral triangle, what is the length of $\overline{AD}$, in units?

A 5

B $5\sqrt{2}$

C 10

D $10\sqrt{2}$



44. REVIEW For which measure of θ is $\theta = \frac{\sqrt{3}}{3}$?

F 135°

G 270°

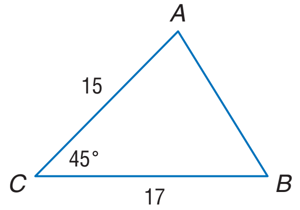
H 1080°

J 1830°

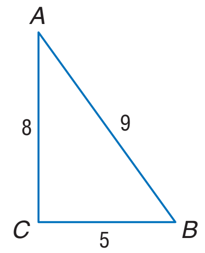
Spiral Review

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

45.



46.



Find the area of $\triangle ABC$. Round to the nearest tenth. (Lesson 13-4)

47. $a = 11$ in., $c = 5$ in., $B = 79°$

48. $b = 4$ m, $c = 7$ m, $A = 63°$

48. BULBS The lifetimes of 10,000 light bulbs are normally distributed. The mean lifetime is 300 days, and the standard deviation is 40 days. How many light bulbs will last between 260 and 340 days? (Lesson 12-7)

Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

49. $a_1 = 3, r = 1.2$

50. $16, 4, 1, \frac{1}{4}, \ldots$

51. $\sum_{n=1}^{\infty} 13(-0.625)^{n-1}$

GET READY for the Next Lesson

PREREQUISITE SKILL Find each value of θ. Round to the nearest degree. (Lesson 13-1)

52. $\sin \theta = 0.3420$

53. $\cos \theta = -0.3420$

54. $\tan \theta = 3.2709$

13-7 Inverse Trigonometric Functions

Main Ideas

- Solve equations by using inverse trigonometric functions.
- Find values of expressions involving trigonometric functions.

New Vocabulary

principal values
Arcsine function
Arccosine function
Arctangent function

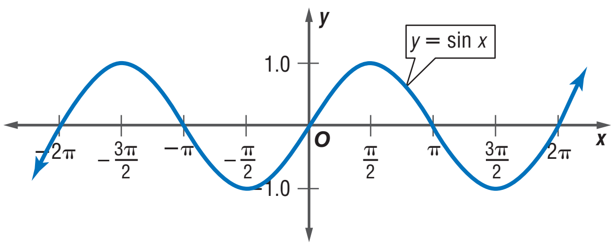
GET READY for the Lesson

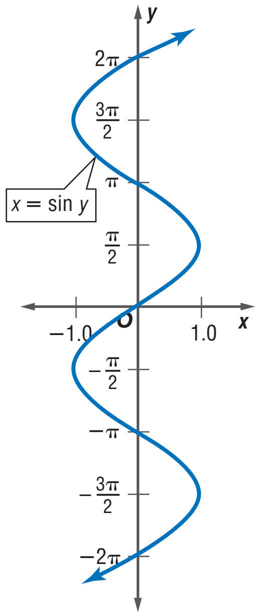
When a car travels a curve on a horizontal road, the friction between the tires and the road keeps the car on the road. Above a certain speed, however, the force of friction will not be great enough to hold the car in the curve. For this reason, civil engineers design banked curves.

The proper banking angle θ for a car making a turn of radius r feet at a velocity v in feet per second is given by the equation $\tan \theta = \frac{v^2}{32r}$. In order to determine the appropriate value of θ for a specific curve, you need to know the radius of the curve, the maximum allowable velocity of cars making the curve, and how to determine the angle θ given the value of its tangent.

Solve Equations Using Inverses Sometimes the value of a trigonometric function for an angle is known and it is necessary to find the measure of the angle. The concept of inverse functions can be applied to find the inverse of trigonometric functions.

In Lesson 8-8, you learned that the inverse of a function is the relation in which all the values of x and y are reversed. The graphs of $y = \sin x$ and its inverse, $x = \sin y$, are shown below.





Notice that the inverse is not a function, since it fails the vertical line test. None of the inverses of the trigonometric functions are functions.

We must restrict the domain of trigonometric functions so that their inverses are functions. The values in these restricted domains are called **principal values.** Capital letters are used to distinguish trigonometric functions with restricted domains from the usual trigonometric functions.

Doug Plummer/Photonica

Concepts in Motion
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KEY CONCEPT — *Principal Values of Sine, Cosine, and Tangent*

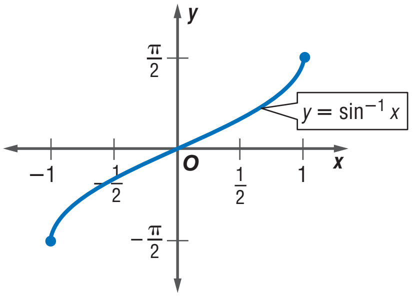
$y = \text{Sin } x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

$y = \text{Cos } x$ if and only if $y = \cos x$ and $0 \le x \le \pi$.

$y = \text{Tan } x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The inverse of the Sine function is called the **Arcsine function** and is symbolized by $\textbf{Sin}^{-1}$ or **Arcsin**. The Arcsine function has the following characteristics.

- Its domain is the set of real numbers from -1 to 1.
- Its range is the set of angle measures from $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $\text{Sin } x = y$ if and only if $\text{Sin}^{-1}\, y = x$.
- $[\text{Sin}^{-1} \circ \text{Sin}](x) = [\text{Sin} \circ \text{Sin}^{-1}](x) = x$.



Study Tip

Look Back
To review **composition and functions**, see Lesson 7-1.

The definitions of the **Arccosine** and **Arctangent** functions are similar to the definition of the Arcsine function.

CONCEPT SUMMARY — *Inverse Sine, Cosine, and Tangent*

- Given $y = \text{Sin } x$, the inverse Sine function is defined by $y = \text{Sin}^{-1}\, x$ or $y = \text{Arcsin } x$.
- Given $y = \text{Cos } x$, the inverse Cosine function is defined by $y = \text{Cos}^{-1}\, x$ or $y = \text{Arccos } x$.
- Given $y = \text{Tan } x$, the inverse Tangent function is defined by $y = \text{Tan}^{-1}\, x$ or $y = \text{Arctan } x$.

The expressions in each row of the table below are equivalent. You can use these expressions to rewrite and solve trigonometric equations.

$y = \text{Sin } x$	$x = \text{Sin}^{-1}\, y$	$x = \text{Arcsin } y$
$y = \text{Cos } x$	$x = \text{Cos}^{-1}\, y$	$x = \text{Arccos } y$
$y = \text{Tan } x$	$x = \text{Tan}^{-1}\, y$	$x = \text{Arctan } y$

EXAMPLE 1 Solve an Equation

Solve $\text{Sin } x = \frac{\sqrt{3}}{2}$ by finding the value of x to the nearest degree.

If $\text{Sin } x = \frac{\sqrt{3}}{2}$, then x is the least value whose sine is $\frac{\sqrt{3}}{2}$. So, $x = \text{Arcsin } \frac{\sqrt{3}}{2}$.

Use a calculator to find x.

KEYSTROKES: [2nd] [SIN^{-1}] [2nd] [$\sqrt{\ }$] 3 [)] [÷] 2 [)] [ENTER] 60

Therefore, $x = 60°$.

CHECK Your Progress

1. Solve $\text{Cos } x = -\frac{\sqrt{2}}{2}$ by finding the value of x to the nearest degree.

Many application problems involve finding the inverse of a trigonometric function.

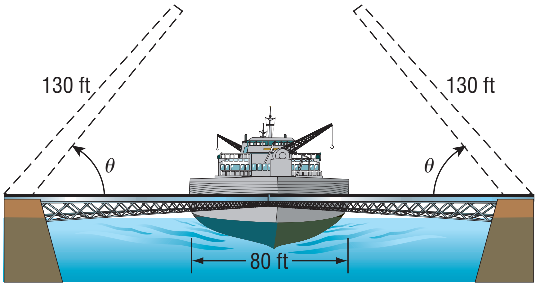
Real-World Link

Bascule bridges have spans (leaves) that pivot upward utilizing gears, motors, and counterweights.

Source: www.multnomah.lib.or.us

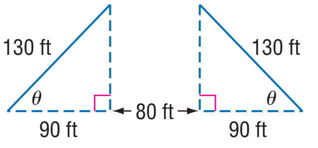
Real-World EXAMPLE Apply an Inverse to Solve a Problem

2 **DRAWBRIDGE** **Each leaf of a certain double-leaf drawbridge is 130 feet long. If an 80-foot wide ship needs to pass through the bridge, what is the minimum angle θ, to the nearest degree, which each leaf of the bridge should open so that the ship will fit?**



When the two parts of the bridge are in their lowered position, the bridge spans 130 + 130 or 260 feet. In order for the ship to fit, the distance between the leaves must be at least 80 feet.

This leaves a horizontal distance of $\frac{260 - 80}{2}$ or 90 feet from the pivot point of each leaf to the ship as shown in the diagram at the right.



To find the measure of angle θ, use the cosine ratio for right triangles.

$\cos \theta = \frac{\text{adj}}{\text{hyp}}$ Cosine ratio

$\cos \theta = \frac{90}{130}$ Replace *adj* with 90 and *hyp* with 130.

$\theta = \cos^{-1}\left(\frac{90}{130}\right)$ Inverse cosine function

$\theta \approx 46.2°$ Use a calculator.

Thus, the minimum angle each leaf of the bridge should open is 47°.

CHECK Your Progress

2. If each leaf of another drawbridge is 150 feet long, what is the minimum angle θ, to the nearest degree, that each leaf should open to allow a 90-foot-wide ship to pass?

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Study Tip

Angle Measure

Remember that when evaluating an inverse trigonometric function the result is an angle measure.

Trigonometric Values You can use a calculator to find the values of trigonometric expressions.

EXAMPLE Find a Trigonometric Value

3 **Find each value. Write angle measures in radians. Round to the nearest hundredth.**

a. ArcSin $\frac{\sqrt{3}}{2}$

KEYSTROKES: [2nd] [SIN^{-1}] [2nd] [$\sqrt{\ }$] 3 [)] [÷] 2 [)] [ENTER] 1.047197551

Therefore, ArcSin $\frac{\sqrt{3}}{2} \approx 1.05$ radians.

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b. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$

KEYSTROKES: [TAN] [2nd] [COS^{-1}] 6 [÷] 7 [)] [ENTER] 0.6009252126

Therefore, $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right) \approx 0.60$.

CHECK Your Progress

3A. $\text{Arccos}\left(\frac{\sqrt{3}}{2}\right)$ **3B.** $\cos\left(\text{Arcsin}\frac{4}{5}\right)$

CHECK Your Understanding

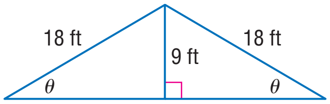
Example 1 (p. 807)

Solve each equation by finding the value of x to the nearest degree.

1. $x = \text{Cos}^{-1}\frac{\sqrt{2}}{2}$ **2.** $\text{Arctan } 0 = x$

Example 2 (p. 808)

3. ARCHITECTURE The support for a roof is shaped like two right triangles as shown at the right. Find θ.



Example 3 (pp. 808–809)

Find each value. Write degree measures in radians. Round to the nearest hundredth.

4. $\text{Tan}^{-1}\left(\frac{\sqrt{3}}{3}\right)$ **5.** $\text{Cos}^{-1}(-1)$ **6.** $\cos\left(\text{Cos}^{-1}\frac{2}{9}\right)$

7. $\sin\left(\text{Sin}^{-1}\frac{3}{4}\right)$ **8.** $\sin\left(\text{Cos}^{-1}\frac{3}{4}\right)$ **9.** $\tan\left(\text{Sin}^{-1}\frac{1}{2}\right)$

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–24	1
25–35	3
36, 37	2

Solve each equation by finding the value of x to the nearest degree.

10. $x = \text{Cos}^{-1}\frac{1}{2}$ **11.** $\text{Sin}^{-1}\frac{1}{2} = x$ **12.** $\text{Arctan } 1 = x$

13. $x = \text{Arctan}\frac{\sqrt{3}}{3}$ **14.** $x = \text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **15.** $x = \text{Cos}^{-1} 0$

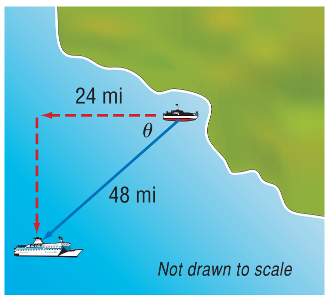
Find each value. Write angle measures in radians. Round to the nearest hundredth.

16. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$ **17.** $\text{Sin}^{-1}\frac{\pi}{2}$ **18.** $\text{Arctan}\frac{\sqrt{3}}{3}$

19. $\text{Arccos}\frac{\sqrt{3}}{2}$ **20.** $\sin\left(\text{Sin}^{-1}\frac{1}{2}\right)$ **21.** $\cot\left(\text{Sin}^{-1}\frac{5}{6}\right)$

22. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$ **23.** $\sin\left(\text{Arctan}\frac{\sqrt{3}}{3}\right)$ **24.** $\cos\left(\text{Arcsin}\frac{3}{5}\right)$

25. TRAVEL The cruise ship *Reno* sailed due west 24 miles before turning south. When the *Reno* became disabled and radioed for help, the rescue boat found that the fastest route to her covered a distance of 48 miles. The cosine of the angle at which the rescue boat should sail is 0.5. Find the angle θ, to the nearest tenth of a degree, at which the rescue boat should travel to aid the *Reno*.



EXTRA PRACTICE
See pages 922, 938.

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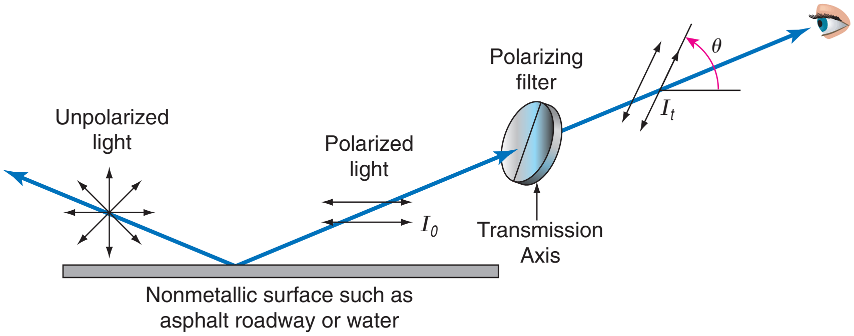
26. **OPTICS** You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose horizontally-polarized light with intensity I_0 strikes a polarizing filter with its axis at an angle of θ with the horizontal. The intensity of the transmitted light I_t and θ are related by the equation $\cos \theta = \sqrt{\frac{I_t}{I_0}}$. If one fourth of the polarized light is transmitted through the lens, what angle does the transmission axis of the filter make with the horizontal?



Real-World Link
The shot is a metal sphere that can be made out of solid iron. Shot putters stand inside a seven-foot circle and must "put" the shot from the shoulder with one hand.

Source: www.coolrunning.com.au

Find each value. Write angle measures in radians. Round to the nearest hundredth.

27. $\cot\left(\mathrm{Sin}^{-1}\frac{7}{9}\right)$

28. $\cos\left(\mathrm{Tan}^{-1}\sqrt{3}\right)$

29. $\tan(\mathrm{Arctan}\ 3)$

30. $\cos\left[\mathrm{Arccos}\left(-\frac{1}{2}\right)\right]$

31. $\mathrm{Sin}^{-1}\left(\tan\frac{\pi}{4}\right)$

32. $\cos\left(\mathrm{Cos}^{-1}\frac{\sqrt{2}}{2} - \frac{\pi}{2}\right)$

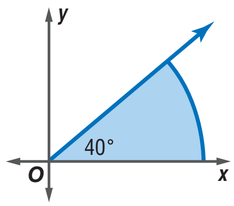
33. $\mathrm{Cos}^{-1}(\mathrm{Sin}^{-1} 90)$

34. $\sin\left(2\,\mathrm{Cos}^{-1}\frac{3}{5}\right)$

35. $\sin\left(2\,\mathrm{Sin}^{-1}\frac{1}{2}\right)$

36. **FOUNTAINS** Architects who design fountains know that both the height and distance that a water jet will project is dependent on the angle θ at which the water is aimed. For a given angle θ, the ratio of the maximum height H of the parabolic arc to the horizontal distance D it travels is given by $\frac{H}{D} = \frac{1}{4}\tan\theta$. Find the value of θ, to the nearest degree, that will cause the arc to go twice as high as it travels horizontally.

37. **TRACK AND FIELD** A shot put must land in a 40° sector. The vertex of the sector is at the origin and one side lies along the x-axis. An athlete puts the shot at a point with coordinates (18, 17), did the shot land in the required region? Explain your reasoning.



For Exercises 38–40, consider $f(x) = \mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x$.

38. Make a table of values, recording x and $f(x)$ for $x = \left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1, -\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}, -1\right\}$.

39. Make a conjecture about $f(x)$.

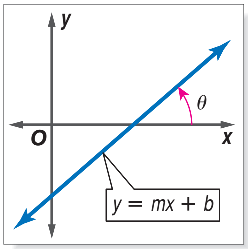
40. Considering only positive values of x, provide an explanation of why your conjecture might be true.

H.O.T. Problems

41. **OPEN ENDED** Write an equation giving the value of the Cosine function for an angle measure in its domain. Then, write your equation in the form of an inverse function.

Steven E. Sutton/PCN Photography

CHALLENGE **For Exercises 42–44, use the following information.**
If the graph of the line $y = mx + b$ intersects the x-axis such that an angle of θ is formed with the positive x-axis, then $\tan \theta = m$.



42. Find the acute angle that the graph of $3x + 5y = 7$ makes with the positive x-axis to the nearest degree.

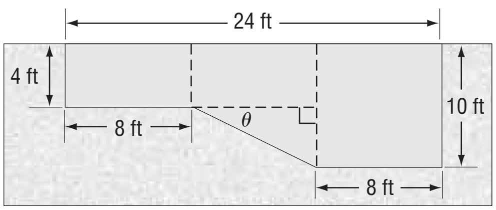
43. Determine the obtuse angle formed at the intersection of the graphs of $2x + 5y = 8$ and $6x - y = -8$. State the measure of the angle to the nearest degree.

44. Explain why this relationship, $\tan \theta = m$, holds true.

45. *Writing in Math* Use the information on page 806 to explain how inverse trigonometric functions are used in road design. Include a few sentences describing how to determine the banking angle for a road and a description of what would have to be done to a road if the speed limit were increased and the banking angle was not changed.

STANDARDIZED TEST PRACTICE

46. **ACT/SAT** To the nearest degree, what is the angle of depression θ between the shallow end and the deep end of the swimming pool?



Side View of Swimming Pool

A 25°
B 37°
C 53°
D 73°

47. **REVIEW** If $\sin \theta = 23$ and $-90° \leq \theta \leq 90°$, then $\cos (2\theta) =$

F $-\frac{1}{9}$.
G $-\frac{1}{3}$.
H $\frac{1}{3}$.
J $\frac{1}{9}$.

Spiral Review

Find the exact value of each function. (Lesson 13-6)

48. $\sin -660°$ **49.** $\cos 25\pi$ **50.** $(\sin 135°)^2 + (\cos -675°)^2$

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

51. $a = 3.1, b = 5.8, A = 30°$ **52.** $a = 9, b = 40, c = 41$

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function. (Lesson 6-7)

53. $f(x) = 5x^2 + 6x - 17$ **54.** $f(x) = -3x^2 + 2x - 1$ **55.** $f(x) = 4x^2 - 10x + 5$

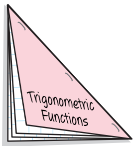
56. **PHYSICS** A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. (Lesson 5-7)

CHAPTER 13 Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Right Triangle Trigonometry (Lesson 13-1)

- $\sin \theta = \frac{\text{opp}}{\text{hyp}}$, $\cos \theta = \frac{\text{adj}}{\text{hyp}}$, $\tan \theta = \frac{\text{opp}}{\text{adj}}$,

  $\csc \theta = \frac{\text{hyp}}{\text{opp}}$, $\sec \theta = \frac{\text{hyp}}{\text{adj}}$, $\cot \theta = \frac{\text{adj}}{\text{opp}}$

Angles and Angle Measure (Lesson 13-2)

- An angle in standard position has its vertex at the origin and its initial side along the positive x-axis.
- The measure of an angle is determined by the amount of rotation from the initial side to the terminal side.

Trigonometric Functions of General Angles (Lesson 13-3)

- You can find the exact values of the six trigonometric functions of θ, given the coordinates of a point $P(x, y)$ on the terminal side of the angle.

Law of Sines and Law of Cosines (Lesson 13-4 and 13-5)

- $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- $a^2 = b^2 + c^2 - 2bc \cos A$
- $b^2 = a^2 + c^2 - 2ac \cos B$
- $c^2 = a^2 + b^2 - 2ab \cos C$

Circular and Inverse Trigonometric Functions (Lesson 13-6 and 13-7)

- If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$.
- $y = \text{Sin } x$ if $y = \sin x$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Key Vocabulary

angle of depression (p. 764)
angle of elevation (p. 764)
arccosine function (p. 807)
arcsine function (p. 807)
arctangent function (p. 807)
circular function (p. 800)
cosecant (p. 759)
cosine (p. 759)
cotangent (p. 759)
coterminal angles (p. 771)
initial side (p. 768)
law of cosines (p. 793)
law of sines (p. 786)
period (p. 801)
periodic (p. 801)
principal values (p. 806)
quadrantal angles (p. 777)
radian (p. 769)
reference angle (p. 778)
secant (p. 759)
sine (p. 759)
solve a right triangle (p. 762)
standard position (p. 768)
tangent (p. 759)
terminal side (p. 768)
trigonometric functions (p. 759)
trigonometry (p. 759)
unit circle (p. 769)

Vocabulary Check

State whether each sentence is *true* or *false*. If false, replace the underlined word(s) or number to make a true sentence.

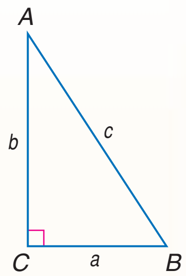
1. When two angles in standard position have the same terminal side, they are called <u>quadrantal</u> angles.
2. The <u>Law of Sines</u> is used to solve a triangle when the measure of two angles and the measure of any side are known.
3. <u>Trigonometric</u> functions can be defined by using a unit circle.
4. For all values of θ, <u>$\csc \theta$</u> $= \frac{1}{\cos \theta}$.
5. A <u>radian</u> is the measure of an angle on the unit circle where the rays of the angle intercept an arc with length 1 unit.
6. In a coordinate plane, the <u>initial</u> side of an angle is the ray that rotates about the center.

Math Online Vocabulary Review at algebra2.com

Lesson-by-Lesson Review

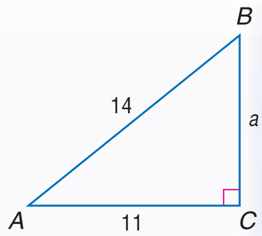
13-1 Right Triangle Trigonometry (pp. 759–767)

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



7. $c = 16, a = 7$
8. $A = 25°, c = 6$
9. $B = 45°, c = 12$
10. $B = 83°, b = \sqrt{31}$
11. $a = 9, B = 49°$
12. $\cos A = \frac{1}{4}, a = 4$
13. **SKATEBOARDING** A skateboarding ramp has an angle of elevation of 15.7°. Its vertical drop is 159 feet. Estimate the length of this ramp.

Example 1 **Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**



Find a.
$$a^2 + b^2 = c^2$$
$$a^2 + 11^2 = 14^2$$
$$a = \sqrt{14^2 - 11^2}$$
$$a \approx 8.7$$

Find A. $\cos A = \frac{11}{14}$

Use a calculator.

To the nearest degree $A \approx 38°$.

Find B.
$$38° + B \approx 90°$$
$$B \approx 52°$$

Therefore, $a \approx 8.7$, $A \approx 38°$, and $B \approx 52°$.

13-2 Angles and Angle Measure (pp. 768–774)

Rewrite each degree measure in radians and each radian measure in degrees.

14. 255°
15. −210°
16. $\frac{7\pi}{4}$
17. -4π

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

18. 205°
19. −40°
20. $\frac{4\pi}{3}$
21. $-\frac{7\pi}{4}$
22. **BICYCLING** A bicycle tire has a 12-inch radius. When riding at a speed of 18 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and nearest radian.

Example 2 **Rewrite the degree measure in radians and the radian measure in degrees.**

a. 240°
$$240° = 240\cancel{°}\left(\frac{\pi \text{ radians}}{180\cancel{°}}\right)$$
$$= \frac{240\pi}{180} \text{ radians or } \frac{4\pi}{3}$$

b. $\frac{\pi}{12}$
$$\frac{\pi}{12} = \left(\frac{\pi}{12}\ \cancel{\text{radians}}\right)\left(\frac{180°}{\pi\ \cancel{\text{radians}}}\right)$$
$$= \frac{180°}{12} \text{ or } 15°$$

CHAPTER 13

Study Guide and Review

13-3 Trigonometric Functions of General Angles (pp. 776–783)

Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

23. $P(2, 5)$ **24.** $P(15, -8)$

Find the exact value of each trigonometric function.

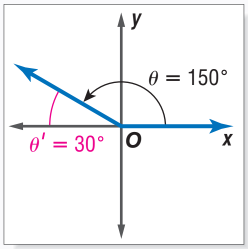
25. $\cos 3\pi$ **26.** $\tan 120°$

27. BASEBALL The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground. If the ball was hit with an initial velocity of 60 feet per second at an angle of 25°, how far was it hit?

Example 3 **Find the exact value of $\cos 150°$.**

Because the terminal side of 150° lies in Quadrant II, the reference angle θ' is $180° - 150°$ or 30°. The cosine function is negative in Quadrant II, so

$\cos 150° = -\cos 30°$ or $-\frac{\sqrt{3}}{2}$.



13-4 Law of Sines (pp. 785–792)

Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

28. $a = 24, b = 36, A = 64°$

29. $A = 40°, b = 10, a = 8$

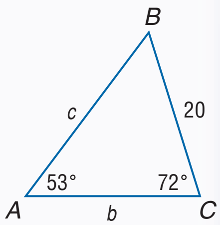
30. $b = 10, c = 15, C = 66°$

31. $A = 82°, a = 9, b = 12$

32. $A = 105°, a = 18, b = 14$

33. NAVIGATION Two fishing boats, A, and B, are anchored 4500 feet apart in open water. A plane flies at a constant speed in a straight path directly over the two boats, maintaining a constant altitude. At one point duing the flight, the angle of depression to A is 85°, and the angle of depression to B is 25°. Ten seconds later the plane has passed over A and spots B at a 35° angle of depression. How fast is the plane flying?

Example 4 **Solve $\triangle ABC$.**



First, find the measure of the third angle.

$$53° + 72° + B = 180°$$
$$B = 55°$$

Now use the law of Sines to find b and c. Write two equations, each with one variable.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin 53°}{20} = \frac{\sin 72°}{c} \qquad \frac{\sin 55°}{b} = \frac{20 \sin 53°}{20}$$

$$c = \frac{20 \sin 72°}{\sin 53°} \qquad b = \frac{20 \sin 55°}{\sin 53°}$$

$$c \approx 23.8 \qquad b \approx 20.5$$

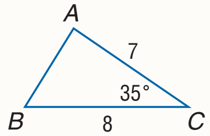
Therefore, $B = 55°$, $b \approx 20.5$, and $c \approx 23.8$.

Mixed Problem Solving
For mixed problem-solving practice, see page 938.

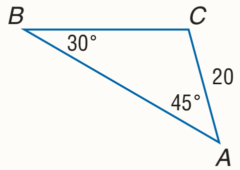
13-5 Law of Cosines (pp. 793–798)

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. 34–38. See Ch. 13 Answer Appendix.

34.



35.



36. $C = 65°, a = 4, b = 7$

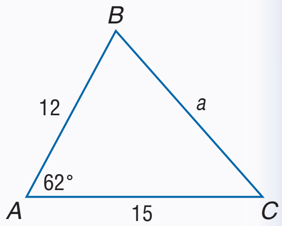
37. $A = 36°, a = 6, b = 8$

38. $b = 7.6, c = 14.1, A = 29°$

39. SURVEYING Two sides of a triangular plot of land have lengths of 320 feet and 455 feet. The measure of the angle between those sides is 54.3°. Find the perimeter of the plot. about 1148.5 ft

Example 5 $\triangle ABC$ **for** $A = 62°$, $b = 15$, **and** $c = 12$.

You are given the measure of two sides and the included angle. Begin by drawing a diagram and using the Law of Cosines to determine a.



$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 15^2 + 12^2 - 2(15)(12) \cos 62°$$

$$a^2 \approx 200$$

$$a \approx 14.1$$

Next, you can use the Law of Sines to find the measure of angle C.

$$\frac{\sin 62°}{14.1} \approx \frac{\sin C}{12}$$

$$\sin C \approx \frac{12 \sin 62°}{14.1} \text{ or about } 48.7°$$

The measure of the angle B is approximately $180 - (62 + 48.7)$ or 69.3°. Therefore, $a \approx 14.1$, $C \approx 48.7°$, $B \approx 69.3°$.

Study Guide and Review

13-6 Circular Functions (pp. 799–805)

Find the exact value of each function.

40. $\sin(-150°)$

41. $\cos 300°$

42. $(\sin 45°)(\sin 225°)$

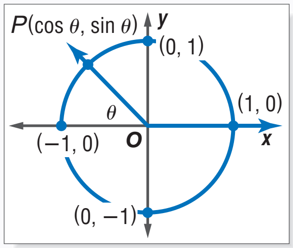
43. $\sin \frac{5\pi}{4}$

44. $(\sin 30°)^2 + (\cos 30°)^2$

45. $\frac{4\cos 150° + 2\sin 300°}{3}$

46. **FERRIS WHEELS** A Ferris wheel with a diameter of 100 feet completes 2.5 revolutions per minute. What is the period of the function that describes the height of a seat on the outside edge of the Ferris wheel as a function of time?

Example 6 Find the exact value of $\cos\left(-\frac{7\pi}{4}\right)$.



$$\cos -\frac{7\pi}{4} = \cos\left(-\frac{7\pi}{4} + 2\pi\right)$$

$$= \cos \frac{\pi}{4} \text{ or } \frac{\sqrt{2}}{2}$$

13-7 Inverse Trigonometric Functions (pp. 806–811)

Find each value. Write angle measures in radians. Round to the nearest hundredth.

47. $\text{Sin}^{-1}(-1)$

48. $\text{Tan}^{-1}\sqrt{3}$

49. $\tan\left(\text{Arcsin}\frac{3}{5}\right)$

50. $\cos(\text{Sin}^{-1} 1)$

51. **FLYWHEELS** The equation $y = \text{Arctan } 1$ describes the counterclockwise angle through which a flywheel rotates in 1 millisecond. Through how many degrees has the flywheel rotated after 25 milliseconds?

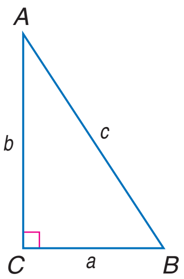
Example 7 Find the value of $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$ in radians. Round to the nearest hundredth.

KEYSTROKES: 2nd [COS^{-1}] TAN (–)
2nd [π] ÷ 6))
ENTER 2.186276035

Therefore, $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] \approx 2.19$ radians.

CHAPTER 13 Practice Test

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



1. $a = 7, A = 49°$
2. $B = 75°, b = 6$
3. $A = 22°, c = 8$
4. $a = 7, c = 16$

Rewrite each degree measure in radians and each radian measure in degrees.

5. $275°$
6. $-\frac{\pi}{6}$
7. $\frac{11\pi}{2}$
8. $330°$
9. $-600°$
10. $-\frac{7\pi}{4}$

Find the exact value of each expression. Write angle measures in degrees.

11. $\cos(-120°)$
12. $\sin \frac{7\pi}{4}$
13. $\cot 300°$
14. $\sec\left(-\frac{7\pi}{6}\right)$
15. $\text{Sin}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
16. Arctan 1
17. $\tan 135°$
18. $\csc \frac{5\pi}{6}$

19. Determine the number of possible solutions for a triangle in which $A = 40°$, $b = 10$, and $a = 14$. If a solution exists, solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

20. Determine whether $\triangle ABC$, with $A = 22°$, $a = 15$, and $b = 18$, has *no* solution, *one* solution, or *two* solutions. Then solve the triangle, if possible. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

21. Suppose θ is an angle in standard position whose terminal side lies in Quadrant II. Find the exact values of the remaining five trigonometric functions for θ for $\cos\theta = -\frac{\sqrt{3}}{2}$.

22. **GEOLOGY** From the top of the cliff, a geologist spots a dry riverbed. The measurement of the angle of depression to the riverbed is 70°. The cliff is 50 meters high. How far is the riverbed from the base of the cliff?

23. **MULTIPLE CHOICE** Triangle ABC has a right angle at C, angle $B = 30°$, and $BC = 6$. Find the area of triangle ABC.

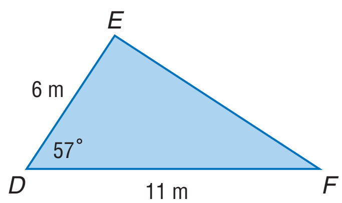
    A 6 units^2

    B $\sqrt{3}$ units^2

    C $6\sqrt{3}$ units^2

    D 12 units^2

24. Find the area of $\triangle DEF$ to the nearest tenth.



25. Determine whether $\triangle ABC$, with $b = 11$, $c = 14$, and $A = 78°$, should be solved by beginning with the Law of Sines or Law of Cosines. Then solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

CHAPTER 13

Standardized Test Practice

Cumulative, Chapters 1–13

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. If $3n + k = 30$ and n is a positive even integer, then which of the following statements must be true?

I. k is divisible by 3.
II. k is an even integer.
III. k is less than 20.

A I only

B II only

C I and II only

D I, II, and III

2. If $4x^2 + 5x = 80$ and $4x^2 - 5y = 30$, then what is the value of $6x + 6y$?

F 10

G 50

H 60

J 110

3. If $a = b + cb$, then what does $\frac{b}{a}$ equal in terms of c?

A $\frac{1}{c}$

B $\frac{1}{1 + c}$

C $1 - c$

D $1 + c$

4. GRIDDABLE What is the value of $\sum_{n=1}^{5} 3n^2$?

5. GRIDDABLE When six consecutive integers are multiplied, their product is 0. What is their greatest possible sum?

6. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

F 4

G 6

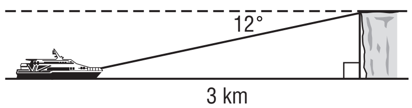
H 8

J 12

TEST-TAKING TIP

Question 6 The answer choices for multiple-choice questions can provide clues to help you solve a problem. In Question 6, you can add the values in the answer choices to the number of yellow marbles and the total number of marbles to find which is the correct answer.

7. From a lookout point on a cliff above a lake, the angle of depression to a boat on the water is 12°. The boat is 3 kilometers from the shore just below the cliff. What is the height of the cliff from the surface of the water to the lookout point?



A $\frac{3}{\sin 12°}$

B $\frac{3}{\tan 12°}$

C $\frac{3}{\cos 12°}$

D $3 \tan 12°$

8. If $x + y = 90°$ and x and y are positive, then $\frac{\cos x}{\sin y} =$

F 0.

G $\frac{1}{2}$.

H 1.

J cannot be determined

Math Online Standardized Test Practice at algebra2.com

Preparing for Standardized Tests
For test-taking strategies and more practice, see pages 941–956.

9. A child flying a kite holds the string 4 feet above the ground. The taut string is 40 feet long and makes an angle of 35° with the horizontal. How high is the kite off the ground?

A $4 + 40 \sin 35°$

B $4 + 40 \cos 35°$

C $4 + 40 \tan 35°$

D $4 + \dfrac{40}{\sin 35°}$

10. If $\sin \theta = -\frac{1}{2}$ and $180° < \theta < 270°$, then $\theta =$

F 200°.

G 210°.

H 225°.

J 240°.

11. If $\cos \theta = \frac{8}{17}$ and the terminal side of the angle is in quadrant IV, then $\sin \theta =$

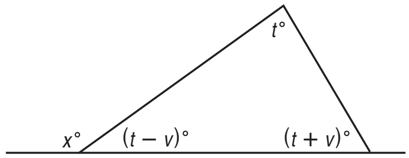
A $-\frac{15}{8}$.

B $-\frac{17}{15}$.

C $-\frac{15}{17}$.

D $-\frac{15}{17}$.

12. GRIDDABLE In the figure, if $t = 2v$, what is the value of x?



13. The variables a, b, c, d, and e are integers in a sequence, where $a = 2$ and $b = 12$. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, $c = 25$, because $2(12) = 24$, $2 - 1 = 1$, and $24 + 1 = 25$. What is the value of e?

F 74

G 144

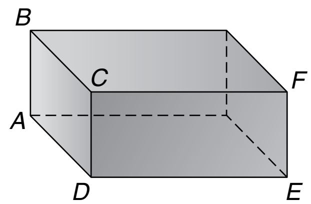
H 146

J 256

Pre-AP

Record your answers on a sheet of paper. Show your work.

14. GEOMETRY The length, width, and height of the rectangular box illustrated below are each integers greater than 1. If the area of $ABCD$ is 18 square units and the area of $CDEF$ is 21 square units, what is the volume of the box?



NEED EXTRA HELP?														
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Go to Lesson...	1-2	5-3	1-3	11-4	Prior Course	12-3	13-1	13-2	13-1	13-3	13-2	11-1	1-3	Prior Course