CHAPTER 14

Trigonometric Graphs and Identities

BIG Ideas

- Graph trigonometric functions and determine period, amplitude, phase shifts, and vertical shifts.
- Use and verify trigonometric identities.
- Solve trigonometric equations.

Key Vocabulary

amplitude (p. 823)
phase shift (p. 829)
vertical shift (p. 831)
trigonometric identity (p. 837)
trigonometric equation (p. 861)

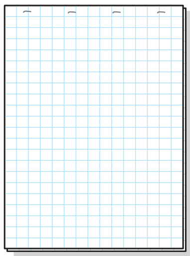
Real-World Link

Music String vibrations produce the sound you hear in stringed instruments such as guitars, violins, and pianos. These vibrations can be modeled using trigonometric functions.

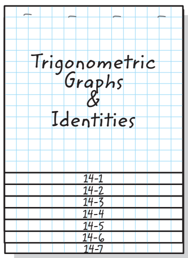
FOLDABLES Study Organizer

Trigonometric Graphs and Identities Make this Foldable to help you organize your notes. Begin with eight sheets of grid paper.

1. **Staple** the stack of grid paper along the top to form a booklet.



2. **Cut** seven lines from the bottom of the top sheet, six lines from the second sheet, and so on. Label with lesson numbers as shown.



Getty Images

GET READY for Chapter 14

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

Math Online Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Find the exact value of each trigonometric function. (Lesson 13-3)

1. $\sin 135°$
2. $\tan 315°$
3. $\cos 90°$
4. $\tan 45°$
5. $\sin \frac{5\pi}{4}$
6. $\cos \frac{7\pi}{6}$
7. $\cos(-150°)$
8. $\cot \frac{9\pi}{4}$
9. $\sec \frac{13\pi}{6}$
10. $\tan\left(-\frac{3\pi}{2}\right)$
11. $\tan \frac{8\pi}{3}$
12. $\csc(-720°)$
13. **AMUSEMENT** The distance from the highest point of a Ferris wheel to the ground can be found by multiplying 60 ft by $\sin 90°$. What is the height of the Ferris wheel at the highest point? (Lesson 13-3)

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

14. $-15x^2 - 5x$
15. $2x^4 - 4x^2$
16. $x^3 + 4$
17. $2x^2 - 3x - 2$
18. **PARKS** The rectangular wooded area of a park covers $x^2 - 6x + 8$ square feet of land. If the area is $(x - 2)$ feet long, what is the width? (Lesson 6-3)

Solve each equation by factoring. (Lesson 5-3)

19. $x^2 - 5x - 24 = 0$
20. $x^2 - 2x - 48 = 0$
21. $x^2 - 12x = 0$
22. $x^2 - 16 = 0$
23. **HOME IMPROVEMENT** You are putting new flooring in your laundry room, which is 40 square feet. The expression $x^2 + 3x$ can be used to represent the product of the length and the width of the room. Find the possible values for x. (Lesson 5-3)

QUICKReview

Example 1 **Find the exact value of $\sin \frac{11\pi}{6}$.**

The terminal side of $\frac{11\pi}{6}$ lies in Quadrant IV, so the reference angle θ is $2\pi - \frac{11\pi}{6}$ or $\frac{\pi}{6}$. The sine function is negative in the Quadrant IV.

$\sin \frac{11\pi}{6} = -\sin \frac{\pi}{6}$

$= -\sin 30°$ $\frac{\pi}{6}$ radians $= 30°$

$= -\frac{1}{2}$ $\sin 30° = \frac{1}{2}$

Example 2 **Factor $x^3 - 4x^2 - 21x$ completely.**

$x^3 - 4x^2 - 21x = x(x^2 - 4x - 21)$

The product of the coefficients of the x-terms must be -21, and their sum must be -4. The product of 7 and 3 is 21 and their difference is 4. Since the sum must be negative, the coefficients of the x-terms are -7 and 3.

$x(x^2 - 4x - 21) = x(x - 7)(x + 3)$

Example 3 **Solve the equation factored in Example 2.**

From Example 2,

$x^3 - 4x^2 - 21x = x(x - 7)(x + 3)$.

Apply the Zero Product Property and solve.

$x = 0$ or $x - 7 = 0$ or $x + 3 = 0$

$x = 7$ $x = -3$

The solution set is $\{-3, 0, 7\}$.

14-1 Graphing Trigonometric Functions

Main Ideas

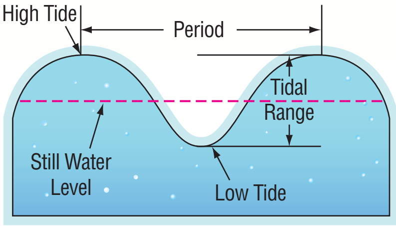
- Graph trigonometric functions.
- Find the amplitude and period of variation of the sine, cosine, and tangent functions.

New Vocabulary

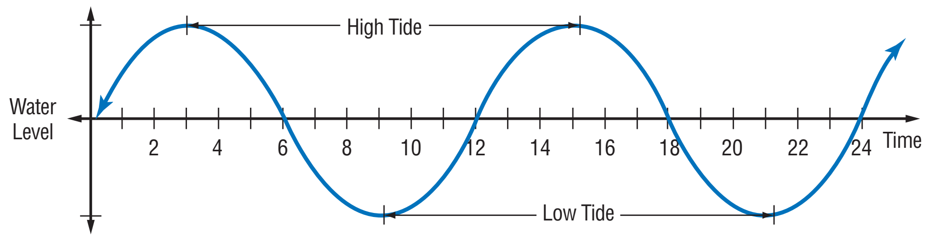
amplitude

GET READY for the Lesson

The rise and fall of tides can have great impact on the communities and ecosystems that depend upon them. One type of tide is a semidiurnal tide. This means that bodies of water, like the Atlantic Ocean, have two high tides and two low tides a day. Because tides are periodic, they behave the same way each day.



Graph Trigonometric Functions The diagram below illustrates the water level as a function of time for a body of water with semidiurnal tides.



Review Vocabulary

Period The least possible value of a for which $f(x) = f(x + a)$.

In each cycle of high and low tides, the pattern repeats itself. Recall that a function whose graph repeats a basic pattern is said to be *periodic.*

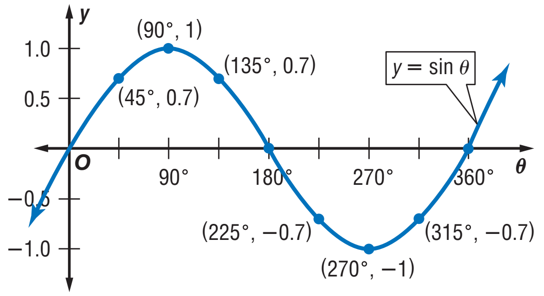
To find the period, start from any point on the graph and proceed to the right until the pattern begins to repeat. The simplest approach is to begin at the origin. Notice that after about 12 hours the graph begins to repeat. Thus, the period of the function is about 12 hours.

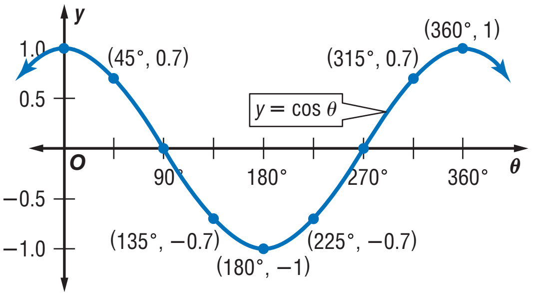
To graph the functions $y = \sin\theta$, $y = \cos\theta$, or $y = \tan\theta$, use values of θ expressed either in degrees or radians. Ordered pairs for points on these graphs are of the form $(\theta, \sin\theta)$, $(\theta, \cos\theta)$, and $(\theta, \tan\theta)$, respectively.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
nearest tenth	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
nearest tenth	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0	0.5	0.7	0.9	1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
nearest tenth	0	0.6	1	1.7	nd	-1.7	-1	-0.6	0	0.6	1	1.7	nd	-1.7	-1	-0.6	0
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

nd = not defined

After plotting several points, complete the graphs of $y = \sin \theta$ and $y = \cos \theta$ by connecting the points with a smooth, continuous curve. Recall from Chapter 13 that each of these functions has a period of 360° or 2π radians. That is, the graph of each function repeats itself every 360° or 2π radians.

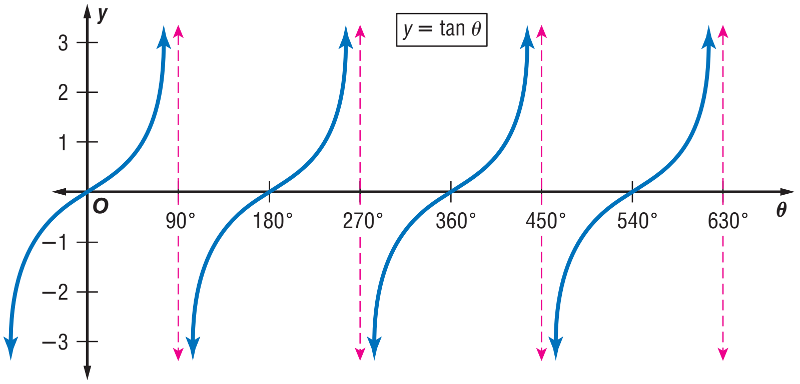




Notice that both the sine and cosine have a maximum value of 1 and a minimum value of −1. The **amplitude** of the graph of a periodic function is the absolute value of half the difference between its maximum value and its minimum value. So, for both the sine and cosine functions, the amplitude of their graphs is $\left|\frac{1-(-1)}{2}\right|$ or 1.

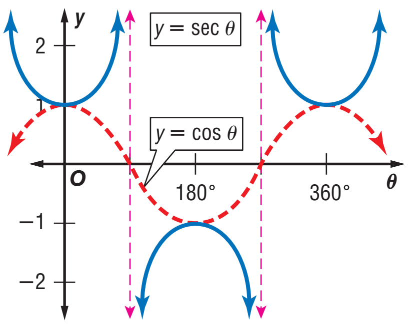
Concepts in Motion
Animation
algebra2.com

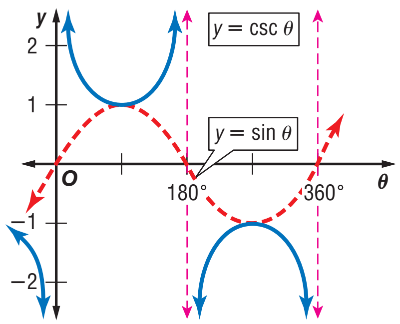
By examining the values for $\tan \theta$ in the table, you can see that the tangent function is not defined for 90°, 270°, …, 90° + $k \cdot 180°$, where k is an integer. The graph is separated by vertical asymptotes whose x-intercepts are the values for which $y = \tan \theta$ is not defined.

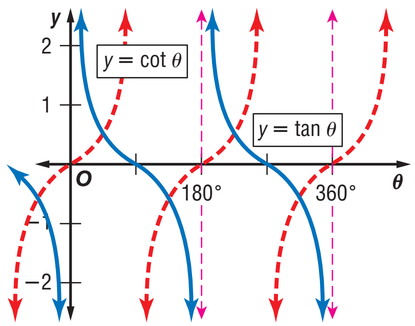


The period of the tangent function is 180° or π radians. Since the tangent function has no maximum or minimum value, it has no amplitude.

Compare the graphs of the secant, cosecant, and cotangent functions to the graphs of the cosine, sine, and tangent functions, shown below.







Notice that the period of the secant and cosecant functions is 360° or 2π radians. The period of the cotangent is 180° or π radians. Since none of these functions have a maximum or minimum value, they have no amplitude.

Variations of Trigonometric Functions Just as with other functions, a trigonometric function can be used to form a family of graphs by changing the period and amplitude.

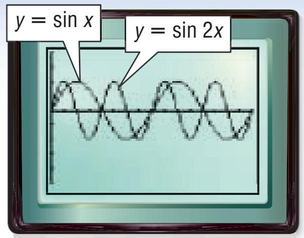
GRAPHING CALCULATOR LAB

Period and Amplitude

On a TI-83/84 Plus, set the MODE to degrees.

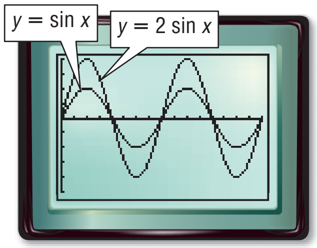
THINK AND DISCUSS

1. Graph $y = \sin x$ and $y = \sin 2x$. What is the maximum value of each function?
2. How many times does each function reach a maximum value?
3. Graph $y = \sin\left(\frac{x}{2}\right)$. What is the maximum value of this function? How many times does this function reach its maximum value?



[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

4. Use the equations $y = \sin bx$ and $y = \cos bx$. Repeat Exercises 1–3 for maximum values and the other values of b. What conjecture can you make about the effect of b on the maximum values and the periods of these functions?
5. Graph $y = \sin x$ and $y = 2 \sin x$. What is the maximum value of each function? What is the period of each function?
6. Graph $y = \frac{1}{2} \sin x$. What is the maximum value of this function? What is the period of this function?



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7. Use the equations $y = a \sin x$ and $y = a \cos x$. Repeat Exercises 5 and 6 for other values of a. What conjecture can you make about the effect of a on the amplitudes and periods of $y = a \sin x$ and $y = a \cos x$?

Study Tip

Amplitude and Period

Note that the amplitude affects the graph along the vertical axis and the period affects it along the horizontal axis.

The results of the investigation suggest the following generalization.

KEY CONCEPT — *Amplitudes and Periods*

Words For functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.
For functions of the form $y = a \tan b$, the amplitude is not defined, and the period is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.

Examples

$y = 3 \sin 4\theta$	amplitude 3 and period $\frac{360°}{4}$ or 90°
$y = -6 \cos 5\theta$	amplitude $|-6|$ or 6 and period $\frac{2\pi}{5}$
$y = 2 \tan \frac{1}{3}\theta$	no amplitude and period 3π

You can use the amplitude and period of a trigonometric function to help you graph the function.

EXAMPLE Graph Trigonometric Functions

1 Find the amplitude, if it exists, and period of each function. Then graph the function.

a. $y = \cos 3\theta$

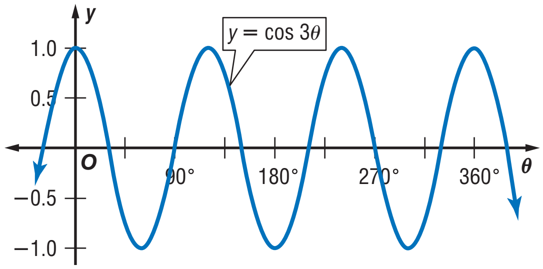
First, find the amplitude.

$|a| = |1|$ The coefficient of $\cos 3\theta$ is 1.

Next, find the period.

$$\frac{360°}{|b|} = \frac{360°}{|3|} \quad b = 3$$
$$= 120°$$

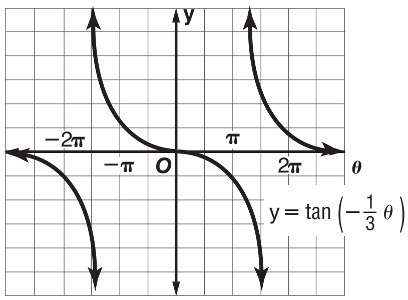
Use the amplitude and period to graph the function.



b. $y = \tan\left(-\frac{1}{3}\theta\right)$

Amplitude: This function does not have an amplitude because it has no maximum or minimum value.

$$\text{Period: } \frac{\pi}{|b|} = \frac{\pi}{\left|-\frac{1}{3}\right|}$$
$$= 3\pi$$



Study Tip

Amplitude

Notice that the graph of the longest function has no amplitude, because the tangent function has no minimum or maximum value.

CHECK Your Progress

1A. $y = \frac{1}{4}\sin\theta$

1B. $y = -2\sec\left(\frac{1}{4}\theta\right)$

Real-World EXAMPLE Use Trigonometric Functions

2 OCEANOGRAPHY Refer to the application at the beginning of the lesson. Suppose the tidal range of a city on the Atlantic coast is 18 feet. A tide is at *equilibrium* when it is at its normal level, halfway between its highest and lowest points. Write a function to represent the height h of the tide. Assume that the tide is at equilibrium at $t = 0$ and that the high tide is beginning. Then graph the function.

Since the height of the tide is 0 at $t = 0$, use the sine function $h = a \sin bt$, where a is the amplitude of the tide and t is time in hours.

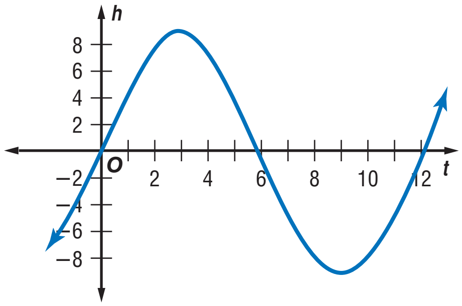
Find the amplitude. The difference between high tide and low tide is the tidal range or 18 feet.

$a = \frac{18}{2}$ or 9

Find the value of b. Each tide cycle lasts about 12 hours.

$\frac{2\pi}{|b|} = 12$ period $= \frac{2\pi}{|b|}$

$b = \frac{2\pi}{12}$ or $\frac{\pi}{6}$ Solve for b.



Thus, an equation to represent the height of the tide is $h = 9 \sin \frac{\pi}{6}t$.

Real-World Link

Lake Superior has one of the smallest tidal ranges. It can be measured in inches, while the tidal range in the Bay of Fundy in Canada measures up to 50 feet.

Source: Office of Naval Research

CHECK Your Progress

2A. Assume that the tidal range is 13 feet. Write a function to represent the height h of the tide. Assume the tide is at equilibrium at $t = 0$ and that the high tide is beginning.

2B. Graph the tide function.

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CHECK Your Understanding

Example 1 (p. 825)

Find the amplitude, if it exists, and period of each function. Then graph each function.

1. $y = \frac{1}{2} \sin \theta$ **2.** $y = 2 \sin \theta$ **3.** $y = \frac{2}{3} \cos \theta$

4. $y = \frac{1}{4} \tan \theta$ **5.** $y = \csc 2\theta$ **6.** $y = 4 \sin 2\theta$

Example 2 (p. 826)

7. $y = 4 \cos \frac{3}{4}\theta$ **8.** $y = \frac{1}{2} \sec 3\theta$ **9.** $y = \frac{3}{4} \cos \frac{1}{2}\theta$

BIOLOGY For Exercises 10 and 11, use the following information.

In a certain wildlife refuge, the population of field mice can be modeled by $y = 3000 + 1250 \sin \frac{\pi}{6}t$, where y represents the number of mice and t represents the number of months past March 1 of a given year.

10. Determine the period of the function. What does this period represent?

11. What is the maximum number of mice, and when does this occur?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–23	1
24–26	2

Find the amplitude, if it exists, and period of each function. Then graph each function.

12. $y = 3 \sin \theta$
13. $y = 5 \cos \theta$
14. $y = 2 \csc \theta$
15. $y = 2 \tan \theta$
16. $y = \frac{1}{5} \sin \theta$
17. $y = \frac{1}{3} \sec \theta$
18. $y = \sin 4\theta$
19. $y = \sin 2\theta$
20. $y = \sec 3\theta$
21. $y = \cot 5\theta$
22. $y = 4 \tan \frac{1}{3}\theta$
23. $y = 2 \cot \frac{1}{2}\theta$

MEDICINE For Exercises 24 and 25, use the following information.
Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.

24. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.

25. How do the periods of the tuning forks compare?

Find the amplitude, if it exists, and period of each function. Then graph each function.

26. $y = 6 \sin \frac{2}{3}\theta$
27. $y = 3 \cos \frac{1}{2}\theta$
28. $y = 3 \csc \frac{1}{2}\theta$
29. $y = \frac{1}{2} \cot 2\theta$
30. $2y = \tan \theta$
31. $\frac{3}{4}y = \frac{2}{3} \sin \frac{3}{5}\theta$

32. Draw a graph of a sine function with an amplitude $\frac{3}{5}$ and a period of 90°. Then write an equation for the function.

33. Draw a graph of a cosine function with an amplitude of $\frac{7}{8}$ and a period of $\frac{2\pi}{5}$. Then write an equation for the function.

34. Graph the functions $f(x) = \sin x$ and $g(x) = \cos x$, where x is measured in radians, for x between 0 and 2π. Identify the points of intersection of the two graphs.

35. Identify all asymptotes to the graph of $g(x) = \sec x$.

BOATING For Exercises 36–38, use the following information.
A marker buoy off the coast of Gulfport, Mississippi, bobs up and down with the waves. The distance between the highest and lowest point is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

36. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

37. Draw a graph showing the height of the buoy as a function of time.

38. What is the height of the buoy after 12 seconds?

EXTRA PRACTICE
See pages 922, 939.
Math Online
Self-Check Quiz at algebra2.com

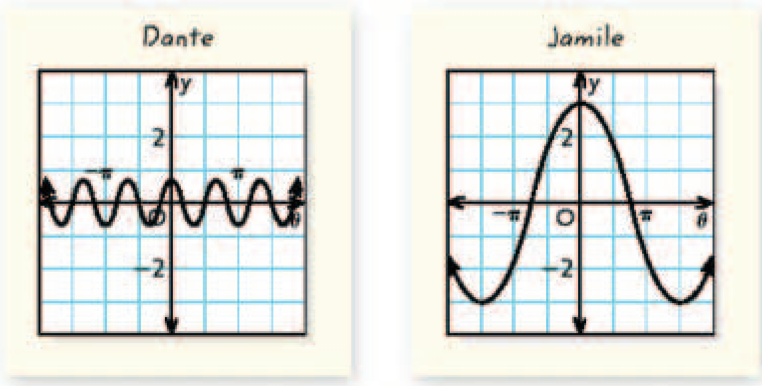
H.O.T. Problems

39. OPEN ENDED Write a trigonometric function that has an amplitude of 3 and a period of π. Graph the function.

40. REASONING Explain what it means to say that the period of a function is 180°.

41. CHALLENGE A function is called *even* if the graphs of $y = f(x)$ and $y = f(-x)$ are exactly the same. Which of the six trigonometric functions are even? Justify your answer with a graph of each function.

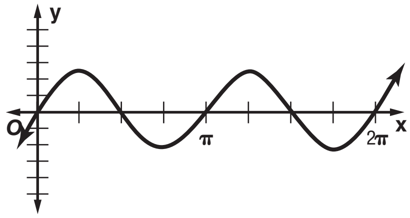
42. FIND THE ERROR Dante and Jamile graphed $y = 3 \cos \frac{2}{3}\theta$. Who is correct? Explain your reasoning.



43. ***Writing in Math*** Use the information on page 822 to explain how you can predict the behavior of tides. Explain why certain tidal characteristics follow the patterns seen in the graph of the sine function.

STANDARDIZED TEST PRACTICE

44. ACT/SAT Identify the equation of the graphed function.



A $y = \frac{1}{2} \sin 4\theta$

B $y = \frac{1}{4} \sin 2\theta$

C $y = 2 \sin \frac{1}{4}\theta$

D $y = 4 \sin \frac{1}{2}\theta$

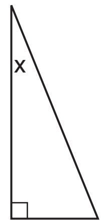
45. REVIEW Refer to the figure below. If $\tan x = \frac{10}{24}$, what are $\sin x$ and $\cos x$?

F $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$

G $\sin x = \frac{10}{26}$ and $\cos x = \frac{24}{26}$

H $\sin x = \frac{26}{10}$ and $\cos x = \frac{26}{24}$

J $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$



Spiral Review

Solve each equation. (Lesson 13-7)

46. $x = \text{Sin}^{-1} 1$

47. $\text{Arcsin}\,(-1) = y$

48. $\text{Arccos}\,\frac{\sqrt{2}}{2} = x$

Find the exact value of each function. (Lesson 13-6)

49. $\sin 390°$

50. $\sin(-315°)$

51. $\cos 405°$

52. PROBABILITY There are 8 girls and 8 boys on the Faculty Advisory Board. Three are juniors. Find the probability of selecting a boy or a girl from the committee who is not a junior. (Lesson 12-5)

53. Find the first five terms of the sequence in which $a_1 = 3$, $a_{n+1} = 2a_n + 5$. (Lesson 11-5)

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each pair of functions on the same set of axes.** (Lesson 5-7)

54. $y = x^2$, $y = 3x^2$

55. $y = 3x^2$, $y = 3x^2 - 4$

56. $y = 2x^2$, $y = 2(x + 1)^2$

14-2 Translations of Trigonometric Graphs

Main Ideas

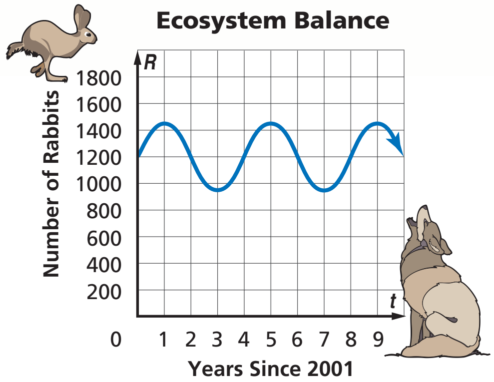
- Graph horizontal translations of trigonometric graphs and find phase shifts.
- Graph vertical translations of trigonometric graphs.

New Vocabulary

phase shift
vertical shift
midline

GET READY for the Lesson

In predator-prey ecosystems, the number of predators and the number of prey tend to vary in a periodic manner. In a certain region with coyotes as predators and rabbits as prey, the rabbit population R can be modeled by the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$, where t is the time in years since January 1, 2001.



Horizontal Translations Recall that a translation is a type of transformation in which the image is identical to the preimage in all aspects except its location on the coordinate plane. A horizontal translation shifts to the left or right, and not upward or downward.

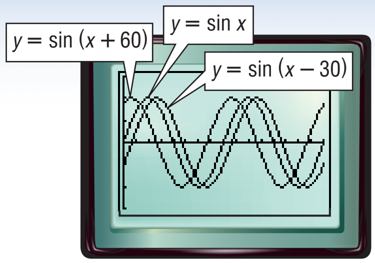
GRAPHING CALCULATOR

Horizontal Translations

On a TI-83/84 Plus, set the MODE to degrees.

THINK AND DISCUSS

1. Graph $y = \sin x$ and $y = \sin (x - 30)$. How do the two graphs compare?
2. Graph $y = \sin (x + 60)$. How does this graph compare to the other two?
3. What conjecture can you make about the effect of h in the function $y = \sin (x - h)$?
4. Test your conjecture on the following pairs of graphs.
   - $y = \cos x$ and $y = \cos (x + 30)$
   - $y = \tan x$ and $y = \tan (x - 45)$
   - $y = \sec x$ and $y = \sec (x + 75)$



[0, 720] scl: 45 by [−1.5, 1.5] scl: 0.5

Notice that when a constant is added to an angle measure in a trigonometric function, the graph is shifted to the left or to the right. If (x, y) are coordinates of $y = \sin x$, then $(x \pm h, y)$ are coordinates of $y = \sin (x \mp h)$. A horizontal translation of a trigonometric function is called a **phase shift.**

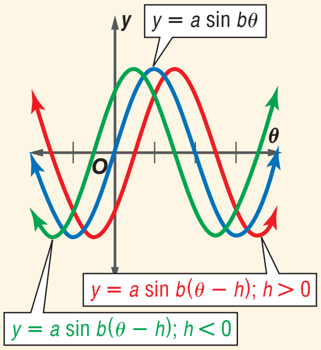
KEY CONCEPT — Phase Shift

Words The phase shift of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is h, where $b > 0$.

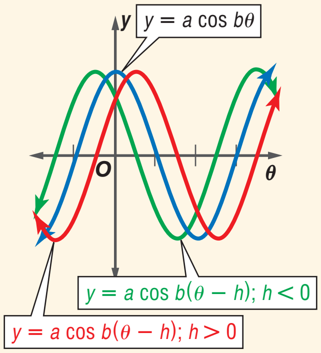
If $h > 0$, the shift is to the right.
If $h < 0$, the shift is to the left.

Models:

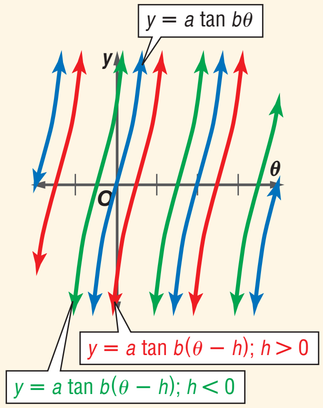
Sine



Cosine



Tangent



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The secant, cosecant, and cotangent can be graphed using the same rules.

Study Tip

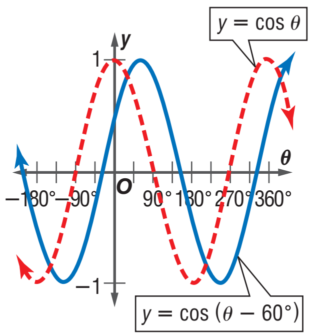
Verifying a Graph
After drawing the graph of a trigonometric function, select values of θ and evaluate them in the equation to verify your graph.

EXAMPLE Graph Horizontal Translations

1 State the amplitude, period, and phase shift for $y = \cos(\theta - 60°)$. Then graph the function.

Since $a = 1$ and $b = 1$, the amplitude and period of the function are the same as $y = \cos\theta$. However, $h = 60°$, so the phase shift is $60°$. Because $h > 0$, the parent graph is shifted to the right.

To graph $y = \cos(\theta - 60°)$, consider the graph of $y = \cos\theta$. Graph this function and then shift the graph $60°$ to the right. The graph $y = \cos(\theta - 60°)$ is the graph of $y = \cos\theta$ shifted to the right.



CHECK Your Progress

1. State the amplitude, period, and phase shift for $y = 2\sin\left(\theta + \frac{\pi}{4}\right)$. Then graph the function.

Study Tip

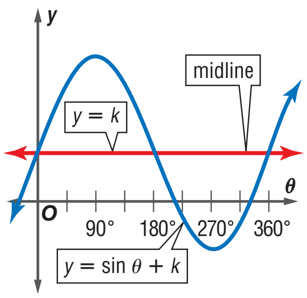
Notation

Pay close attention to trigonometric functions for the placement of parentheses. Note that $\sin(\theta + x) \neq \sin\theta + x$. The first expression represents a phase shift while the second expression represents a vertical shift.

Vertical Translations In Chapter 5, you learned that the graph of $y = x^2 + 4$ is a vertical translation of the parent graph of $y = x^2$. Similarly, graphs of trigonometric functions can be translated vertically through a **vertical shift**.

When a constant is added to a trigonometric function, the graph is shifted upward or downward. If (x, y) are coordinates of $y = \sin x$, then $(x, y \pm k)$ are coordinates of $y = \sin x \pm k$.

A new horizontal axis called the **midline** becomes the reference line about which the graph oscillates. For the graph of $y = \sin\theta + k$, the midline is the graph of $y = k$.

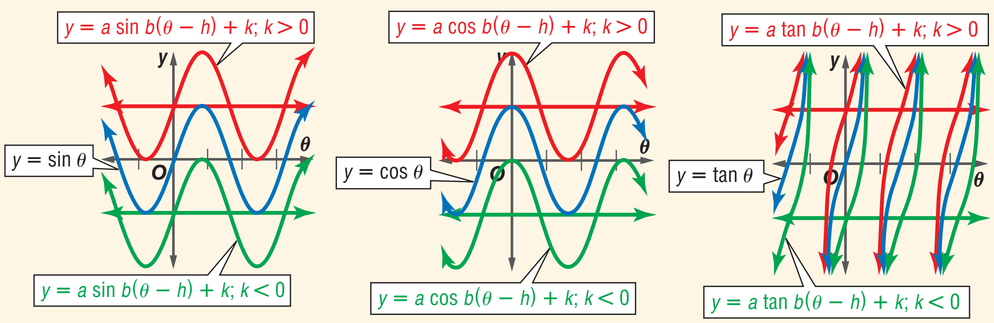


KEY CONCEPT — Vertical Shift

Words The vertical shift of the functions $y = a\sin b(\theta - h) + k$, $y = a\cos b(\theta - h) + k$, and $y = a\tan b(\theta - h) + k$ is k.

If $k > 0$, the shift is up. If $k < 0$, the shift is down. The midline is $y = k$.

Models:



The secant, cosecant, and cotangent can be graphed using the same rules.

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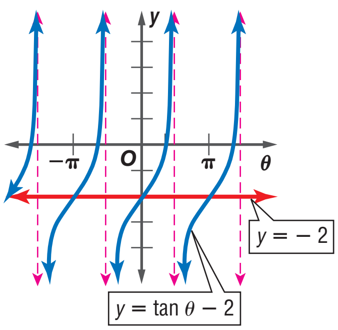
EXAMPLE Graph Vertical Translations

2 **State the vertical shift, equation of the midline, amplitude, and period for $y = \tan\theta - 2$. Then graph the function.**

Since $\tan\theta - 2 = \tan\theta + (-2)$, $k = -2$, and the vertical shift is -2. Draw the midline, $y = -2$. The tangent function has no amplitude and the period is the same as that of $\tan\theta$.

Draw the graph of the function relative to the midline.



Study Tip

Graphing

It may be helpful to first graph the parent graph $y = \sin\theta$ in one color. Then apply the vertical shift and graph the function in another color. Then apply the change in amplitude and graph the function in the final color.

CHECK Your Progress

2. State the vertical shift, equation of the midline, amplitude, and period for $y = \frac{1}{2}\sin\theta + 1$. Then graph the function.

In general, use the following steps to graph any trigonometric function.

CONCEPT SUMMARY — *Graphing Trigonometric Functions*

Step 1 Determine the vertical shift, and graph the midline.

Step 2 Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.

Step 3 Determine the period of the function and graph the appropriate function.

Step 4 Determine the phase shift and translate the graph accordingly.

EXAMPLE Graph Transformations

3 **State the vertical shift, amplitude, period, and phase shift of $y = 4\cos\left[\frac{1}{2}\left(\theta - \frac{\pi}{3}\right)\right] - 6$. Then graph the function.**

The function is written in the form $y = a\cos[b(\theta - h)] + k$. Identify the values of k, a, b, and h.

$k = -6$, so the vertical shift is -6.

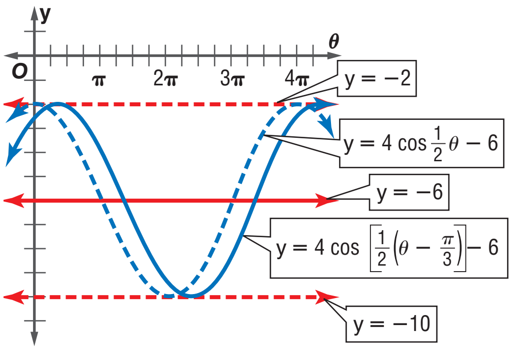
$a = 4$, so the amplitude is $|4|$ or 4.

$b = \frac{1}{2}$, so the period is $\frac{2\pi}{\left|\frac{1}{2}\right|}$ or 4π.

$h = \frac{\pi}{3}$, so the phase shift is $\frac{\pi}{3}$ to the right.

Step 1 The vertical shift is -6. Graph the midline $y = -6$.

Step 2 The amplitude is 4. Draw dashed lines 4 units above and below the midline at $y = -2$ and $y = -10$.



Step 3 The period is 4π, so the graph will be stretched. Graph $y = 4\cos\frac{1}{2}\theta - 6$ using the midline as a reference.

Step 4 Shift the graph $\frac{\pi}{3}$ to the right.

CHECK Your Progress **Graph each equation.**

3. State the vertical shift, amplitude, period, and phase shift of $y = 3\sin\left[\frac{1}{3}\left(\theta - \frac{\pi}{2}\right)\right] + 2$. Then graph the function.

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Real-World EXAMPLE

4 **HEALTH** **Suppose a person's resting blood pressure is 120 over 80. This means that the blood pressure oscillates between a maximum of 120 and a minimum of 80. If this person's resting heart rate is 60 beats per minute, write a sine function that represents the blood pressure at time *t* seconds. Then graph the function.**

Explore You know that the function is periodic and can be modeled using sine.

Plan Let P represent blood pressure and let t represent time in seconds. Use the equation $P = a \sin [b(t - h)] + k$.

Solve

- Write the equation for the midline. Since the maximum is 120 and the minimum is 80, the midline lies halfway between these values.

  $P = \frac{120 + 80}{2}$ or 100

- Determine the amplitude by finding the difference between the midline value and the maximum and minimum values.

  $a = |120 - 100|$ $\qquad$ $a = |80 - 100|$
  $\quad = |20|$ or 20 $\qquad$ $= |-20|$ or 20

  Thus, $a = 20$.

- Determine the period of the function and solve for b. Recall that the period of a function can be found using the expression $\frac{2\pi}{|b|}$.

  Since the heart rate is 60 beats per minute, there is one heartbeat, or cycle, per second. So, the period is 1 second.

  $1 = \frac{2\pi}{|b|}$ Write an equation.

  $|b| = 2\pi$ Multiply each side by $|b|$.

  $b = \pm 2\pi$ Solve.

  For this example, let $b = 2\pi$. The use of the positive or negative value depends upon whether you begin a cycle with a maximum value (positive) or a minimum value (negative).

- There is no phase shift, so $h = 0$. So, the equation is $P = 20 \sin 2\pi t + 100$.

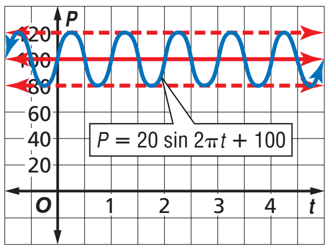
- Graph the function.

  Step 1 Draw the midline $P = 100$.

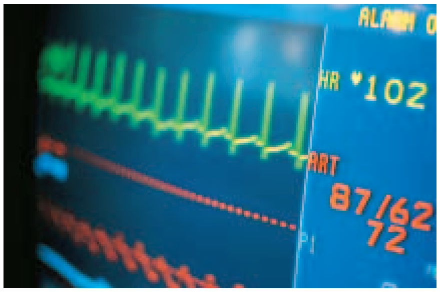
  Step 2 Draw maximum and minimum reference lines.

  Step 3 Use the period to draw the graph of the function.

  Step 4 There is no phase shift.



Check Notice that each cycle begins at the midline, rises to 120, drops to 80, and then returns to the midline. This represents the blood pressure of 120 over 80 for one heartbeat. Since each cycle lasts 1 second, there will be 60 cycles, or heartbeats, in 1 minute. Therefore, the graph accurately represents the information.



Real-World Link

Blood pressure can change from minute to minute and can be affected by the slightest of movements, such as tapping your fingers or crossing your arms.

Source: American Heart Association

Ben Edwards/Getty Images

CHECK Your Progress

4. Suppose that while doing some moderate physical activity, the person's blood pressure is 130 over 90 and that the person has a heart rate of 90 beats per minute. Write a sine function that represents the person's blood pressure at time t seconds. Then graph the function.

CHECK Your Understanding

Example 1 (p. 830)

State the amplitude, period, and phase shift for each function. Then graph the function.

1. $y = \sin\left(\theta - \frac{\pi}{2}\right)$
2. $y = \tan(\theta + 60°)$
3. $y = \cos(\theta - 45°)$
4. $y = \sec\left(\theta + \frac{\pi}{3}\right)$

Example 2 (pp. 831–832)

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

5. $y = \cos\theta + \frac{1}{4}$
6. $y = \sec\theta - 5$
7. $y = \tan\theta + 4$
8. $y = \sin\theta + 0.25$

Example 3 (p. 832)

State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

9. $y = 3\sin[2(\theta - 30°)] + 10$
10. $y = 2\cot(3\theta + 135°) - 6$
11. $y = \frac{1}{2}\sec\left[4\left(\theta - \frac{\pi}{4}\right)\right] + 1$
12. $y = \frac{2}{3}\cos\left[\frac{1}{2}\left(\theta + \frac{\pi}{6}\right)\right] - 2$

Example 4 (p. 833)

PHYSICS For Exercises 13–15, use the following information.

A weight is attached to a spring and suspended from the ceiling. At equilibrium, the weight is located 4 feet above the floor. The weight is pulled down 1 foot and released.

13. Determine the vertical shift, amplitu de, and period of a function that represents the height of the weight above the floor if the weight returns to its lowest position every 4 seconds.
14. Write the equation for the height h of the weight above the floor as a function of time t seconds.
15. Draw a graph of the function you wrote in Exercise 14.

Exercises

HOMEWORK HELP

For Exercises	See Examples
16–21	1
22–27	2
28–35	3
36–38	4

State the amplitude, period, and phase shift for each function. Then graph the function.

16. $y = \cos(\theta + 90°)$
17. $y = \cot(\theta - 30°)$
18. $y = \sin\left(\theta - \frac{\pi}{4}\right)$
19. $y = \cos\left(\theta + \frac{\pi}{3}\right)$
20. $y = \frac{1}{4}\tan(\theta + 22.5°)$
21. $y = 3\sin(\theta - 75°)$

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

22. $y = \sin\theta - 1$
23. $y = \sec\theta + 2$
24. $y = \cos\theta - 5$
25. $y = \csc\theta - \frac{3}{4}$
26. $y = \frac{1}{2}\sin\theta + \frac{1}{2}$
27. $y = 6\cos\theta + 1.5$

State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

28. $y = 2 \sin [3(\theta - 45°)] + 1$

29. $y = 4 \cos [2(\theta + 30°)] - 5$

30. $y = 3 \csc \left[\frac{1}{2}(\theta + 60°)\right] - 3.5$

31. $y = 6 \cot \left[\frac{2}{3}(\theta - 90°)\right] + 0.75$

32. $y = \frac{1}{4} \cos (2\theta - 150°) + 1$

33. $y = \frac{2}{5} \tan (6\theta + 135°) - 4$

34. $y = 3 + 2 \sin \left[\left(2\theta + \frac{\pi}{4}\right)\right]$

35. $y = 4 + 5 \sec \left[\frac{1}{3}\left(\theta + \frac{2\pi}{3}\right)\right]$

ZOOLOGY **For Exercises 36–38, use the following information.**

The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of owls O can be represented by $O = 150 + 30 \sin \left(\frac{\pi}{10}t\right)$ where t is the time in years since January 1, 2001. In that same system, the population of mice M can be represented by $M = 600 + 300 \sin \left(\frac{\pi}{10}t + \frac{\pi}{20}\right)$.

Real-World Link

The average weight of a male Cactus Ferruginous Pygmy-Owl is 2.2 ounces.

Source: www.kidsplanet.org

36. Find the maximum number of owls. After how many years does this occur?

37. What is the minimum number of mice? How long does it take for the population of mice to reach this level?

38. Why would the maximum owl population follow behind the population of mice?

39. Graph $y = 3 - \frac{1}{2} \cos \theta$ and $y = 3 + \frac{1}{2} \cos (\theta + \pi)$. How do the graphs compare?

40. Compare the graphs of $y = -\sin \left[\frac{1}{4}\left(\theta - \frac{\pi}{2}\right)\right]$ and $y = \cos \left[\frac{1}{4}\left(\theta + \frac{3\pi}{2}\right)\right]$.

41. Graph $y = 5 + \tan \left(\theta + \frac{\pi}{4}\right)$. Describe the transformation to the parent graph $y = \tan \theta$.

42. Draw a graph of the function $y = \frac{2}{3}\cos (\theta - 50°) + 2$. How does this graph compare to the graph of $y = \cos \theta$?

43. **MUSIC** When represented on oscilloscope, the note A above middle C has a period of $\frac{1}{440}$. Which of the following can be an equation for an oscilloscope graph of this note? The amplitude of the graph is K.

a. $y = K \sin 220\pi t$ **b.** $y = K \sin 440\pi t$ **c.** $y = K \sin 880\pi t$

EXTRA PRACTICE
See pages 922, 939.

Math Online
Self-Check Quiz at algebra2.com

44. **TIDES** The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M. and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height h of the water t hours after noon on the first day.

H.O.T. Problems

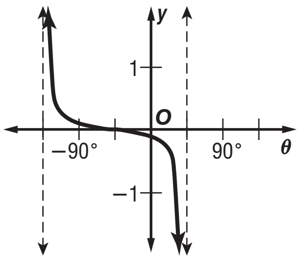
45. **OPEN ENDED** Write the equation of a trigonometric function with a phase shift of $-45°$. Then graph the function, and its parent graph.

46. **CHALLENGE** The graph of $y = \cot \theta$ is a transformation of the graph of $y = \tan \theta$. Determine a, b, and h so that $\cot \theta = a \tan [b(\theta - h)]$ for all values of θ for which each function is defined.

47. *Writing in Math* Use the information on page 829 to explain how translations of trigonometric graphs can be used to show animal populations. Include a description of what each number in the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$ represents.

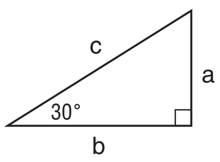
STANDARDIZED TEST PRACTICE

48. ACT/SAT Which equation is represented by the graph?



A $y = \cot(\theta + 45°)$
B $y = \cot(\theta - 45°)$
C $y = \tan(\theta + 45°)$
D $y = \tan(\theta - 45°)$

49. REVIEW Refer to the figure below. If $c = 14$, find the value of b.



F $\frac{\sqrt{3}}{2}$
G $14\sqrt{3}$
H 7
J $7\sqrt{3}$

Spiral Review

Find the amplitude, if it exists, and period of each function. Then graph each function. (Lesson 14-1)

50. $y = 3 \csc \theta$ **51.** $y = \sin \frac{\theta}{2}$ **52.** $y = 3 \tan \frac{2}{3}\theta$

Find each value. (Lesson 13-7)

53. $\sin\left(\text{Cos}^{-1} \frac{2}{3}\right)$ **54.** $\cos\left(\text{Cos}^{-1} \frac{4}{7}\right)$ **55.** $\text{Sin}^{-1}\left(\sin \frac{5}{6}\right)$ **56.** $\cos\left(\text{Tan}^{-1} \frac{3}{4}\right)$

57. GEOMETRY Find the total number of diagonals that can be drawn in a decagon. (Lesson 12-2)

Solve each equation. Round to the nearest hundredth. (Lesson 9-4)

58. $4^x = 24$ **59.** $4.3^{3x+1} = 78.5$ **60.** $7^{x-2} = 53^{-x}$

Simplify each expression. (Lesson 8-4)

61. $\frac{3}{a-2} + \frac{2}{a-3}$ **62.** $\frac{w+12}{4w-16} - \frac{w+4}{2w-8}$ **63.** $\frac{3y+1}{2y-10} + \frac{1}{y^2-2y-15}$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find the value of each function.** (Lessons 13-3)

64. $\cos 150°$ **65.** $\tan 135°$ **66.** $\sin \frac{3\pi}{2}$ **67.** $\cos\left(-\frac{\pi}{3}\right)$

68. $\sin(-\pi)$ **69.** $\tan\left(-\frac{5\pi}{6}\right)$ **70.** $\cos 225°$ **71.** $\tan 405°$

14-3 Trigonometric Identities

Main Ideas

- Use identities to find trigonometric values.
- Use trigonometric identities to simplify expressions.

New Vocabulary

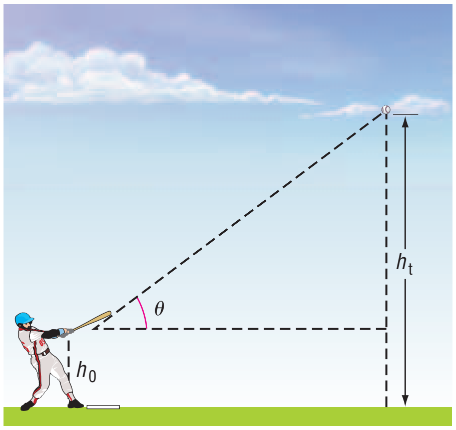
trigonometric identity

GET READY for the Lesson

A model for the height of a baseball after it is hit as a function of time can be determined using trigonometry. If the ball is hit with an initial velocity of v feet per second at an angle of θ from the horizontal, then the height h of the ball after t seconds can be represented by

$$h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0,$$

where h_0 is the height of the ball in feet the moment it is hit.



Find Trigonometric Values In the equation above, the second term $\left(\frac{\sin\theta}{\cos\theta}\right)t$ can also be written as $(\tan\theta)t$. $\left(\frac{\sin\theta}{\cos\theta}\right)t = (\tan\theta)t$ is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

The identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is true except for angle measures such as 90°, 270°, 450°, …, $90° + 180° \cdot k$. The cosine of each of these angle measures is 0, so none of the expressions tan 90°, tan 270°, tan 450°, and so on, are defined. An identity similar to this is $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

These identities are sometimes called *quotient identities*. These and other basic trigonometric identities are listed below.

KEY CONCEPT — *Basic Trigonometric Identities*

Quotient Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$	
Reciprocal Identities	$\csc\theta = \frac{1}{\sin\theta}$ $\sin\theta \neq 0$	$\sec\theta = \frac{1}{\cos\theta}$ $\cos\theta \neq 0$	$\cot\theta = \frac{1}{\tan\theta}$ $\tan\theta \neq 0$
Pythagorean Identities	$\cos^2\theta + \sin^2\theta = 1$ $\cot^2\theta + 1 = \csc^2\theta$	$\tan^2\theta + 1 = \sec^2\theta$	

You can use trigonometric identities to find values of trigonometric functions.

EXAMPLE 1 Find a Value of a Trigonometric Function

a. Find $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $90° < \theta < 180°$.

$\cos^2\theta + \sin^2\theta = 1$ — Trigonometric identity

$\cos^2\theta = 1 - \sin^2\theta$ — Subtract $\sin^2\theta$ from each side.

$\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2$ — Substitute $\frac{3}{5}$ for $\sin\theta$.

$\cos^2\theta = 1 - \frac{9}{25}$ — Square $\frac{3}{5}$.

$\cos^2\theta = \frac{16}{25}$ — Subtract.

$\cos\theta = \pm\frac{4}{5}$ — Take the square root of each side.

Since θ is in the second quadrant, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{4}{5}$.

b. Find $\csc\theta$ if $\cot\theta = -\frac{1}{4}$ and $270° < \theta < 360°$.

$\cot^2\theta + 1 = \csc^2\theta$ — Trigonometric identity

$\left(-\frac{1}{4}\right)^2 + 1 = \csc^2\theta$ — Substitute $-\frac{1}{4}$ for $\cot\theta$.

$\frac{1}{16} + 1 = \csc^2\theta$ — Square $-\frac{1}{4}$.

$\frac{17}{16} = \csc^2\theta$ — Add.

$\pm\frac{\sqrt{17}}{4} = \csc\theta$ — Take the square root of each side.

Since θ is in the fourth quadrant, $\csc\theta$ is negative. Thus, $\csc\theta = -\frac{\sqrt{17}}{4}$.

CHECK Your Progress

1A. Find $\sin\theta$ if $\cos\theta = \frac{1}{3}$ and $270° < \theta < 360°$.

1B. Find $\sec\theta$ if $\sin\theta = -\frac{2}{7}$ and $180° < \theta < 270°$.

SIMPLIFY EXPRESSIONS Trigonometric identities can also be used to simplify expressions containing trigonometric functions. Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

Study Tip

It is often easiest to write all expressions in terms of sine and/or cosine.

EXAMPLE 2 Simplify an Expression

Simplify $\frac{\csc^2\theta - \cot^2\theta}{\cos\theta}$.

$$\frac{\csc^2\theta - \cot^2\theta}{\cos\theta} = \frac{\frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}{\cos\theta}$$ — $\csc^2\theta = \frac{1}{\sin^2\theta}$, $\cot^2\theta = \frac{\cos^2\theta}{\sin^2\theta}$

$$= \frac{\frac{1-\cos^2\theta}{\sin^2\theta}}{\cos\theta}$$ — Add.

Math Online Extra Examples at algebra2.com

$$= \frac{\frac{\sin^2 \theta}{\sin^2 \theta}}{\cos \theta} \qquad 1 - \cos^2 \theta = \sin^2 \theta$$

$$= \frac{1}{\cos \theta} \qquad \frac{\sin^2 \theta}{\sin^2 \theta} = 1$$

$$= \sec \theta \qquad \frac{1}{\cos \theta} = \sec \theta$$

CHECK Your Progress **Simplify each expression.**

2A. $\dfrac{\tan^2 \theta \csc^2 \theta - 1}{\sec^2 \theta}$

2B. $\dfrac{\sec \theta}{\sin \theta}(1 - \cos^2 \theta)$

EXAMPLE 3 Simplify and Use an Expression

BASEBALL **Refer to the application at the beginning of the lesson. Rewrite the equation in terms of tan θ.**

$$h = \left(\frac{-16}{v^2\cos^2 \theta}\right)t^2 + \left(\frac{\sin \theta}{\cos \theta}\right)t + h_0 \qquad \text{Original equation}$$

$$= -\frac{16}{v^2}\left(\frac{1}{\cos^2 \theta}\right)t^2 + \left(\frac{\sin \theta}{\cos \theta}\right)t + h_0 \qquad \text{Factor.}$$

$$= -\frac{16}{v^2}\left(\frac{1}{\cos^2 \theta}\right)t^2 + (\tan \theta)t + h_0 \qquad \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$= -\frac{16}{v^2}(\sec^2 \theta)t^2 + (\tan \theta)t + h_0 \qquad \text{Since } \frac{1}{\cos \theta} = \sec \theta, \frac{1}{\cos^2 \theta} = \sec^2 \theta.$$

$$= -\frac{16}{v^2}(1 + \tan^2 \theta)t^2 + (\tan \theta)t + h_0 \qquad \sec^2 \theta = 1 + \tan^2 \theta$$

Thus, $\left(\dfrac{-16}{v^2\cos^2 \theta}\right)t^2 + \left(\dfrac{\sin \theta}{\cos \theta}\right)t + h_0 = -\dfrac{16}{v^2}(1 + \tan^2 \theta)t^2 + (\tan \theta)t + h_0$.

CHECK Your Progress

3. Rewrite the expression $\cot^2 \theta - \tan^2 \theta$ in terms of $\sin \theta$.

CHECK Your Understanding

Example 1 (p. 838)

Find the value of each expression.

1. $\tan \theta$, if $\sin \theta = \frac{1}{2}$; $90° \le \theta < 180°$

2. $\csc \theta$, if $\cos \theta = -\frac{3}{5}$; $180° \le \theta < 270°$

3. $\cos \theta$, if $\sin \theta = \frac{4}{5}$; $0° \le \theta < 90°$

4. $\sec \theta$, if $\tan \theta = -1$; $270° < \theta < 360°$

Example 2 (pp. 838–839)

Simplify each expression.

5. $\csc \theta \cos \theta \tan \theta$

6. $\sec^2 \theta - 1$

7. $\dfrac{\tan \theta}{\sin \theta}$

8. $\sin \theta (1 + \cot^2 \theta)$

Example 3 (p. 839)

9. PHYSICAL SCIENCE When a person moves along a circular path, the body leans away from a vertical position. The nonnegative acute angle that the body makes with the vertical is called the *angle of inclination* and is represented by the equation $\tan \theta = \frac{v^2}{gR}$, where R is the radius of the circular path, v is the speed of the person in meters per second, and g is the acceleration due to gravity, 9.8 meters per second squared. Write an equivalent expression using $\sin \theta$ and $\cos \theta$.

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–17	1
18–26	2
27, 28	3

Find the value of each expression.

10. $\tan\theta$, if $\cot\theta = 2$; $0° \le \theta < 90°$

11. $\sin\theta$, if $\cos\theta = \frac{2}{3}$; $0° \le \theta < 90°$

12. $\sec\theta$, if $\tan\theta = -2$; $90° < \theta < 180°$

13. $\tan\theta$, if $\sec\theta = -3$; $180° < \theta < 270°$

14. $\csc\theta$, if $\cos\theta = -\frac{3}{5}$; $90° < \theta < 180°$

15. $\cos\theta$, if $\sec\theta = \frac{5}{3}$; $270° < \theta < 360°$

16. $\cos\theta$, if $\sin\theta = \frac{1}{2}$; $0° \le \theta < 90°$

17. $\csc\theta$, if $\cos\theta = -\frac{2}{3}$; $180° < \theta < 270°$

Simplify each expression.

18. $\cos\theta\csc\theta$

19. $\tan\theta\cot\theta$

20. $\sin\theta\cot\theta$

21. $\cos\theta\tan\theta$

22. $2(\csc^2\theta - \cot^2\theta)$

23. $3(\tan^2\theta - \sec^2\theta)$

24. $\dfrac{\cos\theta\csc\theta}{\tan\theta}$

25. $\dfrac{\sin\theta\csc\theta}{\cot\theta}$

26. $\dfrac{1-\cos^2\theta}{\sin^2\theta}$

ELECTRONICS **For Exercises 27 and 28, use the following information.**

When an alternating current of frequency f and a peak current I pass through a resistance R, then the power delivered to the resistance at time t seconds is $P = I^2R - I^2R\cos^2 2ft\pi$.

27. Write an expression for the power in terms of $\sin^2 2ft\pi$.

28. Write an expression for the power in terms of $\tan^2 2ft\pi$.

Find the value of each expression.

29. $\tan\theta$, if $\cos\theta = \frac{4}{5}$; $0° \le \theta < 90°$

30. $\cos\theta$, if $\csc\theta = -\frac{5}{3}$; $270° < \theta < 360°$

31. $\sec\theta$, if $\sin\theta = \frac{3}{4}$; $90° < \theta < 180°$

32. $\sin\theta$, if $\tan\theta = 4$; $180° < \theta < 270°$

Simplify each expression.

33. $\dfrac{1-\sin^2\theta}{\sin^2\theta}$

34. $\dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}$

35. $\dfrac{\tan^2\theta - \sin^2\theta}{\tan^2\theta\sin^2\theta}$

Real-World Link

The oldest operational carousel in the United States is the Flying Horse Carousel at Martha's Vineyard, Massachusetts.

Source: Martha's Vineyard Preservation Trust

AMUSEMENT PARKS **For Exercises 36–38, use the following information.**

Suppose a child is riding on a merry-go-round and is seated on an outside horse. The diameter of the merry-go-round is 16 meters.

36. Refer to Exercise 9. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child?

37. What is the velocity of the merry-go-round?

38. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

LIGHTING **For Exercises 39 and 40, use the following information.**

The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface is related to the distance R in feet from the light source. The formula $\sec\theta = \frac{I}{ER^2}$, where I is the intensity of the light source measured in candles and θ is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important.

39. Solve the formula in terms of E.

40. Is the equation in Exercise 39 equivalent to $R^2 = \frac{I\tan\theta\cos\theta}{E}$? Explain.

EXTRA PRACTICE

See pages 923, 939.

Math Online

Self-Check Quiz at algebra2.com

H.O.T. Problems

41. REASONING Describe how you can determine the quadrant in which the terminal side of angle α lies if $\sin \alpha = -\frac{1}{4}$.

42. OPEN ENDED Write two expressions that are equivalent to $\tan \theta \sin \theta$.

43. REASONING If $\cot(x) = \cot\left(\frac{\pi}{3}\right)$ and $3\pi < x < 4\pi$, find x.

44. CHALLENGE If $\tan \beta = \frac{3}{4}$, find $\frac{\sin \beta \sec \beta}{\cot \beta}$.

45. *Writing in Math* Use the information on page 837 to explain how trigonometry can be used to model the path of a baseball. Include an explanation of why the equation at the beginning of the lesson is the same as $y = -\frac{16 \sec^2 \theta}{v^2}x^2 + (\tan \theta)x + h_0$.

STANDARDIZED TEST PRACTICE

46. ACT/SAT If $\sin x = m$ and $0 < x < 90°$, then $\tan x =$

A $\frac{1}{m^2}$.

B $\frac{1 - m^2}{m}$.

C $\frac{m}{\sqrt{1 - m^2}}$.

D $\frac{m}{1 - m^2}$.

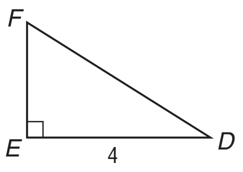
47. REVIEW Refer to the figure below. If $\cos D = 0.8$, what is length $\overline{DF}$?

F 5

G 4

H 3.2

J $\frac{4}{5}$



Spiral Review

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function. (Lesson 14-2)

48. $y = \sin \theta - 1$

49. $y = \tan \theta + 12$

Find the amplitude, if it exists, and period of each function. Then graph each function. (Lesson 14-1)

50. $y = \csc 2\theta$

51. $y = \cos 3\theta$

52. $y = \frac{1}{3} \cot 5\theta$

53. Find the sum of a geometric series for which $a_1 = 48$, $a_n = 3$, and $r = \frac{1}{2}$. (Lesson 11-4)

54. Write an equation of a parabola with focus at $(11, -1)$ and directrix $y = 2$. (Lesson 10-2)

55. TEACHING Ms. Granger has taught 288 students at this point in her career. If she has 30 students each year from now on, the function $S(t) = 288 + 30t$ gives the number of students $S(t)$ she will have taught after t more years. How many students will she have taught after 7 more years? (Lesson 2-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Name the property illustrated by each statement.** (Lesson1-3)

56. If $4 + 8 = 12$, then $12 = 4 + 8$.

57. If $7 + s = 21$, then $s = 14$.

58. If $4x = 16$, then $12x = 48$.

59. If $q + (8 + 5) = 32$, then $q + 13 = 32$.

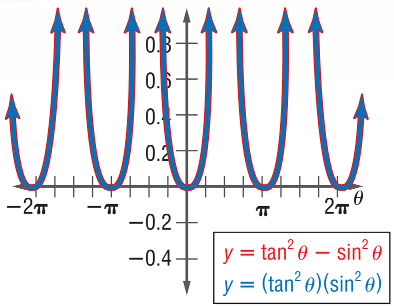
14-4 Verifying Trigonometric Identities

Main Ideas

- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

GET READY for the Lesson

Examine the graphs at the right. Recall that when the graphs of two functions coincide, the functions are equivalent. However, the graphs only show a limited range of solutions. It is not sufficient to show some values of θ and conclude that the statement is true for all values of θ. In order to show that the equation $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$ for all values of θ, you must consider the general case.



Transform One Side of an Equation You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. For example, if you wish to show that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$ is an identity, you need to show that it is true for all values of θ.

Verifying an identity is like checking the solution of an equation. You must simplify one or both sides of an equation *separately* until they are the same. In many cases, it is easier to work with only one side of an equation. You may choose either side, but it is often easier to begin with the more complicated side of the equation. Transform that expression into the form of the simpler side.

Study Tip

Common Misconception
You cannot perform operations to the quantities from each side of an unverified identity as you do with equations. Until an identity is verified it is not considered an equation, so the properties of equality do not apply.

EXAMPLE 1 Transform One Side of an Equation

Verify that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$ is an identity.

Transform the left side.

$$\tan^2\theta - \sin^2\theta \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Original equation}$$

$$\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} - \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Rewrite using the LCD, } \cos^2\theta.$$

$$\frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Subtract.}$$

$$\frac{\sin^2\theta\,(1 - \cos^2\theta)}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Factor.}$$

$$\frac{\sin^2\theta\sin^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \qquad 1-\cos^2\theta=\sin^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\sin^2\theta}{1} \stackrel{?}{=} \tan^2\theta\sin^2\theta \qquad \frac{ab}{c}=\frac{a}{c}\cdot\frac{b}{1}$$

$$\tan^2\theta\sin^2\theta = \tan^2\theta\sin^2\theta \qquad \frac{\sin^2\theta}{\cos^2\theta}=\tan\theta$$

CHECK Your Progress

1. Verify that $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$ is an identity.

STANDARDIZED TEST EXAMPLE **Find an Equivalent Expression**

2 $\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\cot\theta}\right)=$

A $\cos\theta$ **B** $\sin\theta$ **C** $\cos^2\theta$ **D** $\sin^2\theta$

Read the Test Item

Find an expression that is equal to the given expression.

Solve the Test Item

Transform the given expression to match one of the choices.

$$\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\cot\theta}\right)=\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}}\right) \qquad \cot\theta=\frac{\cos\theta}{\sin\theta}$$

$$=\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta\sin\theta}{\cos\theta}\right) \qquad \text{Simplify.}$$

$$=\sin\theta\left(\frac{1}{\sin\theta}-\sin\theta\right) \qquad \text{Simplify.}$$

$$=1-\sin^2\theta \qquad \text{Distributive Property}$$

$$=\cos^2\theta$$

The answer is C.

Test-Taking Tip

Verify your answer by choosing values for θ. Then evaluate the original expression and compare to your answer choice.

CHECK Your Progress

2. $\tan^2\theta(\cot^2\theta-\cos^2\theta)=$

F $\cot^2\theta$ **G** $\tan^2\theta$ **H** $\cos^2\theta$ **J** $\sin^2\theta$

Transform Both Sides of an Equation Sometimes it is easier to transform both sides of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

- Substitute one or more basic trigonometric identities to simplify an expression.
- Factor or multiply to simplify an expression.
- Multiply both the numerator and denominator by the same trigonometric expression.
- Write both sides of the identity in terms of sine and cosine only. Then simplify each side as much as possible.

EXAMPLE 3 Verify by Transforming Both Sides

Verify that $\sec^2 \theta - \tan^2 \theta = \tan \theta \cot \theta$ is an identity.

$\sec^2 \theta - \tan^2 \theta \stackrel{?}{=} \tan \theta \cot \theta$	Original equation
$\dfrac{1}{\cos^2 \theta} - \dfrac{\sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \dfrac{\sin \theta}{\cos \theta} \cdot \dfrac{\cos \theta}{\sin \theta}$	Express all terms using sine and cosine.
$1 - \dfrac{\sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} 1$	Subtract on the left. Multiply on the right.
$\dfrac{\cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} 1$	$1 - \sin^2 \theta = \cos^2 \theta$
$1 = 1$	Simplify the left side.

CHECK Your Progress

3. Verify that $\csc^2 \theta - \cot^2 \theta = \cot \theta \tan \theta$ is an identity.

CHECK Your Understanding

Examples 1, 3 (pp. 842, 844)

Verify that each of the following is an identity.

1. $\tan \theta (\cot \theta + \tan \theta) = \sec^2 \theta$

2. $\tan^2 \theta \cos^2 \theta = 1 - \cos^2 \theta$

3. $\dfrac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$

4. $\dfrac{1 + \tan^2 \theta}{\csc^2 \theta} = \tan^2 \theta$

5. $\dfrac{\sin \theta}{\sec \theta} = \dfrac{1}{\tan \theta + \cot \theta}$

6. $\dfrac{\sec \theta + 1}{\tan \theta} = \dfrac{\tan \theta}{\sec \theta - 1}$

Example 2 (p. 843)

7. STANDARDIZED TEST PRACTICE Which expression can be used to form an identity with $\dfrac{\sec \theta + \csc \theta}{1 + \tan \theta}$?

A $\sin \theta$ **B** $\cos \theta$ **C** $\tan \theta$ **D** $\csc \theta$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8–21	1–3

Verify that each of the following is an identity.

8. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1$

9. $\cot \theta (\cot \theta + \tan^2 \theta) = \csc^2 \theta$

10. $1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta$

11. $\sin \theta \sec \theta \cot \theta = 1$

12. $\dfrac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2$

13. $\dfrac{1 - 2\cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta$

14. $\cot \theta \csc \theta = \dfrac{\cot \theta + \csc \theta}{\sin \theta + \tan \theta}$

15. $\sin \theta + \cos \theta = \dfrac{1 + \tan \theta}{\sec \theta}$

16. $\dfrac{\sec \theta}{\sin \theta} - \dfrac{\sin \theta}{\cos \theta} = \cot \theta$

17. $\dfrac{\sin \theta}{1 - \cos \theta} + \dfrac{1 - \cos \theta}{\sin \theta} = 2 \csc \theta$

18. Verify that $\tan \theta \sin \theta \cos \theta \csc^2 \theta = 1$ is an identity.

19. Show that $1 + \cos \theta$ and $\dfrac{\sin^2 \theta}{1 - \cos \theta}$ form an identity.

Real-World Link

Model rocketry was developed during the "space-race" era. The rockets are constructed of cardboard, plastic, and balsa wood, and are fueled by single-use rocket motors.

PHYSICS **For Exercises 20 and 21, use the following information.**
If an object is propelled from ground level, the maximum height that it reaches is given by $h = \frac{v^2 \sin^2 \theta}{2g}$, where θ is the angle between the ground and the initial path of the object, v is the object's initial velocity, and g is the acceleration due to gravity, 9.8 meters per second squared.

20. Verify the identity $\frac{v^2 \sin^2 \theta}{2g} = \frac{v^2 \tan^2 \theta}{2g \sec^2 \theta}$.

21. A model rocket is launched with an initial velocity of 110 meters per second at an angle of 80° with the ground. Find the maximum height of the rocket.

Verify that each of the following is an identity.

22. $\frac{1 + \sin \theta}{\sin \theta} = \frac{\cot^2 \theta}{\csc \theta - 1}$

23. $\frac{1 + \tan \theta}{1 + \cot \theta} = \frac{\sin \theta}{\cos \theta}$

24. $\frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta} = 1$

25. $1 + \frac{1}{\cos \theta} = \frac{\tan^2 \theta}{\sec \theta - 1}$

26. $1 - \tan^4 \theta = 2 \sec^2 \theta - \sec^4 \theta$

27. $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta$

28. $\frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$

29. $\frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta$

Graphing Calculator

VERIFYING TRIGONOMETRIC IDENTITIES You can determine whether or not an equation may be a trigonometric identity by graphing the expressions on either side of the equals sign as two separate functions. If the graphs do not match, then the equation is not an identity. If the two graphs do coincide, the equation *might* be an identity. The equation has to be verified algebraically to ensure that it is an identity.

Determine whether each of the following *may be* or *is not* an identity.

30. $\cot x + \tan x = \csc x \cot x$

31. $\sec^2 x - 1 = \sin^2 x \sec^2 x$

32. $(1 + \sin x)(1 - \sin x) = \cos^2 x$

33. $\frac{1}{\sec x \tan x} = \csc x - \sin x$

34. $\frac{\sec^2 x}{\tan x} = \sec x \csc x$

35. $\frac{1}{\sec x} + \frac{1}{\csc x} = 1$

EXTRA PRACTICE
See pages 923, 939.

Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

36. OPEN ENDED Write a trigonometric equation that is *not* an identity. Explain how you know it is not an identity.

37. Which One Doesn't Belong? Identify the equation that does not belong with the other three. Explain your reasoning.

$\sin^2 \theta + \cos^2 \theta = 1$ $\qquad$ $1 + \cot^2 \theta = \csc^2 \theta$

$\sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta$ $\qquad$ $\tan^2 \theta + 1 = \sec^2 \theta$

38. CHALLENGE Present a logical argument for why the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ is true when $0 \le x \le 1$.

39. *Writing in Math* Use the information on pages 842 and 843 to explain why you cannot perform operations to each side of an unverified identity and explain why you cannot use the graphs of two expressions to verify an identity.

STANDARDIZED TEST PRACTICE

40. ACT/SAT Which of the following is not equivalent to $\cos\theta$?

A $\dfrac{\cos\theta}{\cos^2\theta + \sin^2\theta}$

B $\dfrac{1 - \sin^2\theta}{\cos\theta}$

C $\cot\theta \sin\theta$

D $\tan\theta \csc\theta$

41. REVIEW Which of the following is equivalent to $\sin\theta + \cot\theta\cos\theta$?

F $2\sin\theta$

G $\dfrac{1}{\sin\theta}$

H $\cos^2\theta$

J $\dfrac{\sin\theta + \cos\theta}{\sin^2\theta}$

Spiral Review

Find the value of each expression. (Lesson 14-3)

42. $\sec\theta$, if $\tan\theta = \frac{1}{2}$; $0° < \theta < 90°$

43. $\cos\theta$, if $\sin\theta = -\frac{2}{3}$; $180° < \theta < 270°$

44. $\csc\theta$, if $\cot\theta = -\frac{7}{12}$; $90° < \theta < 180°$

45. $\sin\theta$, if $\cos\theta = \frac{3}{4}$; $270° < \theta < 360°$

State the amplitude, period, and phase shift of each function. Then graph each function. (Lesson 14-2)

46. $y = \cos(\theta - 30°)$

47. $y = \sin(\theta - 45°)$

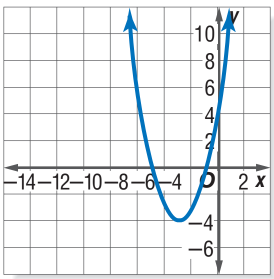
48. $y = 3\cos\left(\theta + \frac{\pi}{2}\right)$

49. COMMUNICATIONS The carrier wave for a certain FM radio station can be modeled by the equation $y = A\sin(10^7 \cdot 2\pi t)$, where A is the amplitude of the wave and t is the time in seconds. Determine the period of the carrier wave. (Lesson 14-1)

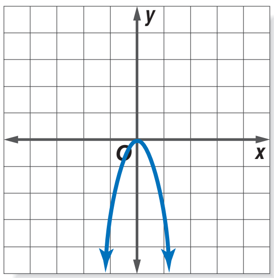
50. BUSINESS A company estimates that it costs $0.03x^2 + 4x + 1000$ dollars to produce x units of a product. Find an expression for the average cost per unit. (Lesson 6-3)

Use the related graph of each equation to determine its solutions. (Lesson 5-2)

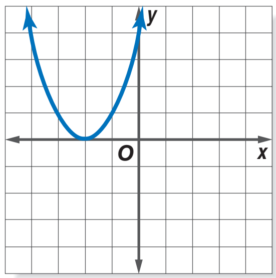
51. $y = x^2 + 6x + 5$



52. $y = -3x^2$



53. $y = x^2 + 4x - 4$



GET READY for the Next Lesson

PREREQUISITE SKILL Simplify each expression. (Lessons 7-5)

54. $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

55. $\frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

56. $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}$

57. $\frac{1}{2} - \frac{\sqrt{3}}{4}$

CHAPTER 14

Mid-Chapter Quiz

Lessons 14-1 through 14-4

1. Find the amplitude and period of $y = \frac{3}{4}\sin\frac{1}{2}\theta$. Then graph the function. (Lesson 14-1)

POPULATION **For Exercises 2–4 use the following information.**

The population of a certain species of deer can be modeled by the function $p = 30{,}000 + 20{,}000\cos\left(\frac{\pi}{10}t\right)$, where p is the population and t is the time in years. (Lesson 14-1)

2. What is the amplitude of the population and what does it represent?

3. What is the period of the function and what does it represent?

4. Graph the function.

5. **MULTIPLE CHOICE** Find the amplitude, if it exists, and period of $y = 3\cot\left(-\frac{1}{4}\theta\right)$. (Lesson 14-1)

A $3; \frac{\pi}{4}$

B $3; 4\pi$

C not defined; 4π

D not defined; $\frac{\pi}{4}$

For Exercises 6–9, consider the function $y = 2\cos\left[\frac{1}{4}\left(\theta - \frac{\pi}{4}\right)\right] - 5$. (Lesson 14-2)

6. State the vertical shift.

7. State the amplitude and period.

8. State the phase shift.

9. Graph the function.

10. **PENDULUM** The position of the pendulum on a particular clock can be modeled using a sine equation. The period of the pendulum is 2 seconds and the phase shift is 0.5 second. The pendulum swings 6 inches to either side of the center position. Write an equation to represent the position of the pendulum p at time t seconds. Assume that the x-axis represents the center line of the pendulum's path, that the area above the x-axis represents a swing to the right, and that the pendulum swings to the right first. (Lesson 14-2)

Find the value of each expression. (Lesson 14-3)

11. $\cos\theta$, if $\sin\theta = \frac{4}{5}$; $90^\circ < \theta < 180^\circ$

12. $\csc\theta$, if $\cot\theta = -\frac{2}{3}$; $270^\circ < \theta < 360^\circ$

13. $\sec\theta$, if $\tan\theta = \frac{1}{2}$; $0^\circ < \theta < 90^\circ$

14. **SWINGS** Amy takes her cousin to the park to swing while she is babysitting. The horizontal force that Amy uses to push her cousin can be found using the formula $F = Mg\tan\theta$, where F is the force, M is the mass of the child, g is gravity, and θ is the angle that the swing makes with it's resting position. Write an equivalent expressing using $\sin\theta$ and $\sec\theta$. (Lesson 14-3)

15. **MULTIPLE CHOICE** Which of the following is equivalent to $\frac{1 - \sin^2\theta}{1 - \cos^2\theta} \cdot \tan\theta$? (Lesson 14-3)

F $\tan\theta$

G $\cot\theta$

H $\sin\theta$

J $\cos\theta$

Verify that each of the following is an identity. (Lesson 14-4)

16. $\tan^2\theta + 1 = \frac{\tan\theta}{\cos\theta \cdot \sin\theta}$

17. $\frac{\sin\theta \cdot \sec\theta}{\sec\theta - 1} = (\sec\theta + 1)\cot\theta$

18. $\sin^2\theta \cdot \tan^2\theta = \tan^2\theta - \sin^2\theta$

19. $\cot\theta(1 - \cos\theta) = \frac{\cos\theta \cdot \sin\theta}{1 + \cos\theta}$

20. **OPTICS** If two prisms of the same power are placed next to each other, their total power can be determined using the formula $z = 2p\cos\theta$ where z is the combined power of the prisms, p is the power of the individual prisms, and θ is the angle between the two prisms. Verify the identity $2p\cos\theta = 2p(1 - \sin^2\theta)\sec\theta$. (Lesson 14-4)

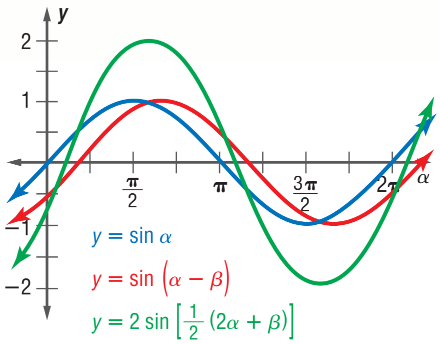
14-5 Sum and Differences of Angles Formulas

Main Ideas

- Find values of sine and cosine involving sum and difference formulas.
- Verify identities by using sum and difference formulas.

GET READY for the Lesson

Have you ever been talking on a cell phone and temporarily lost the signal? Radio waves that pass through the same place at the same time cause interference. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude.



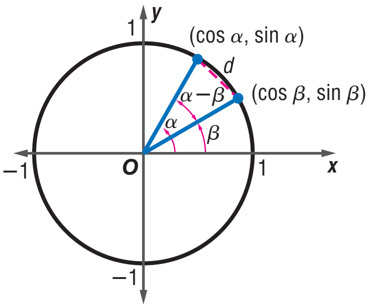
Sum and Difference Formulas Notice that the third equation shown above involves the sum of α and β. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find sin 15° by evaluating sin (60° − 45°). Formulas can be developed that can be used to evaluate expressions like $\sin(\alpha - \beta)$ or $\cos(\alpha + \beta)$.

Reading Math

Greek Letters The Greek letter *beta*, β, can be used to denote the measure of an angle.

It is important to realize that $\sin(\alpha \pm \beta)$ is not the same as $\sin \alpha \pm \sin \beta$.

The figure at the right shows two angles α and β in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos \beta, \sin \beta)$ and $(x_2, y_2) = (\cos \alpha, \sin \alpha)$.



$$d = \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2}$$

$$d^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$$

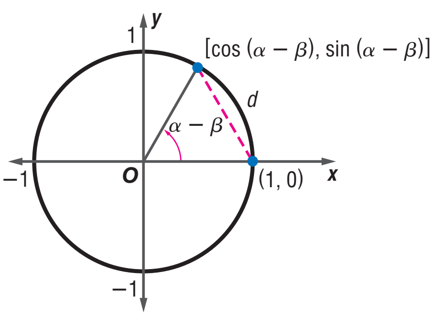
$$d^2 = (\cos^2 \alpha - 2\cos \alpha \cos \beta + \cos^2 \beta) + (\sin^2 \alpha - 2\sin \alpha \sin \beta + \sin^2 \beta)$$

$$d^2 = \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta$$

$$d^2 = 1 + 1 - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \quad \sin^2 \alpha + \cos^2 \alpha = 1 \text{ and}$$

$$d^2 = 2 - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \quad \sin^2 \beta + \cos^2 \beta = 1$$

Now find the value of d^2 when the angle having measure $\alpha - \beta$ is in standard position on the unit circle, as shown in the figure at the left.



$$d = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$$

$$\begin{aligned} d^2 &= [\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2 \\ &= [\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1] + \sin^2(\alpha - \beta) \\ &= \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 \\ &= 1 - 2\cos(\alpha - \beta) + 1 \\ &= 2 - 2\cos(\alpha - \beta) \end{aligned}$$

By equating the two expressions for d^2, you can find a formula for $\cos(\alpha - \beta)$.

$$\begin{aligned} d^2 &= d^2 \\ 2 - 2\cos(\alpha - \beta) &= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\ -1 + \cos(\alpha - \beta) &= -1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Divide each side by } -2. \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Add 1 to each side.} \end{aligned}$$

Use the formula for $\cos(\alpha - \beta)$ to find a formula for $\cos(\alpha + \beta)$.

$$\begin{aligned} \cos(\alpha - \beta) &= \cos[\alpha - (-\beta)] \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \cos(-\beta) = \cos\beta;\ \sin(-\beta) = -\sin\beta \end{aligned}$$

You can use a similar method to find formulas for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

KEY CONCEPT *Sum and Difference of Angles Formulas*

The following identities hold true for all values of α and β.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Notice the symbol $\mp$ in the formula for $\cos(\alpha \pm \beta)$. It means "minus or plus." In the cosine formula, when the sign on the left side of the equation is plus, the sign on the right side is minus; when the sign on the left side is minus, the sign on the right side is plus. The signs match each other in the sine formula.

EXAMPLE 1 Use Sum and Difference of Angles Formulas

Find the exact value of each expression.

a. $\cos 75°$

Use the formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\begin{aligned} \cos 75° &= \cos(30° + 45°) && \alpha = 30°, \beta = 45° \\ &= \cos 30°\cos 45° - \sin 30°\sin 45° \\ &= \left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right) && \text{Evaluate each expression.} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} && \text{Multiply.} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} && \text{Simplify.} \end{aligned}$$

b. $\sin(-210°)$

Use the formula $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$$\begin{aligned} \sin(-210°) &= \sin(60° - 270°) && \alpha = 60°, \beta = 270° \\ &= \sin 60°\cos 270° - \cos 60°\sin 270° \\ &= \left(\frac{\sqrt{3}}{2}\right)(0) - \left(\frac{1}{2}\right)(-1) && \text{Evaluate each expression.} \\ &= 0 - \left(-\frac{1}{2}\right) \text{ or } \frac{1}{2} && \text{Simplify.} \end{aligned}$$

Extra Examples at algebra2.com

CHECK Your Progress

1A. $\sin 15°$

1B. $\cos(-15°)$

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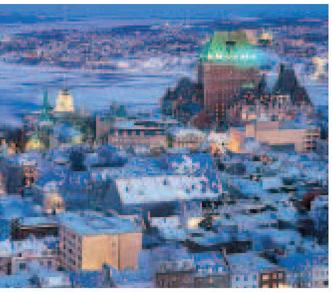
Reading Math

Greek Letters The symbol ϕ is the lowercase Greek letter *phi*.



Real-World Link

In the northern hemisphere, the day with the least number of hours of daylight is December 21 or 22, the first day of winter.

Source: www.infoplease.com

Real-World EXAMPLE

2 **PHYSICS** **On June 22, the maximum amount of light energy falling on a square foot of ground at a location in the northern hemisphere is given by $E \sin(113.5° - \phi)$, where ϕ is the latitude of the location and E is the amount of light energy when the Sun is directly overhead. Use the difference of angles formula to determine the amount of light energy in Rochester, New York, located at a latitude of 43.1° N.**

Use the difference formula for sine.

$$\begin{aligned}\sin(113.5° - \phi) &= \sin 113.5° \cos\phi - \cos 113.5° \sin\phi \\ &= \sin 113.5° \cos 43.1° - \cos 113.5° \sin 43.1° \\ &= 0.9171 \cdot 0.7302 - (-0.3987) \cdot 0.6833 \\ &= 0.9420\end{aligned}$$

In Rochester, New York, the maximum light energy per square foot is $0.9420E$.

CHECK Your Progress

2. Determine the amount of light energy in West Hollywood, California, which is located at a latitude of 34.1° N.

Verify Identities You can also use the sum and difference formulas to verify identities.

EXAMPLE Verify Identities

3 **Verify that each of the following is an identity.**

a. $\sin(180° + \theta) = -\sin\theta$

$$\begin{aligned}\sin(180° + \theta) &\stackrel{?}{=} -\sin\theta && \text{Original equation} \\ \sin 180° \cos\theta + \cos 180° \sin\theta &\stackrel{?}{=} -\sin\theta && \text{Sum of angles formula} \\ 0\cos\theta + (-1)\sin\theta &\stackrel{?}{=} -\sin\theta && \text{Evaluate each expression.} \\ -\sin\theta &= -\sin\theta && \text{Simplify.}\end{aligned}$$

b. $\cos(180° + \theta) = -\cos\theta$

$$\begin{aligned}\cos(180° + \theta) &\stackrel{?}{=} -\cos\theta && \text{Original equation} \\ \cos 180° \cos\theta - \sin 180° \sin\theta &\stackrel{?}{=} -\cos\theta && \text{Sum of angles formula} \\ (-1)\cos\theta - 0\sin\theta &\stackrel{?}{=} -\cos\theta && \text{Evaluate each expression.} \\ -\cos\theta &= -\cos\theta && \text{Simplify.}\end{aligned}$$

CHECK Your Progress

3A. $\sin(90° - \theta) = \cos\theta$

3B. $\cos(90° + \theta) = -\sin\theta$

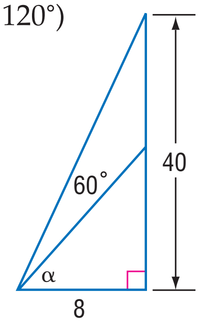
CHECK Your Understanding

Example 1 (pp. 849–850)

Find the exact value of each expression.

1. sin 75° **2.** sin 165° **3.** cos 255°

4. cos (–30°) **5.** sin (–240°) **6.** cos (–120°)



Example 2 (p. 850)

7. GEOMETRY Determine the exact value of tan α in the figure.

Example 3 (p. 850)

Verify that each of the following is an identity.

8. $\cos(270° - \theta) = -\sin\theta$

9. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

10. $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos\theta$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–24	1
25–28	2
29–36	3

Find the exact value of each expression.

11. sin 135° **12.** cos 105° **13.** sin 285° **14.** cos 165°

15. cos 195° **16.** sin 255° **17.** cos 225° **18.** sin 315°

19. sin (–15°) **20.** cos (–45°) **21.** cos (–150°) **22.** sin (–165°)

PHYSICS For Exercises 23–26, use the following information.

On December 22, the maximum amount of light energy that falls on a square foot of ground at a certain location is given by $E \sin(113.5° + \phi)$, where ϕ is the latitude of the location. Find the amount of light energy, in terms of E, for each location.

23. Salem, OR (Latitude: 44.9° N) **24.** Chicago, IL (Latitude: 41.8° N)

25. Charleston, SC (Latitude: 28.5° N) **26.** San Diego, CA (Latitude 32.7° N)

Verify that each of the following is an identity.

27. $\sin(270° - \theta) = -\cos\theta$ **28.** $\cos(90° + \theta) = -\sin\theta$

29. $\cos(90° - \theta) = \sin\theta$ **30.** $\sin(90° - \theta) = \cos\theta$

31. $\sin\left(\theta + \frac{3\pi}{2}\right) = -\cos\theta$ **32.** $\cos(\pi - \theta) = -\cos\theta$

33. $\cos(2\pi + \theta) = \cos\theta$ **34.** $\sin(\pi - \theta) = \sin\theta$

COMMUNICATION For Exercises 35 and 36, use the following information.

A radio transmitter sends out two signals, one for voice communication and another for data. Suppose the equation of the voice wave is $v = 10\sin(2t - 30°)$ and the equation of the data wave is $d = 10\cos(2t + 60°)$.

35. Draw a graph of the waves when they are combined.

36. Refer to the application at the beginning of the lesson. What type of interference results? Explain.

EXTRA PRACTICE
See pages 923, 939.

Math Online
Self-Check Quiz at algebra2.com

Verify that each of the following is an identity.

37. $\sin(60° + \theta) + \sin(60° - \theta) = \sqrt{3}\cos\theta$ **38.** $\sin\left(\theta + \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{6}\right) = \sin\theta$

39. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$ **40.** $\cos(\alpha + \beta) = \dfrac{1 - \tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

H.O.T. Problems

41. OPEN ENDED Give a counterexample to the statement that $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$ is an identity.

42. REASONING Determine whether $\cos(\alpha - \beta) < 1$ is *sometimes*, *always*, or *never* true. Explain your reasoning.

43. CHALLENGE Use the sum and difference formulas for sine and cosine to derive formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$.

44. *Writing in Math* Use the information on page 848 to explain how the sum and difference formulas are used to describe communication interference. Include an explanation of the difference between constructive and destructive interference.

STANDARDIZED TEST PRACTICE

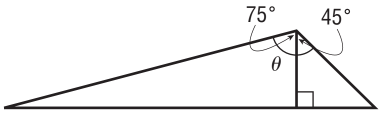
45. ACT/SAT Find the exact value of $\sin \theta$.

A $\frac{\sqrt{3}}{2}$

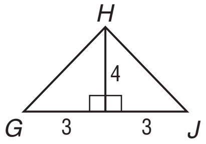
B $\frac{\sqrt{2}}{2}$

C $\frac{1}{2}$

D $\frac{\sqrt{3}}{3}$



46. REVIEW Refer to the figure below. Which equation could be used to find $m\angle G$?



F $\sin G = \frac{3}{4}$

G $\cos G = \frac{3}{4}$

H $\cot G = \frac{3}{4}$

J $\tan G = \frac{3}{4}$

Spiral Review

Verify that each of the following is an identity. (Lesson 14-4)

47. $\cot \theta + \sec \theta = \frac{\cos^2 \theta + \sin \theta}{\sin \theta \cos \theta}$

48. $\sin^2 \theta + \tan^2 \theta = (1 - \cos^2 \theta) + \frac{\sec^2 \theta}{\csc^2 \theta}$

49. $\sin \theta(\sin \theta + \csc \theta) = 2 - \cos^2 \theta$

50. $\frac{\sec \theta}{\tan \theta} = \csc \theta$

Simplify each expression. (Lesson 14-3)

51. $\frac{\tan \theta \csc \theta}{\sec \theta}$

52. $4\left(\sec^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta}\right)$

53. $(\cot \theta + \tan \theta)\sin \theta$

54. $\csc \theta \tan \theta + \sec \theta$

55. AVIATION A pilot is flying from Chicago to Columbus, a distance of 300 miles. In order to avoid an area of thunderstorms, she alters her initial course by 15° and flies on this course for 75 miles. How far is she from Columbus? (Lesson 13-5)

56. Write $6y^2 - 34x^2 = 204$ in standard form. (Lesson 10-6)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 5-5)

57. $x^2 = \frac{20}{16}$

58. $x^2 = \frac{9}{25}$

59. $x^2 = \frac{5}{25}$

60. $x^2 = \frac{18}{32}$

14-6 Double-Angle and Half-Angle Formulas

Main Ideas

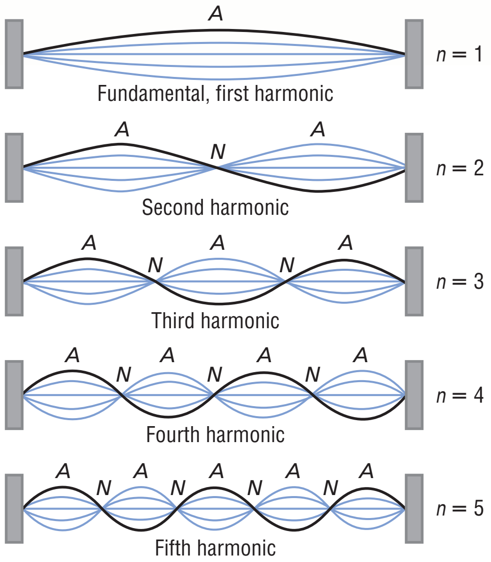
- Find values of sine and cosine involving double-angle formulas.
- Find values of sine and cosine involving half-angle formulas.

New Vocabulary

double-angle formulas
half-angle formula

GET READY for the Lesson

Stringed instruments such as a piano, guitar, or violin rely on waves to produce the tones we hear. When the strings are struck or plucked, they vibrate. If the motion of the strings were observed in slow motion, you could see that there are places on the string, called *nodes*, that do not move under the vibration. Halfway between each pair of consecutive nodes are *antinodes* that undergo the maximum vibration. The nodes and antinodes form *harmonics*. These harmonics can be represented using variations of the equations $y = \sin 2\theta$ and $y = \sin \frac{1}{2}\theta$.



Double-Angle Formulas You can use the formula for $\sin(\alpha + \beta)$ to find the sine of twice an angle θ, $\sin 2\theta$, and the formula for $\cos(\alpha + \beta)$ to find the cosine of twice an angle θ, $\cos 2\theta$.

$$\begin{aligned} \sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta \end{aligned}$$

$$\begin{aligned} \cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ &= \cos^2\theta - \sin^2\theta \end{aligned}$$

You can find alternate forms for $\cos 2\theta$ by making substitutions into the expression $\cos^2\theta - \sin^2\theta$.

$$\begin{aligned} \cos^2\theta - \sin^2\theta &= (1 - \sin^2\theta) - \sin^2\theta && \text{Substitute } 1 - \sin^2\theta \text{ for } \cos^2\theta. \\ &= 1 - 2\sin^2\theta && \text{Simplify.} \end{aligned}$$

$$\begin{aligned} \cos^2\theta - \sin^2\theta &= \cos^2\theta - (1 - \cos^2\theta) && \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta. \\ &= 2\cos^2\theta - 1 && \text{Simplify.} \end{aligned}$$

These formulas are called the **double-angle formulas**.

KEY CONCEPT — *Double-Angle Formulas*

The following identities hold true for all values of θ.

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\sin^2\theta - 1$

EXAMPLE 1 Double-Angle Formulas

Find the exact value of each expression if $\sin \theta = \frac{4}{5}$ and θ is between 90° and 180°.

a. $\sin 2\theta$

Use the identity $\sin 2\theta = 2 \sin \theta \cos \theta$.

First, find the value of $\cos \theta$.

$\cos^2 \theta = 1 - \sin^2 \theta$ — $\cos^2 \theta + \sin^2 \theta = 1$

$\cos^2 \theta = 1 - \left(\frac{4}{5}\right)^2$ — $\sin \theta = \frac{4}{5}$

$\cos^2 \theta = \frac{9}{25}$ — Subtract.

$\cos \theta = \pm\frac{3}{5}$ — Find the square root of each side.

Since θ is in the second quadrant, cosine is negative. Thus, $\cos \theta = -\frac{3}{5}$.

Now find $\sin 2\theta$.

$\sin 2\theta = 2 \sin \theta \cos \theta$ — Double-angle formula

$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$ — $\sin \theta = \frac{4}{5}$, $\cos \theta = -\frac{3}{5}$

$= -\frac{24}{25}$ — Multiply.

The value of $\sin 2\theta$ is $-\frac{24}{25}$.

b. $\sin 2\theta$

Use the identity $\cos 2\theta = 1 - 2 \sin^2 \theta$.

$\cos 2\theta = 1 - 2 \sin^2 \theta$ — Double-angle formula

$= 1 - 2\left(\frac{4}{5}\right)^2$ — $\sin \theta = \frac{4}{5}$

$= -\frac{7}{25}$ — Simplify.

The value of $\cos 2\theta$ is $-\frac{7}{25}$.

CHECK Your Progress

Find the exact value of each expression if $\cos = -\frac{1}{3}$ and $90° < \theta < 180°$.

1A. $\sin 2\theta$ **1B.** $\cos 2\theta$

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Half-Angle Formulas You can derive formulas for the sine and cosine of half a given angle using the double-angle formulas.

Find $\sin \frac{\alpha}{2}$.

$1 - 2 \sin^2 \theta = \cos 2\theta$ — Double-angle formula

$1 - 2 \sin^2 \frac{\alpha}{2} = \cos \alpha$ — Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}$ — Solve for $\sin^2 \frac{\alpha}{2}$.

$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$ — Take the square root of each side.

Find $\cos \frac{\alpha}{2}$.

$2 \cos^2 \theta - 1 = \cos 2\theta$ — Double-angle formula

$2 \cos^2 \frac{\alpha}{2} - 1 = \cos \alpha$ — Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}$ — Solve for $\cos^2 \frac{\alpha}{2}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ — Take the square root of each side.

These are called the **half-angle formulas**. The signs are determined by the function of $\frac{\alpha}{2}$.

KEY CONCEPT — *Half-Angle Formulas*

The following identities hold true for all values of α.

$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$ $\qquad$ $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$

EXAMPLE Half-Angle Formulas

2 Find $\cos \frac{\alpha}{2}$ if $\sin \alpha = -\frac{3}{4}$ and α is in the third quadrant.

Since $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$, we must find $\cos \alpha$ first.

$\cos^2 \alpha = 1 - \sin^2 \alpha$ — $\cos^2 \alpha + \sin^2 \alpha = 1$

$\cos^2 \alpha = 1 - \left(-\frac{3}{4}\right)^2$ — $\sin \alpha = -\frac{3}{4}$

$\cos^2 \alpha = \frac{7}{16}$ — Simplify.

$\cos \alpha = \pm\frac{\sqrt{7}}{4}$ — Take the square root of each side.

Since α is in the third quadrant, $\cos \alpha = \frac{\sqrt{7}}{4}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ — Half-angle formula

$= \pm\sqrt{\frac{1 - \frac{\sqrt{7}}{4}}{2}}$ — $\cos \alpha = -\frac{\sqrt{7}}{4}$

$= \pm\sqrt{\frac{4 - \sqrt{7}}{8}}$ — Simplify the radicand.

$= \pm\frac{\sqrt{4 - \sqrt{7}}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ — Rationalize.

$= \pm\frac{\sqrt{8 - 2\sqrt{7}}}{4}$ — Multiply.

Since α is between 180° and 270°, $\frac{\alpha}{2}$ is between 90° and 135°. Thus, $\cos \frac{\alpha}{2}$ is negative and equals $-\frac{\sqrt{8 - 2\sqrt{7}}}{4}$.

CHECK Your Progress

2. Find $\sin \frac{\alpha}{2}$ if $\sin \alpha = \frac{2}{3}$ and α is in the 2nd quadrant.

Study Tip

Choosing the Sign

You may want to determine the quadrant in which the terminal side of $\frac{\alpha}{2}$ will lie in the first step of the solution. Then you can use the correct sign from the beginning.

EXAMPLE 3 Evaluate Using Half-Angle Formulas

Find the exact value of each expression by using the half-angle formulas.

a. $\sin 105°$

$$\sin 105° = \sin \frac{210°}{2}$$

$$= \sqrt{\frac{1 - \cos 210°}{2}} \qquad \sin \frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \sqrt{\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{2}} \qquad \cos 210° = -\frac{\sqrt{3}}{2}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}} \qquad \text{Simplify the radicand.}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2} \qquad \text{Simplify the denominator.}$$

b. $\cos \frac{\pi}{8}$

$$\cos \frac{\pi}{8} = \frac{\frac{\pi}{4}}{2}$$

$$= \sqrt{\frac{1 + \cos\frac{\pi}{4}}{2}} \qquad \cos \frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$= \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}} \qquad \text{Simplify the radicand.}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2} \qquad \text{Simplify the denominator.}$$

CHECK Your Progress

3A. $\sin 135°$

3B. $\cos \frac{7\pi}{8}$

Recall that you can use the sum and difference formulas to verify identities. Double- and half-angle formulas can also be used to verify identities.

EXAMPLE 4 Verify Identities

Verify that $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$ is an identity.

$$(\sin\theta + \cos\theta)^2 \stackrel{?}{=} 1 + \sin 2\theta \qquad \text{Original equation}$$

$$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \stackrel{?}{=} 1 + \sin 2\theta \qquad \text{Multiply.}$$

$$1 + 2\sin\theta\cos\theta \stackrel{?}{=} 1 + \sin 2\theta \qquad \sin^2\theta + \cos^2\theta = 1$$

$$1 + \sin 2\theta = 1 + \sin 2\theta \qquad \text{Double-angle formula}$$

CHECK Your Progress

4. Verify that $4\cos^2 x - \sin^2 2x = 4\cos^4 x$

Check Your Understanding

Examples 1, 2 (pp. 854–855)

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

1. $\cos \theta = \frac{3}{5}$; $0° < \theta < 90°$

2. $\cos \theta = -\frac{2}{3}$; $180° < \theta < 270°$

3. $\sin \theta = \frac{1}{2}$; $0° < \theta < 90°$

4. $\sin \theta = -\frac{3}{4}$; $270° < \theta < 360°$

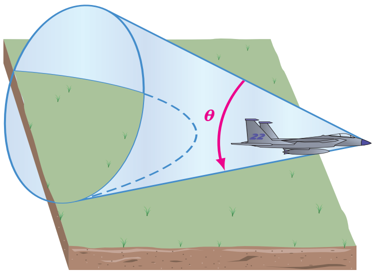
Example 3 (p. 856)

Find the exact value of each expression by using the half-angle formulas.

5. $\sin 195°$

6. $\cos \frac{19\pi}{12}$

7. AVIATION When a jet travels at speeds greater than the speed of sound, a sonic boom is created by the sound waves forming a cone behind the jet. If θ is the measure of the angle at the vertex of the cone, then the Mach number M can be determined using the formula $\sin \frac{\theta}{2} = \frac{1}{M}$. Find the Mach number of a jet if a sonic boom is created by a cone with a vertex angle of 75°.



Example 4 (p. 856)

Verify that each of the following is an identity.

8. $\cot x = \frac{\sin 2x}{1 - \cos 2x}$

9. $\cos^2 2x + 4 \sin^2 x \cos^2 x = 1$

Exercises

HOMEWORK	
For Exercises	**See Examples**
10–15	1, 2
16–21	3
22–27	4

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

10. $\sin \theta = \frac{5}{13}$; $90° < \theta < 180°$

11. $\cos \theta = \frac{1}{5}$; $270° < \theta < 360°$

12. $\cos \theta = -\frac{1}{3}$; $180° < \theta < 270°$

13. $\sin \theta = -\frac{3}{5}$; $180° < \theta < 270°$

14. $\sin \theta = -\frac{3}{8}$; $270° < \theta < 360°$

15. $\cos \theta = -\frac{1}{4}$; $90° < \theta < 180°$

Find the exact value of each expression by using the half-angle formulas.

16. $\cos 165°$

17. $\sin 22\frac{1}{2}°$

18. $\cos 157\frac{1}{2}°$

19. $\sin 345°$

20. $\sin \frac{7\pi}{8}$

21. $\cos \frac{7\pi}{12}$

Verify that each of the following is an identity.

22. $\sin 2x = 2 \cot x \sin^2 x$

23. $2 \cos^2 \frac{x}{2} = 1 + \cos x$

24. $\sin^4 x - \cos^4 x = 2 \sin^2 x - 1$

25. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

26. $\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$

27. $\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} = \tan x$

PHYSICS **For Exercises 28 and 29, use the following information.**

An object is propelled from ground level with an initial velocity of v at an angle of elevation θ.

28. The horizontal distance d it will travel can be determined using the formula $d = \frac{v^2 \sin 2\theta}{g}$, where g is the acceleration due to gravity. Verify that this expression is the same as $\frac{2}{g}v^2(\tan \theta - \tan \theta \sin^2 \theta)$.

29. The maximum height h the object will reach can be determined using the formula $d = \frac{v^2 \sin^2 \theta}{2g}$. Find the ratio of the maximum height attained to the horizontal distance traveled.

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

30. $\cos \theta = \frac{1}{6}$; $0° < \theta < 90°$

31. $\cos \theta = -\frac{12}{13}$; $180° < \theta < 270°$

32. $\sin \theta = -\frac{1}{3}$; $270° < \theta < 360°$

33. $\sin \theta = -\frac{1}{4}$; $180° < \theta < 270°$

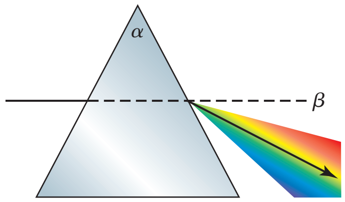
34. $\cos \theta = \frac{2}{3}$; $0° < \theta < 90°$

35. $\sin \theta = \frac{2}{5}$; $90° < \theta < 180°$

Real-World Link

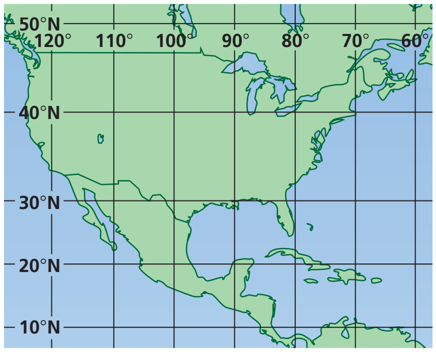
A rainbow appears when the sun shines through water droplets that act as a prism.

36. **OPTICS** If a glass prism has an apex angle of measure α and an angle of deviation of measure β, then the index of refraction n of the prism is given by $n = \frac{\sin\left[\frac{1}{2}(\alpha + \beta)\right]}{\sin \frac{\alpha}{2}}$. What is the angle of deviation of a prism with an apex angle of 40° and an index of refraction of 2?



GEOGRAPHY **For Exercises 37 and 38, use the following information.**

A Mercator projection map uses a flat projection of Earth in which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection uses the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.



37. Write this expression in terms of a trigonometric function of L.

38. Find the exact value of the expression if $L = 60°$.

EXTRA PRACTICE
See pages 924, 939.

Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. **REASONING** Explain how to find $\cos \frac{x}{2}$ if x is in the third quadrant.

40. **REASONING** Describe the conditions under which you would use each of the three identities for $\cos 2\theta$.

41. **OPEN ENDED** Find a counterexample to show that $\cos 2\theta = 2 \cos \theta$ is not an identity.

42. ***Writing in Math*** Use the information on page 853 to explain how trigonometric functions can be used to describe music. Include a description of what happens to the graph of the function of a vibrating string as it moves from one harmonic to the next and an explanation of what happens to the period of the function as you move from the nth harmonic to the $(n + 1)$th harmonic.

STANDARDIZED TEST PRACTICE

43. **ACT/SAT** Find the exact value of $\cos 2\theta$ if $\sin \theta = \frac{-\sqrt{5}}{3}$ and $180° < \theta < 270°$.

A $\frac{-\sqrt{6}}{6}$

B $\frac{-\sqrt{30}}{6}$

C $\frac{-4\sqrt{5}}{9}$

D $\frac{-1}{9}$

44. **REVIEW** Which of the following is equivalent to $\frac{\cos \theta (\cot^2 \theta + 1)}{\csc \theta}$?

F $\tan \theta$

G $\cot \theta$

H $\sec \theta$

J $\csc \theta$

Spiral Review

Find the exact value of each expression. (Lesson 14-5)

45. $\cos 15°$ **46.** $\sin 15°$ **47.** $\sin (-135°)$

48. $\cos 150°$ **49.** $\sin 105°$ **50.** $\cos (-300°)$

Verify that each of the following is an identity. (Lesson 14-4)

51. $\cot^2 \theta - \sin^2 \theta = \frac{\cos^2 \theta \csc^2 \theta - \sin^2 \theta}{\sin^2 \theta \csc^2 \theta}$

52. $\cos \theta (\cos \theta + \cot \theta) = \cot \theta \cos \theta (\sin \theta + 1)$

ANALYZE TABLES **For Exercises 53 and 54, use the following information.**

The magnitude of an earthquake M measured on the Richter scale is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion. (Lesson 9-2)

Strongest Earthquakes in 20th Century

Location, Year	Magnitude
Chile, 1960	9.5
Alaska, 1964	9.2
Russia, 1952	9.0
Ecuador, 1906	8.8
Alaska, 1957	8.8
Kuril Islands, 1958	8.7
Alaska, 1965	8.7
India, 1950	8.6
Chile, 1922	8.5
Indonesia, 1938	8.5

Source: U.S. Geological Survey

53. How many times as great was the 1960 Chile earthquake as the 1938 Indonesia earthquake?

54. The largest aftershock of the 1964 Alaskan earthquake was 6.7 on the Richter scale. How many times as great was the main earthquake as this aftershock?

Write each expression in quadratic form, if possible. (Lesson 6-6)

55. $a^8 - 7a^4 + 13$ **56.** $5n^7 + 3n - 3$ **57.** $d^6 + 2d^3 + 10$

Find each value if $f(x) = x^2 - 7x + 5$. (Lesson 2-1)

58. $f(2)$ **59.** $f(0)$ **60.** $f(-3)$ **61.** $f(n)$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 5-3)

62. $(x + 6)(x - 5) = 0$ **63.** $(x - 1)(x + 1) = 0$

64. $x(x + 2) = 0$ **65.** $(2x - 5)(x + 2) = 0$

66. $(2x + 1)(2x - 1) = 0$ **67.** $x^2(2x + 1) = 0$

EXPLORE 14-7

Graphing Calculating Lab

Solving Trigonometric Equations

The graph of a trigonometric function is made up of points that represent all values that satisfy the function. To solve a trigonometric equation, you need to find all values of the variable that satisfy the equation. You can use a TI-83/84 Plus to solve trigonometric equations by graphing each side of the equation as a function and then locating the points of intersection.

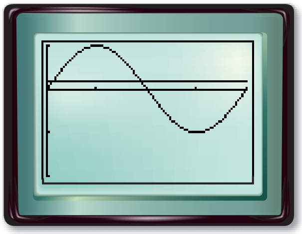
ACTIVITY 1

Use a graphing calculator to solve sin x = 0.2 if 0° ≤ x < 360°.

Rewrite the equation as two functions, $y = \sin x$ and $y = 0.2$. Then graph the two functions. Look for the point of intersection.

Make sure that your calculator is in degree mode to get the correct viewing window.

KEYSTROKES: MODE ▼ ▼ ▶ ENTER WINDOW 0 ENTER 360 ENTER 90 ENTER –2 ENTER 1 ENTER 1 ENTER Y= SIN X,T,θ,n ENTER 0.2 ENTER GRAPH



[0, 360] scl: 90 by [−2, 1] scl: 1

Based on the graph, you can see that there are two points of intersection in the interval 0°≤ x < 360°. Use ZOOM or 2nd [CALC] 5 to approximate the solutions. The approximate solutions are 168.5° and 11.5°.

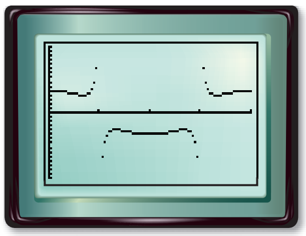
Like other equations you have studied, some trigonometric equations have no real solutions. Carefully examine the graphs over their respective periods for points of intersection. If there are no points of intersection, then the trigonometric equation has no real solutions.

ACTIVITY 2

Use a graphing calculator to solve $\tan^2 x \cos x + 5 \cos x = 0$ if 0° ≤ x < 360°.

Because the tangent function is not continuous, place the calculator in Dot mode. The related functions to be graphed are $y = \tan^2 x \cos x + 5 \cos x$ and $y = 0$.

These two functions do not intersect. Therefore, the equation $\tan^2 x \cos x + 5 \cos x = 0$ has no real solutions.



[0, 360] scl: 90 by [−15, 15] scl: 1

Exercises

Use a graphing calculator to solve each equation for the values of x indicated.

1. $\sin x = 0.8$ if $0° \le x < 360°$

2. $\tan x = \sin x$ if $0° \le x < 360°$

3. $2 \cos x + 3 = 0$ if $0° \le x < 360°$

4. $0.5 \cos x = 1.4$ if $-720° \le x < 720°$

5. $\sin 2x = \sin x$ if $0° \le x < 360°$

6. $\sin 2x - 3 \sin x = 0$ if $-360° \le x < 360°$

Other Calculator Keystrokes at algebra2.com

14-7 Solving Trigonometric Equations

Main Ideas

- Solve trigonometric equations.
- Use trigonometric equations to solve real-world problems.

New Vocabulary

trigonometric equations

GET READY for the Lesson

The average daily high temperature for a region can be described by a trigonometric function. For example, the average daily high temperature for each month in Orlando, Florida, can be modeled by the function $T = 11.56 \sin (0.4516x - 1.641) + 80.89$, where T represents the average daily high temperature in degrees Fahrenheit and x represents the month of the year. This equation can be used to predict the months in which the average temperature in Orlando will be at or above a desired temperature.

Solve Trigonometric Equations You have seen that trigonometric identities are true for *all* values of the variable for which the equation is defined. However, most **trigonometric equations**, like some algebraic equations, are true for *some* but not *all* values of the variable.

EXAMPLE 1 Solve Equations for a Given Interval

Find all solutions of $\sin 2\theta = 2 \cos \theta$ for the interval $0 \le \theta < 360°$.

$\sin 2\theta = 2 \cos \theta$ — Original equation

$2 \sin \theta \cos \theta = 2 \cos \theta$ — $\sin 2\theta = 2 \sin \theta \cos \theta$

$2 \sin \theta \cos \theta - 2 \cos \theta = 0$ — Solve for 0.

$2 \cos \theta (\sin \theta - 1) = 0$ — Factor.

Use the Zero Product Property.

$2 \cos \theta = 0$ or $\sin \theta - 1 = 0$

$\cos \theta = 0$ $\qquad$ $\sin \theta = 1$

$\theta = 90°$ or $270°$ $\qquad$ $\theta = 90°$

The solutions are 90° and 270°.

CHECK Your Progress

1. Find all solutions of $\cos^2 \theta = 1$ for the interval $0° \le \theta < 360°$.

SuperStock

Trigonometric equations are usually solved for values of the variable between 0° and 360° or 0 radians and 2π radians. There are solutions outside that interval. These other solutions differ by integral multiples of the period of the function.

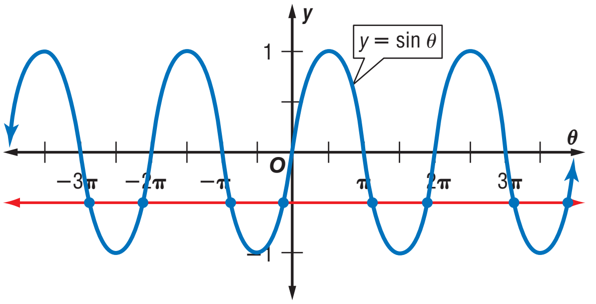
EXAMPLE Solve Trigonometric Equations

2 **Solve $2 \sin \theta = -1$ for all values of θ if θ is measured in radians.**

$2 \sin \theta = -1$ Original equation

$\sin \theta = -\frac{1}{2}$ Divide each side by 2.

Look at the graph of $y = \sin \theta$ to find solutions of $\sin \theta = -\frac{1}{2}$.



The solutions are $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$, and so on, and $\frac{-7\pi}{6}, \frac{-11\pi}{6}, \frac{-19\pi}{6}, \frac{-23\pi}{6}$, and so on. The only solutions in the interval 0 to 2π are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The period of the sine function is 2π radians. So the solutions can be written as $\frac{7\pi}{6} + 2k\pi$ and $\frac{11\pi}{6} + 2k\pi$, where k is any integer.

Study Tip

Expressing Solutions as Multiples

The expression $\frac{\pi}{2} + k \cdot \pi$ includes $\frac{3\pi}{2}$ and its multiples, so it is not necessary to list them separately,

CHECK Your Progress

2. Solve for $\cos 2\theta + \cos \theta + 1 = 0$ for all values of θ if θ is measured in degrees.

If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

EXAMPLE Solve Trigonometric Equations Using Identities

3 **Solve $\cos \theta \tan \theta - \sin^2 \theta = 0$.**

$\cos \theta \tan \theta - \sin^2 \theta = 0$ Original equation

$\cos \theta \left(\frac{\sin \theta}{\cos \theta}\right) - \sin^2 \theta = 0$ $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$\sin \theta - \sin^2 \theta = 0$ Multiply.

$\sin \theta (1 - \sin \theta) = 0$ Factor.

$\sin \theta = 0$ or $1 - \sin \theta = 0$

$\theta = 0°, 180°, \text{ or } 360°$ $\quad \sin \theta = 1$

$\theta = 90°$

CHECK

$$\cos\theta\tan\theta-\sin^2\theta=0$$
$$\cos 0°\tan 0°-\sin^2 0° \stackrel{?}{=} 0 \quad \theta=0°$$
$$1\cdot 0-0 \stackrel{?}{=} 0$$
$$0=0 \quad \text{true}$$

$$\cos\theta\tan\theta-\sin^2\theta=0$$
$$\cos 180°\tan 180°-\sin^2 180° \stackrel{?}{=} 0 \quad \theta=180°$$
$$-1\cdot 0-0 \stackrel{?}{=} 0$$
$$0=0 \quad \text{true}$$

$$\cos\theta\tan\theta-\sin^2\theta=0$$
$$\cos 360°\tan 360°-\sin^2 360° \stackrel{?}{=} 0 \quad \theta=360°$$
$$1\cdot 0-0 \stackrel{?}{=} 0$$
$$0=0 \quad \text{true}$$

$$\cos\theta\tan\theta-\sin^2\theta=0$$
$$\cos 90°\tan 90°-\sin^2 90° \stackrel{?}{=} 0 \quad \theta=90°$$

tan 90° is undefined.

Thus, 90° is not a solution.

The solution is $0° + k \cdot 180°$.

CHECK Your Progress

Solve each equation.

3A. $\sin\theta\cot\theta-\cos^2\theta=0$ **3B.** $\dfrac{\cos\theta}{\cot\theta}+2\sin^2\theta=0$

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Some trigonometric equations have no solution. For example, the equation $\cos x = 4$ has no solution since all values of $\cos x$ are between -1 and 1, inclusive. Thus, the solution set for $\cos x = 4$ is empty.

EXAMPLE 4 Determine Whether a Solution Exists

Solve $3\cos 2\theta - 5\cos\theta = 1$.

$3\cos 2\theta - 5\cos\theta = 1$ Original equation

$3(2\cos^2\theta-1) - 5\cos\theta = 1$ $\cos 2\theta = 2\cos^2\theta - 1$

$6\cos^2\theta - 3 - 5\cos\theta = 1$ Multiply.

$6\cos^2\theta - 5\cos\theta - 4 = 0$ Subtract 1 from each side.

$(3\cos\theta-4)(2\cos\theta+1) = 0$ Factor.

$3\cos\theta - 4 = 0$ or $2\cos\theta + 1 = 0$

$3\cos\theta = 4$ $\quad 2\cos\theta = -1$

$\cos\theta = \frac{4}{3}$ $\quad \cos\theta = -\frac{1}{2}$

Not possible since $\cos\theta$ cannot be greater than 1. $\quad \theta = 120°$ or $240°$

Thus, the solutions are $120° + k\cdot 360°$ and $240° + k\cdot 360°$.

CHECK Your Progress **Solve each equation.**

4A. $\sin^2\theta + 2\cos^2\theta = 4$ **4B.** $\cos^2\theta - 3 = 4 - \sin^2\theta$

Use Trigonometric Equations Trigonometric equations are often used to solve real-world situations.

Real-World EXAMPLE

5 **GARDENING** **Rhonda wants to wait to plant her flowers until there are at least 14 hours of daylight. The number of hours of daylight H in her town can be represented by $H = 11.45 + 6.5 \sin (0.0168d - 1.333)$, where d is the day of the year and angle measures are in radians. On what day is it safe for Rhonda to plant her flowers?**

$H = 11.45 + 6.5 \sin (0.0168d - 1.333)$	Original equation
$14 = 11.45 + 6.5 \sin (0.0168d - 1.333)$	$H = 14$
$2.55 = 6.5 \sin (0.0168d - 1.333)$	Subtract 11.45 from each side.
$0.392 = \sin (0.0168d - 1.333)$	Divide each side by 6.5.
$0.403 = 0.0168d - 1.333$	$\sin^{-1} 0.392 = 0.403$
$1.736 = 0.0168d$	Add 1.333 to each side.
$103.333 = d$	Divide each side by 0.0168.

Rhonda can safely plant her flowers around the 104th day of the year, or around April 14.

CHECK Your Progress

5. If Rhonda decides to wait only until there are 12 hours of daylight, on what day is it safe for her to plant her flowers?

CHECK Your Understanding

Example 1 (p. 861)

Find all solutions of each equation for the given interval.

1. $4 \cos^2 \theta = 1; 0° \leq \theta < 360°$

2. $2 \sin^2 \theta - 1 = 0; 90° < \theta < 270°$

3. $\sin 2\theta = \cos \theta; 0 \leq \theta < 2\pi$

4. $3 \sin^2 \theta - \cos^2 \theta = 0; 0 \leq \theta < \frac{\pi}{2}$

Example 2 (p. 862)

Solve each equation for all values of θ if θ is measured in radians.

5. $\cos 2\theta = \cos \theta$

6. $\sin \theta + \sin \theta \cos \theta = 0$

Solve each equation for all values of θ if θ is measured in degrees.

7. $\sin \theta = 1 + \cos \theta$

8. $2 \cos^2 \theta + 2 = 5 \cos \theta$

Examples 3, 4 (pp. 862–863)

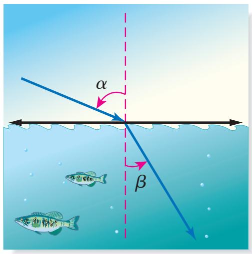
Solve each equation for all values of θ.

9. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

10. $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$

Example 5 (p. 864)

11. **PHYSICS** According to Snell's law, the angle at which light enters water α is related to the angle at which light travels in water β by the equation $\sin \alpha = 1.33 \sin \beta$. At what angle does a beam of light enter the water if the beam travels at an angle of 23° through the water?



Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–15	1
16–23	2
24–27	3, 4
28, 29	5

Find all solutions of each equation for the given interval.

12. $2 \cos \theta - 1 = 0$; $0° \leq \theta < 360°$

13. $2 \sin \theta = -\sqrt{3}$; $180° < \theta < 360°$

14. $4 \sin^2 \theta = 1$; $180° < \theta < 360$

15. $4 \cos^2 \theta = 3$; $0° \leq \theta < 360°$

Solve each equation for all values of θ if θ is measured in radians.

16. $\cos 2\theta + 3 \cos \theta - 1 = 0$

17. $2 \sin^2 \theta - \cos \theta - 1 = 0$

18. $\cos^2 \theta - \frac{5}{2} \cos \theta - \frac{3}{2} = 0$

19. $\cos \theta = 3 \cos \theta - 2$

Solve each equation for all values of θ if θ is measured in degrees.

20. $\sin \theta = \cos \theta$

21. $\tan \theta = \sin \theta$

22. $\sin^2 \theta - 2 \sin \theta - 3 = 0$

23. $4 \sin^2 \theta - 4 \sin \theta + 1 = 0$

Solve each equation for all values of θ.

24. $\sin^2 \theta + \cos 2\theta - \cos \theta = 0$

25. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

26. $\sin^2 \theta = \cos^2 \theta - 1$

27. $2 \cos^2 \theta + \cos \theta = 0$

WAVES **For Exercises 28 and 29, use the following information.**

After a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$, where h is the maximum height of the wave in feet, P is the period in seconds, and t is the propagation of the wave in seconds.

28. If $h = 3$ and $P = 2$, write the equation for the wave and draw its graph over a 10-second interval.

29. How many times over the first 10 seconds does the graph predict the wave to be one foot high?

Find all solutions of each equation for the given interval.

30. $2 \cos^2 \theta = \sin \theta + 1$; $0 \leq \theta < 2\pi$

31. $\sin^2 \theta - 1 = \cos^2 \theta$; $0 \leq \theta < \pi$

32. $2 \sin^2 \theta + \sin \theta = 0$; $\pi < \theta < 2\pi$

33. $2 \cos^2 \theta = -\cos \theta$; $0 \leq \theta < 2\pi$

Solve each equation for all values of θ if θ is measured in radians.

34. $4 \cos^2 \theta - 4 \cos \theta + 1 = 0$

35. $\cos 2\theta = 1 - \sin \theta$

36. $(\cos \theta)(\sin 2\theta) - 2 \sin \theta + 2 = 0$

37. $2 \sin^2 \theta + (\sqrt{2} - 1) \sin \theta = \frac{\sqrt{2}}{2}$

Solve each equation for all values of θ if θ is measured in degrees.

38. $\tan^2 \theta - \sqrt{3} \tan \theta = 0$

39. $\cos^2 \theta - \frac{7}{2} \cos \theta - 2 = 0$

40. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3} \sin \theta + \cos \theta$

41. $1 - \sin^2 \theta - \cos \theta = \frac{3}{4}$

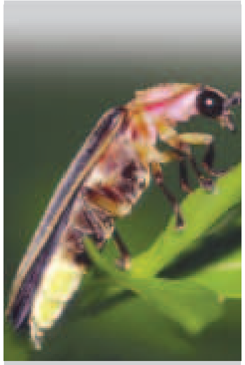
Solve each equation for all values of θ.

42. $\sin \frac{\theta}{2} + \cos \theta = 1$

43. $\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}$

44. $2 \sin \theta = \sin 2\theta$

45. $\tan^2 \theta + \sqrt{3} = \left(1 + \sqrt{3}\right) \tan \theta$



Real-World Link

Fireflies are bioluminescent, which means that they produce light through a biochemical reaction. Almost 100% of a firefly's energy is given off as light.

Source: www.nfs.gov

LIGHT **For Exercises 46 and 47, use the following information.**

The height of the International Peace Memorial at Put-in-Bay, Ohio, is 352 feet.

46. The length of the shadow S of the Memorial depends upon the angle of inclination of the Sun, θ. Express S as a function of θ.

47. Find the angle of inclination θ that will produce a shadow 560 feet long.

H.O.T. Problems

EXTRA PRACTICE
See pages 924, 939.

Math Online
Self-Check Quiz at
algebra2.com

48. OPEN ENDED Write an example of a trigonometric equation that has no solution.

49. REASONING Explain why the equation $\sec \theta = 0$ has no solutions.

50. CHALLENGE Computer games often use transformations to distort images on the screen. In one such transformation, an image is rotated counterclockwise using the equations $x' = x \cos \theta - y \sin \theta$ and $y' = x \sin \theta + y \cos \theta$. If the coordinates of an image point are (3, 4) after a 60° rotation, what are the coordinates of the preimage point?

51. REASONING Explain why the number of solutions to the equation $\sin \theta = \frac{\sqrt{3}}{2}$ is infinite.

52. ***Writing in Math*** Use the information on page 861 to explain how trigonometric equations can be used to predict temperature. Include an explanation of why the sine function can be used to model the average daily temperature and an explanation of why, during one period, you might find a specific average temperature twice.

STANDARDIZED TEST PRACTICE

53. ACT/SAT Which of the following is *not* a possible solution of $0 = \sin \theta + \cos \theta \tan^2 \theta$?

A $\frac{3\pi}{4}$

B $\frac{7\pi}{4}$

C 2π

D $\frac{5\pi}{2}$

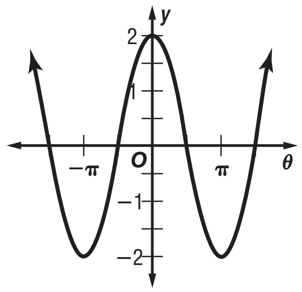
54. REVIEW The graph of the equation $y = 2 \cos \theta$ is shown. Which is a solution for $2 \cos \theta = 1$?

F $\frac{8\pi}{3}$

G $\frac{13\pi}{3}$

H $\frac{10\pi}{3}$

J $\frac{15\pi}{3}$



Spiral Review

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following. (Lesson 14-6)

55. $\sin \theta = \frac{3}{5}$; $0° < \theta < 90°$

56. $\cos \theta = \frac{1}{2}$; $0° < \theta < 90°$

57. $\cos \theta = \frac{5}{6}$; $0° < \theta < 90°$

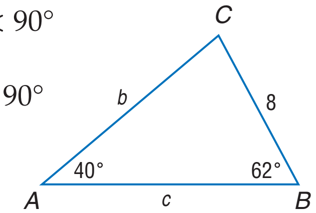
58. $\sin \theta = \frac{4}{5}$; $0° < \theta < 90°$

Find the exact value of each expression. (Lesson 14-5)

59. $\sin 240°$

60. $\cos 315°$

61. $\sin 150°$



62. Solve $\triangle ABC$. Round measures of sides and angles to the nearest tenth. (Lesson 13-4)

Cross-Curricular Project

Algebra and Physics

So you want to be a rocket scientist? It is time to complete your project. Use the information and data you have gathered about the applications of trigonometry to prepare a poster, report, or Web page. Be sure to include graphs, tables, or diagrams in the presentation.

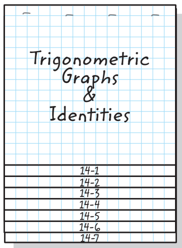
Math Online **Cross-Curricular Project at** algebra2.com

CHAPTER 14 Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer — GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Graphing Trigonometric Functions (Lesson 14-1)

- For trigonometric functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.
- The period of $y = a \tan b\theta$ is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.

Translations of Trigonometric Graphs (Lesson 14-2)

- For trigonometric functions of the form $y = a \sin (\theta - h) + k$, $y = a \cos (\theta - h) + k$, $y = a \tan (\theta - h) + k$, the phase shift is to the right when h is positive and to the left when h is negative. The vertical shift is up when k is positive and down when k is negative.

Trigonometric Identities

(Lessons 14-3, 14-4, and 14-7)

- Trigonometric identities describe the relationships between trigonometric functions.
- Trigonometric identities can be used to simplify, verify, and solve trigonometric equations and expressions.

Sum and Difference of Angles Formulas

(Lesson 14-5)

- For all values of α and β:

$$\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$
$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

Double-Angle and Half-Angle Formulas

(Lesson 14-6)

- Double-angle formulas:

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$
$$\cos 2\theta = 1 - 2 \sin^2 \theta$$
$$\cos 2\theta = 2 \cos^2 \theta - 1$$

- Half-angle formulas:

$$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$$
$$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$$

Key Vocabulary

amplitude (p. 823)
difference of angles formula (p. 849)
double-angle formula (p. 853)
half-angle formula (p. 855)
midline (p. 831)
phase shift (p. 829)
sum of angles formula (p. 849)
trigonometric equation (p. 861)
trigonometric identity (p. 837)
vertical shift (p. 831)

Vocabulary Check

Choose the correct term from the list above to compete each sentence.

1. The horizontal translation of a trigonometric function is a(n) ________.
2. A reference line about which a graph oscillates is a(n) ________.
3. The vertical translation of a trigonometric function is called a(n) ________.
4. The ________ formula can be used to find $\cos 22\frac{1}{2}°$.
5. The ________ can be used to find sin 60° using 30° as a reference.
6. The ________ can be used to find the sine or cosine of 75° if the sine and cosine of 45° and 30° are known.
7. A(n) ________ is an equation that is true for all values for which every expression in the equation is defined
8. The ________ can be used to find the sine or cosine of 65° if the sine and cosine of 90° and 25° are known.
9. The absolute value of half the difference between the maximum value and the minimum value of a periodic function is called the ________.

Math Online Vocabulary Review at algebra2.com

CHAPTER 14

Study Guide and Review

Lesson-by-Lesson Review

14–1 Graphing Trigonometric Functions (pp. 822–828)

Find the amplitude, if it exists, and period of each function. Then graph each function.

10. $y = -\frac{1}{2}\cos\theta$

11. $y = 4\sin 2\theta$

12. $y = \sin\frac{1}{2}\theta$

13. $y = 5\sec\theta$

14. $y = \frac{1}{2}\csc\frac{2}{3}\theta$

15. $y = \tan 4\theta$

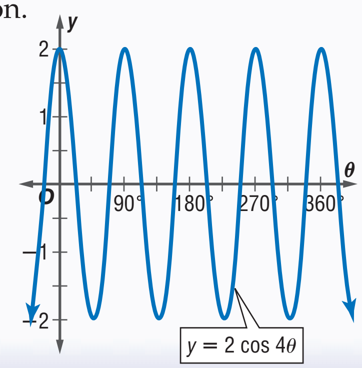
16. MECHANICS The position of a piston can be modeled using the equation $y = A\sin\left(\frac{1}{4}\cdot 2\pi t\right)$ where A is the amplitude of oscillation and t is the time in seconds. Determine the period of oscillation.

Example 1 **Find the amplitude and period of $y = 2\cos 4\theta$. Then graph.**

The amplitude is $|2|$ or 2.

The period is $\frac{360°}{|4|}$ or 90°.

Use the amplitude and period to graph the function.



14–2 Translations of Trigonometric Graphs (pp. 829–836)

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

17. $y = \frac{1}{2}\sin[2(\theta - 60°)] - 1$

18. $y = 2\tan\left[\frac{1}{4}(\theta - 90°)\right] + 3$

19. $y = 3\sec\left[\frac{1}{2}\left(\theta + \frac{\pi}{4}\right)\right] + 1$

20. $y = \frac{1}{3}\cos\left[\frac{1}{3}\left(\theta - \frac{2\pi}{3}\right)\right] - 2$

21. BIOLOGY The population of a species of bees varies periodically over the course of a year. The maximum population of bees occurs in March, and is 50,000. The minimum population of bees occurs in September and is 20,000. Assume that the population can be modeled using the sine function. Write an equation to represent the population of bees p, t months after January.

Example 2 **State the vertical shift, amplitude, period, and phase shift of $y = 3\sin\left[2\left(\theta - \frac{\pi}{2}\right)\right] - 2$. Then graph the function.**

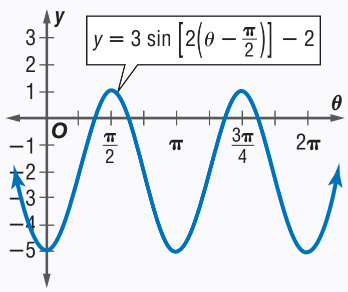
Identify the values of k, a, b, and h.

$k = -2$, so the vertical shift is –2.

$a = 3$, so the amplitude is 3.

$b = 2$, so the period is $\frac{2\pi}{|2|}$ or π.

$h = \frac{\pi}{2}$, so the phase shift is $\frac{\pi}{2}$ to the right.



14-3 Trigonometric Identities (pp. 837–841)

Find the value of each expression.

22. $\cot\theta$, if $\csc\theta = -\frac{5}{3}$; $270^\circ < \theta < 360^\circ$

23. $\sec\theta$, if $\sin\theta = \frac{1}{2}$; $0^\circ \le \theta < 90^\circ$

Simplify each expression.

24. $\sin\theta\csc\theta - \cos^2\theta$

25. $\cos^2\theta\sec\theta\csc\theta$

26. $\cos\theta + \sin\theta\tan\theta$

27. $\sin\theta(1 + \cot^2\theta)$

28. **PHYSICS** The magnetic force on a particle can be modeled by the equation $F = qvB\sin\theta$, where F is the magnetic force, q is the charge of the particle, B is the m agnetic field strength, and θ is the angle between the particle's path and the direction of the magnetic field. Write an equation for the magnetic force in terms of $\tan\theta$ and $\sec\theta$.

Example 3 **Find $\cos\theta$ if $\sin\theta = -\frac{3}{4}$ and $90^\circ < \theta < 180^\circ$.**

$\cos^2\theta + \sin^2\theta = 1$	Trigonometric identity
$\cos^2\theta = 1 - \sin^2\theta$	Subtract $\sin^2\theta$ from each side.
$\cos^2\theta = 1 - \left(\frac{3}{4}\right)^2$	Substitute $\frac{3}{4}$ for $\sin\theta$.
$\cos^2\theta = 1 - \frac{9}{16}$	Square $\frac{3}{4}$.
$\cos^2\theta = \frac{7}{16}$	Subtract.
$\cos\theta = \pm\frac{\sqrt{7}}{4}$	Take the square root of each side.

Since θ is in the second quadrant, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{\sqrt{7}}{4}$.

Example 4 **Simplify $\sin\theta\cot\theta\cos\theta$.**

$\sin\theta\cot\theta\cos\theta$

$= \frac{\sin\theta}{1}\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{1}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$= \cos^2\theta$	Multiply.

14-4 Verifying Trigonometric Identities (pp. 842–846)

Verify that each of the following is an identity.

29. $\frac{\sin\theta}{\tan\theta} + \frac{\cos\theta}{\cot\theta} = \cos\theta + \sin\theta$

30. $\frac{\sin\theta}{1 - \cos\theta} = \csc\theta + \cot\theta$

31. $\cot^2\theta\sec^2\theta = 1 + \cot^2\theta$

32. $\sec\theta(\sec\theta - \cos\theta) = \tan^2\theta$

33. **OPTICS** The amount of light passing through a polarization filter can be modeled using the equation $I = I_m\cos^2\theta$, where I is the amount of light passing through the filter, I_m is the amount of light shined on the filter, and θ is the angle of rotation between the light source and the filter. Verify the identity $I_m\cos^2\theta = I_m - \frac{I_m}{\cos^2\theta + 1}$.

Example 5 **Verify that $\tan\theta + \cot\theta = \sec\theta\csc\theta$ is an identity.**

$\tan\theta + \cot\theta \stackrel{?}{=} \sec\theta\csc\theta$	Original equation
$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$	Rewrite using the LCD, $\cos\theta\sin\theta$.
$\frac{1}{\cos\theta\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$	$\sin^2\theta + \cos^2\theta = 1$
$\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$	Rewrite as the product of two expressions.
$\sec\theta\csc\theta = \sec\theta\csc\theta$	$\frac{1}{\cos\theta} = \sec\theta$, $\frac{1}{\sin\theta} = \csc\theta$

CHAPTER 14

Study Guide and Review

14–5 Sum and Difference of Angles Formula (pp. 848–852)

Find the exact value of each expression.

34. $\cos 15°$
35. $\cos 285°$
36. $\sin 150°$
37. $\sin 195°$
38. $\cos(-210°)$
39. $\sin(-105°)$

Verify that each of the following is an identity.

40. $\cos(90° + \theta) = -\sin\theta$
41. $\sin(30° - \theta) = \cos(60° + \theta)$
42. $\sin(\theta + \pi) = -\sin\theta$
43. $-\cos\theta = (\cos\pi + \theta)$

Example 6 Find the exact value of $\sin 195°$.

$\sin 195° = \sin(150° + 45°)$ $\quad$ $195° = 150° + 45°$

$= \sin 150° \cos 45° + \cos 150° \sin 45°$ $\quad$ $\alpha = 150°, \beta = 45°$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ $\quad$ Evaluate each expression.

$= \frac{\sqrt{2} - \sqrt{6}}{4}$ $\quad$ Simplify.

14–6 Double-Angle and Half-Angle Formulas (pp. 853–859)

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

44. $\sin\theta = \frac{1}{4}$; $0° < \theta < 90°$
45. $\sin\theta = -\frac{5}{13}$; $180° < \theta < 270°$
46. $\cos\theta = -\frac{5}{17}$; $90° < \theta < 180°$
47. $\cos\theta = \frac{12}{13}$; $270° < \theta < 360°$

Example 7 Verify that $\csc 2\theta = \frac{\sec\theta}{2\sin\theta}$ is an identity.

$\csc 2\theta \stackrel{?}{=} \frac{\sec\theta}{2\sin\theta}$ $\quad$ Original equation

$\frac{1}{\sin 2\theta} \stackrel{?}{=} \frac{\frac{1}{\cos\theta}}{2\sin\theta}$ $\quad$ $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$

$\frac{1}{\sin 2\theta} \stackrel{?}{=} \frac{1}{2\sin\theta\cos\theta}$ $\quad$ Simplify the complex fraction.

$\frac{1}{\sin 2\theta} = \frac{1}{\sin 2\theta}$ $\quad$ $2\sin\theta\cos\theta = \sin 2\theta$

14–7 Solving Trigonometric Equations (pp. 861–866)

Find all solutions of each equation for the interval $0° \le \theta < 360°$.

48. $2\sin 2\theta = 1$
49. $\cos^2\theta + \sin^2\theta = 2\cos\theta$ $0°$

50. PRISMS The horizontal and vertical components of an oblique prism can be modeled using the equations $Z_x = P\cos\theta$ and $Z_y = P\sin\theta$ where Z_x is the horizontal component, Z_y is the vertical component, P is the power of the prism, and θ is the angle between the prism and the horizontal. For what values of θ will the vertical and horizontal components be equivalent?

Example 8 Solve $\sin 2\theta + \sin\theta = 0$ if $0° \le \theta < 360°$.

$\sin 2\theta + \sin\theta = 0$ $\quad$ Original equation

$2\sin\theta\cos\theta + \sin\theta = 0$ $\quad$ $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta(2\cos\theta + 1) = 0$ $\quad$ Factor.

$\sin\theta = 0$ $\quad$ or $\quad$ $2\cos\theta + 1 = 0$

$\theta = 0°$ or $180°$ $\quad$ $\cos\theta = -\frac{1}{2}$

$\theta = 120°$ or $240°$

The solutions are 0°, 120°, 180°, and 240°.

CHAPTER 14 Practice Test

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

1. $y = \frac{2}{3} \sin 2\theta + 5$

2. $y = 4 \cos \left[\frac{1}{2}(\theta + 30°)\right] - 1$

3. $y = 7 \cos \left[4\left(\theta + \frac{\pi}{6}\right)\right]$

4. **AUTOMOTIVE** The pistons in a car oscillate according to a sine function. The amplitude of the oscillation is 2, the period is 6π, and the phase shift is $\frac{\pi}{2}$ to the left. Write a formula to model the position of the piston, p, at time t seconds. Graph the equation.

Find the value of each expression.

5. $\tan \theta$, if $\sin \theta = \frac{1}{2}$; $90° < \theta < 180°$

6. $\sec \theta$, if $\cot \theta = \frac{3}{4}$; $180° < \theta < 270°$

7. $\csc \theta$, if $\sec \theta = \frac{\sqrt{5}}{2}$; $270° < \theta < 360°$

Verify that each of the following is an identity.

8. $(\sin \theta - \cos \theta)^2 = 1 - \sin 2\theta$

9. $\frac{\cos \theta}{1 - \sin^2 \theta} = \sec \theta$

10. $\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta$

11. $\frac{1 + \tan^2 \theta}{\cos^2 \theta} = \sec^4 \theta$

12. **RACING** Race tracks are designed based on the average car velocity so that the angle of the track prevents sliding in the curves. The equation for the banking angle is $\tan \theta = \frac{v^2}{gr}$ where v is velocity, g is gravity, and r is the radius of the curve. Write an equivalent expression using $\sec \theta$ and $\csc \theta$.

Find the exact value of each expression.

13. $\cos 165°$

14. $\sin 255°$

15. $\sin (-225°)$

16. $\cos 480°$

17. $\cos 67.5°$

18. $\sin 75°$

Solve each equation for all values of θ if θ is measured in degrees.

19. $\sec \theta = 1 + \tan \theta$

20. $\cos 2\theta = \cos \theta$

21. $\cos 2\theta + \sin \theta = 1$

22. $\sin \theta = \tan \theta$

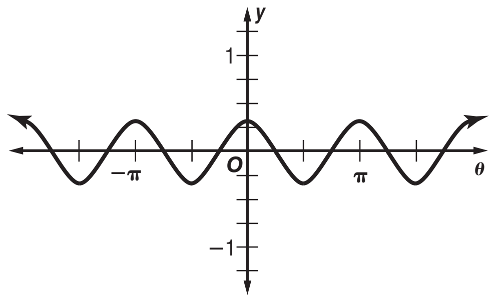
GOLF **For Exercises 23 and 24, use the following information.**

A golf ball leaves the club with an initial velocity of 100 feet per second. The distance the ball travels is found by the formula $d = \frac{v_0^2}{g} \sin 2\theta$, where v_0 is the initial velocity, g is the acceleration due to gravity, and θ is the measurement of the angle that the path of the ball makes with the ground. The acceleration due to gravity is 32 feet per second squared.

23. Find the distance that the ball travels if the angle between the path of the ball and the ground measures 60°.

24. If a ball travels 312.5 feet, what was the angle the path of the ball made with the ground to the nearest degree?

25. **MULTIPLE CHOICE** Identify the equation of the graphed function.



A $y = 3 \cos 2\theta$ **C** $y = 3 \cos \frac{1}{2}\theta$

B $y = \frac{1}{3} \cos 2\theta$ **D** $y = \frac{1}{3} \cos \frac{1}{2}\theta$

Standardized Test Practice

Cumulative, Chapters 1–14

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. A small business owner must hire seasonal workers as the need arises. The following list shows the number of employees hired monthly for a 5-month period.

5, 14, 6, 8, 12

If the mean of this data is 9, what is the population standard deviation for these data? (Round to the nearest tenth.)

A 2.6

B 5.7

C 8.6

D 12.3

2. If $f(x) = 2x^3 + 5x - 8$, find $f(2a^2)$.

F $f(2a^2) = 16a^5 + 10a^2 - 8$

G $f(2a^2) = 64a^5 + 10a^2 - 8$

H $f(2a^2) = 16a^6 + 10a^2 - 8$

J $f(2a^2) = 64a^6 + 10a^2 - 8$

3. Simplify $128^{\frac{1}{4}}$.

A $2\sqrt[4]{2}$

B $2\sqrt[4]{8}$

C 4

D $4\sqrt[4]{2}$

4. Lisa is 6 years younger than Petra. Stella is twice as old as Petra. The total of their ages is 54. Which equation can be used to find Petra's age?

F $x + (x - 6) + 2(x - 6) = 54$

G $x - 6x + (x + 2) = 54$

H $x - 6 + 2x = 54$

J $x + (x - 6) + 2x = 54$

5. **GRIDDABLE** The mean of seven numbers is 0. The sum of three of the numbers is −9. What is the sum of the remaining four numbers?

6. Which of the following functions represents exponential decay?

A $y = 0.2(7)^x$

B $y = (0.5)^x$

C $y = 4(9)^x$

D $y = 5\left(\frac{4}{3}\right)^x$

7. Solve the following system of equations.

$$3y = 4x + 1$$
$$2y - 3x = 2$$

F (−4, −5)

G (−2, −3)

H (2, 3)

J (4, 5)

8. **GRIDDABLE** If k is a positive integer and $7k + 3$ equals a prime number that is less than 50, then what is one possible value of $7k + 3$?

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Preparing for Standardized Tests
For test-taking strategies and more practice, see pages 941–956.

9. Find the center and radius of the circle with the equation $(x - 4)^2 + y^2 - 16 = 0$.

A $C(-4, 0)$; $r = 4$ units

B $C(-4, 0)$; $r = 16$ units

C $C(4, 0)$; $r = 4$ units

D $C(4, 0)$; $r = 16$ units

10. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

F 4

G 6

H 8

J 12

11. What is the effect of the graph on the equation $y = 3x^2$ when the equation is changed to $y = 2x^2$?

A The graph of $y = 2x^2$ is a reflection of the graph of $y = 3x^2$ across the y-axis.

B The graph is rotated 90 degress about the origin.

C The graph is narrower.

D The graph is wider.

TEST-TAKING TIP

Question 11 If the question involves a graph but does not include the graph, draw one. A diagram can help you see relationships among the given values that will help you answer the question.

12. GEOMETRY The perimeter of a right triangle is 36 inches. Twice the length of the longer leg minus twice the length of the shorter leg exceeds the hypotenuse by 6 inches. What are the lengths of all three sides?

F 3 in., 4 in., 5 in.

G 6 in., 8 in., 10 in.

H 9 in., 12 in., 15 in.

J 12 in., 16 in., 20 in.

Pre-AP/Anchor Problem

13. The table below shows the cost of a pizza depending on the diameter of the pizza.

	Dimensions	Cost ($)
Round	10″ diameter	8.10
Round	20″ diameter	15.00
Square	10″ side	10.00
Square	20″ side	20.00

Which pizza should you buy if you want to get the most pizza per dollar?

NEED EXTRA HELP?													
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13
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